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# Integral Means of Analytic Functions

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**Abstract**

For analytic functions $f(z)$ and $g(z)$ which satisfy the subordination $f(z) \prec g(z)$, J. E. Littlewood(Proc. London Math. Soc. **23**(1925), 481-519) has shown some interesting results for integral means of $f(z)$ and $g(z)$. The object of the present paper is to derive some applications of integral means by J. E. Littlewood. We also show interesting examples for our theorems.

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## 1. Introduction

Let $\mathcal{A}_n$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N} := \{1,2,3,\dots\}) \tag{1.1}$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} \,|\, |z| < 1\}$. Let $\mathcal{S}_n^*(\alpha)$ be the subclass of $\mathcal{A}_n$ consistng of all functions $f(z)$ satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}) \tag{1.2}$$

for some $\alpha(0 \leqq \alpha < 1)$. A function $f(z)$ in $\mathcal{S}_n^*(\alpha)$ is said to be starlike of order $\alpha$ in $\mathbb{U}$. Let $\mathcal{K}_n(\alpha)$ denote the subclass of $\mathcal{A}_n$ consisting of functions $f(z)$ which satisfy

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U}) \tag{1.3}$$

for some $\alpha(0 \leqq \alpha < 1)$. A function $f(z)$ belonging to $\mathcal{K}_n(\alpha)$ is called as a convex function of order $\alpha$ in $\mathbb{U}$. Note that $f(z) \in \mathcal{K}_n(\alpha)$ if and only if $zf'(z) \in \mathcal{S}_n^*(\alpha)$.

For the classes $\mathcal{S}_n^*(\alpha)$ and $\mathcal{K}_n(\alpha)$, Chatterjea [1](also see Srivastava, Owa and Chatterjea [9]) has given the following results.

**Theorem A.** *If a function $f(z) \in \mathcal{A}_n$ satisfies*

$$\sum_{k=n+1}^{\infty} (k-\alpha)\ |a_k| \leqq 1-\alpha \tag{1.4}$$

*for some* $\alpha(0 \leqq \alpha < 1)$, *then* $f(z) \in \mathcal{S}_n^*(\alpha)$.

**Theorem B.** *If a function* $f(z) \in \mathcal{A}_n$ satisfies

$$\sum_{k=n+1}^{\infty} k(k-\alpha)\ |a_k| \leqq 1-\alpha. \tag{1.5}$$

*for some* $\alpha(0 \leqq \alpha < 1)$, *then* $f(z) \in \mathcal{K}_n(\alpha)$.

When $n = 1$ in Theorem A and Theorem B, the results for $\mathcal{S}_1^*(\alpha)$ and $\mathcal{K}_1(\alpha)$ above were given by Silverman [7].

For anlytic functions $f(z)$ and $g(z)$, the function $f(z)$ is said to be subordinate to $g(z)$ in $\mathbb{U}$ if there exists a function $w(z)$ analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. We denote this subordination by

$$f(z) \prec g(z) \quad \text{(cf. Duren[2])}.$$

For subordinations, Littltewood [4] has given the following integral mean.

**Theorem C.** *If* $f(z)$ *and* $g(z)$ *are analytic in* $\mathbb{U}$ *with* $f(z) \prec g(z)$, *then, for* $\mu > 0$ *and* $z = re^{i\theta}(0 < r < 1)$

$$\int_0^{2\pi} |f(z)|^{\mu}\, d\theta \leqq \int_0^{2\pi} |g(z)|^{\mu}\, d\theta.$$

Applying the Theorem C by Littlewood [4] above, Silvermann [8], Kim and Choi [3], Sekine,Tsurumi and Srivastava [6], and Owa, Tsurumi, Nunokawa and Sekine [5] have considered some interesting properties for integral means of analytic functions. In the present paper, we discuss some conditions of coefficients for integral means.

## 2. Integral means for $f(z)$ and $g(z)$

In this section, we discuss the integral means for $f(z) \in \mathcal{A}_n$ and $g(z)$ defined by

$$g(z) = z + b_j z^j + b_{2j-1} z^{2j-1} \quad (j \geqq n+1). \tag{2.1}$$

Our first reult for integral means is contained in

**Theorem 2.1** *Let* $f(z) \in \mathcal{A}_n$ *and* $g(z)$ *be given by* (2.1). *If* $f(z)$ *satisfies*

$$\sum_{k=n+1}^{\infty} |a_k| \leqq |b_{2j-1}| - |b_j| \quad (|b_j| < |b_{2j-1}|), \tag{2.2}$$

*then, for* $\mu > 0$ *and* $z = re^{i\theta}(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^{\mu}\, d\theta \leqq \int_0^{2\pi} |g(z)|^{\mu}\, d\theta. \tag{2.3}$$

*Proof.* By putting $z = re^{i\theta}(0 < r < 1)$, we see that

$$\int_0^{2\pi} |f(z)|^\mu d\theta = r^\mu \int_0^{2\pi} \left|1 + \sum_{k=n+1}^{\infty} a_k z^{k-1}\right|^\mu d\theta$$

and

$$\int_0^{2\pi} |g(z)|^\mu d\theta = r^\mu \int_0^{2\pi} \left|1 + b_j z^{j-1} + b_{2j-1} z^{2j-2}\right|^\mu d\theta.$$

Applying Theorem C, we have to show that

$$1 + \sum_{k=n+1}^{\infty} a_k z^{k-1} \prec 1 + b_j z^{j-1} + b_{2j-1} z^{2j-2}.$$

Let us define the function $w(z)$ by

$$1 + \sum_{k=n+1}^{\infty} a_k z^{k-1} = 1 + b_j w(z)^{j-1} + b_{2j-1} w(z)^{2j-2},$$

or, by

$$b_{2j-1} w(z)^{2j-2} + b_j w(z)^{j-1} = \sum_{k=n+1}^{\infty} a_k z^{k-1}. \tag{2.4}$$

Since, for $z = 0$,

$$w(0)^{j-1}\left(b_{2j-1} w(0)^{j-1} + b_j\right) = 0,$$

there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $w(0) = 0$.

Next, we prove the analytic function $w(z)$ satidfies $|w(z)| < 1 (z \in \mathbb{U})$ for

$$\sum_{k=n+1}^{\infty} |a_k| \leqq |b_{2j-1}| - |b_j| \quad (|b_j| < |b_{2j-1}|).$$

By the equality (2.4), we know that

$$\left|b_{2j-1} w(z)^{2j-2} + b_j w(z)^{j-1}\right| \leqq \left|\sum_{k=n+1}^{\infty} a_k z^{k-1}\right| < \sum_{k=n+1}^{\infty} |a_k|,$$

for $z \in \mathbb{U}$, hence,

$$|b_{2j-1}|\,|w(z)|^{2j-2} - |b_j|\,|w(z)|^{j-1} - \sum_{k=n+1}^{\infty} |a_k| < 0. \tag{2.5}$$

Letting $t = |w(z)|^{j-1}$ $(t \geqq 0)$ in (2.5), we define the function $G(t)$ by

$$G(t) = |b_{2j-1}|\,t^2 - |b_j|\,t - \sum_{k=n+1}^{\infty} |a_k| \quad (t \geqq 0).$$

If $G(1) \geqq 0$, then we have $t < 1$ for $G(t) < 0$. Therefore, for $|w(z)| < 1$ $(z \in \mathbb{U})$, we need

$$G(1) = |b_{2j-1}| - |b_j| - \sum_{k=n+1}^{\infty} |a_k| \geqq 0,$$

that is,

$$\sum_{k=n+1}^{\infty} |a_k| \leqq |b_{2j-1}| - |b_j|.$$

Consequently, if the inequality (2.2) holds true, there exists an analytic function $w(z)$ with $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{U})$ such that $f(z) = g(w(z))$. This completes the proof of Theorem 2.1.

**Corollary 2.1.** *Let $f(z) \in \mathcal{A}_n$ and $g(z)$ be given by* (2.1). *If $f(z)$ satisfies* (2.2), *then , for $0 < \mu \leqq 2$ and $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\int_0^{2\pi} |f(z)|^{\mu}\, d\theta \leqq 2\pi r^{\mu} \left\{ 1 + |b_j|^2 r^{2(j-1)} + |b_{2j-1}|^2 r^{4(j-1)} \right\}^{\frac{\mu}{2}}$$
$$< 2\pi \left\{ 1 + |b_j|^2 + |b_{2j-1}|^2 \right\}^{\frac{\mu}{2}}. \tag{2.6}$$

*Further, we have that $f(z) \in \mathcal{H}^p(\mathbb{U})$ for $0 < p \leqq 2$, where $\mathcal{H}^p$ denotes the Hardy space* (cf. Duren [2]).

*Proof.* Since,

$$\int_0^{2\pi} |g(z)|^{\mu} d\theta = \int_0^{2\pi} |z|^{\mu} \left|1 + b_j z^{j-1} + b_{2j-1} z^{2j-2}\right|^{\mu} d\theta,$$

applying Hölder inequality for $0 < \lambda < 2$, we obtain that

$$\begin{aligned}
&\int_0^{2\pi} |g(z)|^{\mu} d\theta \\
&\leqq \left(\int_0^{2\pi} (|z|^{\mu})^{\frac{2}{2-\mu}} d\theta\right)^{\frac{2-\mu}{2}} \left\{\int_0^{2\pi} \left( \left|1 + b_j z^{j-1} + b_{2j-1} z^{2j-2}\right|^{\mu} \right)^{\frac{2}{\mu}} d\theta\right\}^{\frac{\mu}{2}} \\
&= \left(r^{\frac{2\mu}{2-\mu}} \int_0^{2\pi} d\theta\right)^{\frac{2-\mu}{2}} \left(\int_0^{2\pi} \left|1 + b_j z^{j-1} + b_{2j-1} z^{2j-2}\right|^2 d\theta\right)^{\frac{\mu}{2}} \\
&= \left(2\pi r^{\frac{2\mu}{2-\mu}}\right)^{\frac{2-\mu}{2}} \left\{2\pi \left( 1 + |b_j|^2 r^{2(j-1)} + |b_{2j-1}|^2 r^{4(j-1)} \right)\right\}^{\frac{\mu}{2}} \\
&= 2\pi r^{\mu} \left( 1 + |b_j|^2 r^{2(j-1)} + |b_{2j-1}|^2 r^{4(j-1)} \right)^{\frac{\mu}{2}} \\
&< 2\pi \left( 1 + |b_j|^2 + |b_{2j-1}|^2 \right)^{\frac{\mu}{2}}.
\end{aligned}$$

Further, it is easy to see that, for $\mu = 2$,

$$\begin{aligned}
\int_0^{2\pi} |f(z)|^2\, d\theta &\leqq 2\pi r^2 \left( 1 + |b_j|^2 r^{2j-1} + |b_{2j-1}|^2 r^{4(j-1)} \right) \\
&< 2\pi \left( 1 + |b_j|^2 + |b_{2j-1}|^2 \right).
\end{aligned}$$

From the above, we also have that, for $0 < \mu \leqq 2$,

$$\sup_{z \in \mathbb{U}} \frac{1}{2\pi} \int_0^{2\pi} |f(z)|^{\mu} \, d\theta < \left( 1 + |b_j|^2 + |b_{2j-1}|^2 \right)^{\frac{\mu}{2}} < \infty,$$

which observe that $f(z) \in \mathcal{H}^2(\mathbb{U})$. Noting that $\mathcal{H}^q \subset \mathcal{H}^p$ $(0 < p < q < \infty)$, we complete the proof.

**Example 2.1.** Let $f(z) \in \mathcal{A}_n$ satisfy the cefficient inequality (1.4) in Theorem A and

$$g(z) = z + \frac{n}{n+1-\alpha} \epsilon z^j + \delta z^{2j-1} \quad (|\epsilon| = |\delta| = 1) \tag{2.7}$$

with $0 \leqq \alpha < 1$. Then $b_j = (n\epsilon)/(n+1-\alpha)$ and $b_{2j-1} = \delta$.
By virtue of (1.4), we observe that

$$\sum_{k=n+1}^{\infty} |a_k| \leqq \frac{1-\alpha}{n+1-\alpha} = 1 - \frac{n}{n+1-\alpha} = |b_{2j-1}| - |b_j|.$$

Therefore, $f(z)$ and $g(z)$ satisfy the conditions in Theorem 2.1. Thus, we have, for $0 < \mu \leqq 2$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\begin{aligned} &\int_0^{2\pi} |f(z)|^{\mu} d\theta \\ &= 2\pi r^{\mu} \left\{ 1 + \left( \frac{n}{n+1-\alpha} \right)^2 r^{2(j-1)} + r^{4(j-1)} \right\}^{\frac{\mu}{2}} \\ &< 2\pi \left\{ 2 + \left( \frac{n}{n+1-\alpha} \right)^2 \right\}^{\frac{\mu}{2}}. \end{aligned}$$

Using the same technique as in the proof of Theorem 2.1, we derive the following theorem.

**Theorem 2.2.** *Let $f(z) \in \mathcal{A}_n$ and $g(z)$ be given by (2.1). If $f(z)$ saisfies*

$$\sum_{k=n+1}^{\infty} k\,|a_k| \leqq (2j-1)|b_{2j-1}| - j|b_j| \quad (j|b_j| < (2j-1)|b_{2j-1}|), \tag{2.8}$$

*then, for $\mu > 0$ and $z = re^{i\theta}(0 < r < 1)$,*

$$\int_0^{2\pi} |f'(z)|^{\mu} \, d\theta \leqq \int_0^{2\pi} |g'(z)|^{\mu} \, d\theta. \tag{2.9}$$

Further, with the help of Hölder inequality, we have

**Corollary 2.2.** *Let $f(z) \in \mathcal{A}_n$ and $g(z)$ be given by (2.1). If $f(z)$ satisfies (2.8), then , for $0 < \mu \leqq 2$ and $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\begin{aligned} \int_0^{2\pi} |f'(z)|^{\mu} \, d\theta &\leqq 2\pi \left\{ 1 + j^2 |b_j|^2 r^{2(j-1)} + (2j-1)^2 |b_{2j-1}|^2 r^{4(j-1)} \right\}^{\frac{\mu}{2}} \\ &< 2\pi \left\{ 1 + j^2 |b_j|^2 + (2j-1)^2 |b_{2j-1}|^2 \right\}^{\frac{\mu}{2}}. \end{aligned} \tag{2.10}$$

**Example 2.2.** Let $f(z) \in \mathcal{A}_n$ satisfy the cefficient inequality (1.5) in Theorem B and

$$g(z) = z + \frac{n\epsilon}{j(n+1-\alpha)} z^j + \frac{\delta}{2j-1} z^{2j-1} \quad (|\epsilon| = |\delta| = 1) \tag{2.11}$$

with $0 \leqq \alpha < 1$. Then,

$$b_j = \frac{n\epsilon}{j(n+1-\alpha)} \quad \text{and} \quad b_{2j-1} = \frac{\delta}{2j-1}.$$

Since

$$\sum_{k=n+1}^{\infty} k|a_k| \leqq \frac{1-\alpha}{n+1-\alpha} = 1 - \frac{n}{n+1-\alpha} = (2j-1)|b_{2j-1}| - j|b_j|,$$

$f(z)$ and $g(z)$ satisfy the conditions in Theorem 2.2. Thus, by Corollary 2.2, we have, for $0 < \mu \leqq 2$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\begin{aligned} \int_0^{2\pi} |f'(z)|^\mu d\theta &= 2\pi \left\{ 1 + \left( \frac{n}{n+1-\alpha} \right)^2 r^{2(j-1)} + r^{4(j-1)} \right\}^{\frac{\mu}{2}} \\ &< 2\pi \left\{ 2 + \left( \frac{n}{n+1-\alpha} \right)^2 \right\}^{\frac{\mu}{2}}. \end{aligned}$$

## 3. Integral means for $f(z)$ and $h(z)$

In this section, we introduce an analytic function $h(z)$ given by

$$h(z) = z + b_j z^j + b_{2j-1} z^{2j-1} + b_{3j-2} z^{3j-2} \quad (j \geqq n+1) \tag{3.1}$$

**Theorem 3.1.** *Let $f(z) \in \mathcal{A}_n$ and $h(z)$ be given by (3.1), if $f(z)$ satisfies*

$$\sum_{k=n+1}^{\infty} |a_k| \leqq |b_{3j-2}| - |b_{2j-1}| - |b_j| \quad (|b_j| + |b_{2j-1}| < |b_{3j-2}|), \tag{3.2}$$

*then, for $\mu > 0$ and $z = re^{i\theta}(0 < r < 1)$,*

$$\int_0^{2\pi} |f(z)|^\mu \, d\theta \leqq \int_0^{2\pi} |h(z)|^\mu \, d\theta \quad (\mu > 0). \tag{3.3}$$

*Proof.* In a same way with the proof of Theorem 2.1, we have to show that there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that $f(z) = h(w(z))$. Note that this function $w(z)$ is defined by

$$b_{3j-2} w(z)^{3j-3} + b_{2j-1} w(z)^{2j-2} + b_j w(z)^{j-1} = \sum_{k=n+1}^{\infty} a_k z^{k-1}. \tag{3.4}$$

Since, for $z = 0$,

$$w(0)^{j-1}\left(b_{3j-2}w(0)^{2j-2}+b_{2j-1}w(0)^{j-1}+b_j\right)=0,$$

we consider $w(z)$ such as $w(0)=0$.

On the other hand, we have that

$$|b_{3j-2}|\,|w(z)|^{3(j-1)}-|b_{2j-1}|\,|w(z)|^{2(j-1)}-|b_j|\,|w(z)|^{j-1}-\sum_{k=n+1}^{\infty}|a_k|<0. \tag{3.5}$$

Putting $t=|w(z)|^{j-1}$ $(t\geqq 0)$, we define the function $H(t)$ by ,

$$H(t)=|b_{3j-2}|\,t^3-|b_{2j-1}|\,t^2-|b_j|\,t-\sum_{k=n+1}^{\infty}|a_k| \quad (t\geqq 0).$$

It follows that $H(0)\leqq 0$, and

$$H'(t)=3\,|b_{3j-2}|\,t^2-2\,|b_{2j-1}|\,t-|b_j|\,.$$

Since the discriminant of $H'(t)=0$ is greater than 0, if $H'(1)\geqq 0$, then $t<1$ for $H(t)<0$. Therefore, we need the following inequality

$$H(1)=|b_{3j-2}|-|b_{2j-1}|-|b_j|-\sum_{k=n+1}^{\infty}|a_k|\geqq 0,$$

or

$$\sum_{k=n+1}^{\infty}|a_k|\leqq|b_{3j-2}|-|b_{2j-1}|-|b_j|.$$

This completes the proof of Theorem 3.1.

**Corollary 3.1.** *Let $f(z)\in\mathcal{A}_n$ and $h(z)$ be given by* (3.1). *If $f(z)$ satisfies* (3.2), *then , for $0<\mu\leqq 2$ and $z=re^{i\theta}$ $(0<r<1)$,*

$$\int_0^{2\pi}|f(z)|^{\mu}\,d\theta\leqq 2\pi r^{\mu}\left(1+|b_j|^2r^{2(j-1)}+|b_{2j-1}|^2r^{4(j-1)}+|b_{3j-2}|^2r^{6(j-1)}\right)^{\frac{\mu}{2}}$$
$$<2\pi\left(1+|b_j|^2+|b_{2j-1}|^2+|b_{3j-2}|^2\right)^{\frac{\mu}{2}}. \tag{3.6}$$

*Further, we have that $f(z)\in\mathcal{H}^p(\mathbb{U})$ for $0<p\leqq 2$.*

**Example 3.1.** Let $f(z)\in\mathcal{A}_n$ satisfy the coefficient inequality (1.4) in Theorem A and $h(z)$ be biven by

$$h(z)=z+\frac{nt}{n+1-\alpha}\epsilon z^j+\frac{n(1-t)}{n+1-\alpha}\delta z^{2j-1}+\sigma z^{3j-2}$$
$$(0\leqq t\leqq 1,|\epsilon|=|\delta|=|\sigma|=1) \tag{3.7}$$

with $0\leqq\alpha<1$. Then

$$b_j=\frac{nt}{n+1-\alpha}\epsilon,\quad b_{2j-1}=\frac{n(1-t)}{n+1-\alpha}\delta,\quad\text{and}\quad b_{3j-2}=\sigma.$$

In view of (1.4), we see that

$$\sum_{k=n+1}^{\infty} |a_k| \leqq \frac{1-\alpha}{n+1-\alpha} = 1 - \frac{n(1-t)}{n+1-\alpha} - \frac{nt}{n+1-\alpha}$$
$$= |b_{3j-2}| - |b_{2j-1}| - |b_j|.$$

This shows us that $f(z)$ and $h(z)$ satisfy the conditions in Theorem 3.1. Therefore, applying Corollary 3.1, we have, for $0 < \mu \leqq 2$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^{\mu} d\theta$$
$$= 2\pi r^{\mu} \left\{ 1 + \left( \frac{nt}{n+1-\alpha} \right)^2 r^{2(j-1)} + \frac{n(1-t)}{(n+1-\alpha)} r^{4(j-1)} + r^{6(j-1)} \right\}^{\frac{\mu}{2}}$$
$$< 2\pi \left\{ 2 + (2t^2 - 2t + 1) \left( \frac{n}{n+1-\alpha} \right)^2 \right\}^{\frac{\mu}{2}}.$$

Finally, for the integral means of $f'(z)$ and $h'(z)$, we derive the following theorem.

**Theorem 3.2.** *Let $f(z) \in \mathcal{A}_n$ and $h(z)$ be given by* (3.1). *If $f(z)$ satisfies*

$$\sum_{k=n+1}^{\infty} k\,|a_k| \leqq (3j-2)|b_{3j-2}| - (2j-1)|b_{2j-1}| - j\,|b_j|$$
$$(j|b_j| + (2j-1)|b_{2j-1}(2j-1)| < (3j-2)|b_{3j-2}|), \quad (3.8)$$

*then, for $\mu > 0$ and $z = re^{i\theta}(0 < r < 1)$,*

$$\int_0^{2\pi} |f'(z)|^{\mu} d\theta \leqq \int_0^{2\pi} |h'(z)|^{\mu} d\theta. \quad (3.9)$$

The proof of this theorem is similar to one of Theorm 2.2. Therefore, we omit the proof of the theorem.

**Corollary 3.2.** *Let $f(z) \in \mathcal{A}_n$ and $h(z)$ be given by* (3.1). *If $f(z)$ satisfies* (3.8), *then , for $0 < \mu \leqq 2$ and $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\int_0^{2\pi} |f'(z)|^{\mu} d\theta$$
$$\leqq 2\pi \left\{ 1 + j^2|b_j|^2 r^{2j-1} + (2j-1)^2|b_{2j-1}|^2 r^{4(j-1)} + (3j-2)^2|b_{3j-2}|^2 r^{6(j-1)} \right\}^{\frac{\mu}{2}}$$
$$< 2\pi \left\{ 1 + j^2|b_j|^2 + (2j-1)^2|b_{2j-1}|^2 + (3j-2)^2|b_{3j-2}|^2 \right\}^{\frac{\mu}{2}}. \quad (3.10)$$

**Example 3.2.** Let $f(z) \in \mathcal{A}_n$ satisfy the coefficient inequality (1.5) in Theorem B and $h(z)$ be biven by

$$h(z) = z + \frac{nt}{j(n+1-\alpha)} \epsilon z^j + \frac{n(1-t)}{n+1-\alpha} \delta z^{2j-1} + \frac{\sigma}{3j-2} z^{3j-2}$$
$$(0 \leqq t \leqq 1, |\epsilon| = |\delta| = |\sigma| = 1) \quad (3.11)$$

with $0 \leqq \alpha < 1$. It follows that

$$b_j = \frac{nt\epsilon}{j(n+1-\alpha)}, \quad b_{2j-1} = \frac{n(1-t)\delta}{(2j-1)(n+1-\alpha)}, \quad \text{and} \quad b_{3j-2} = \frac{\sigma}{3j-2}.$$

By the coefficient inequality (1.5), we obtain that

$$\begin{aligned} \sum_{k=n+1}^{\infty} k|a_k| &\leqq \frac{1-\alpha}{n+1-\alpha} = 1 - \frac{n}{n+1-\alpha} \\ &= (3j-2)|b_{3j-2}| - (2j-1)|b_{2j-1}| - j|b_j|. \end{aligned}$$

This gives us that $f(z)$ and $h(z)$ satisfy the conditions in Theorem 3.2. Thus, applying Corollary 3.2, we see, for $0 < \mu \leqq 2$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\begin{aligned} &\int_0^{2\pi} |f'(z)|^\mu d\theta \\ &= 2\pi r^\mu \left\{ 1 + \left( \frac{nt}{n+1-\alpha} \right)^2 r^{2(j-1)} + \frac{n(1-t)}{(n+1-\alpha)} r^{4(j-1)} + r^{6(j-1)} \right\}^{\frac{\mu}{2}} \\ &< 2\pi \left\{ 2 + (2t^2 - 2t + 1) \left( \frac{n}{n+1-\alpha} \right)^2 \right\}^{\frac{\mu}{2}}. \end{aligned}$$

## 4. Appendix

Applying the Hölder inequality for analytic functions $F(z)$ and $G(z)$, we obtain, for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_a^b |F(z)G(z)|\, d\theta \leqq \left( \int_a^b |F(z)|^p d\theta \right)^{\frac{1}{p}} \left( \int_a^b |G(z)|^q d\theta \right)^{\frac{1}{q}}. \tag{4.1}$$

with $p > 1$ and $1/p + 1/q = 1$. Note that the inequality (4.1) gives

$$\int_0^{2\pi} |F(z)|^p d\theta \geqq \frac{\left( \int_0^{2\pi} |F(z)G(z)| d\theta \right)^p}{\left( \int_0^{2\pi} |G(z)|^q \, dz \right)^{\frac{p}{q}}}. \tag{4.2}$$

Considering $p = \mu/2$, $q = \mu/(\mu - 2)$, and $\mu > 2$ in in (4.2), we have, for $f(z)$ in the class $\mathcal{A}_n$,

$$
\begin{aligned}
\int_0^{2\pi} |f(z)|^{\mu} d\theta &= \int_0^{2\pi} \left( |f(x)|^2 \right)^{\frac{\mu}{2}} d\theta \\
&\geqq \frac{\left(\int_0^{2\pi} |f(z)|^2 d\theta\right)^{\frac{\mu}{2}}}{\left(\int_0^{2\pi} d\theta\right)^{\frac{\mu-2}{2}}} \\
&= (2\pi)^{\frac{2-\mu}{2}} \left\{ 2\pi \left( r^2 + \sum_{k=n+1}^{\infty} |a_k|^2 r^{2k} \right) \right\}^{\frac{\mu}{2}} \\
&= 2\pi r^{\mu} \left( 1 + \sum_{k=n+1}^{\infty} |a_k|^2 r^{2(k-1)} \right)^{\frac{\mu}{2}}.
\end{aligned}
$$

When $\mu = 2$, we also have that, for $z = re^{i\theta}$ $(0 < r < 1)$,

$$
\begin{aligned}
\int_0^{2\pi} |f(z)|^2 d\theta &= 2\pi r^2 \left( 1 + \sum_{k=n+1}^{\infty} |a_k|^2 r^{2(k-1)} \right) \\
&< 2\pi \left( 1 + \sum_{k=n+1}^{\infty} |a_k|^2 \right).
\end{aligned}
$$

Thus, we conclude that

**Theorem 4.1** *Let $f(z) \in \mathcal{A}_n$ and $\mu \geqq 2$. Then, for $z = re^{i\theta}$ $(0 < r < 1)$,*

$$
\int_0^{2\pi} |f(z)|^{\mu} d\theta \geqq 2\pi r^{\mu} \left( 1 + \sum_{k=n+1}^{\infty} |a_k|^2 r^{2(k-1)} \right)^{\frac{\mu}{2}}.
$$

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