数理解析研究所講究録 1363 巻 2004 年 115-125

Generalization Properties of Integral Means by H.Silverman

Shigeyoshi Owa, Daisuke Yagi and Nicolae N. Pascu

Abstract

For analytic and univalent functions $f(z)$ with negative coefficients in the open unit disk $\mathbb{U}$, H.Silverman (Houston J. math. **23**(1997)) has given some interesting results for integral means of $f(z)$. In the present paper, we discuss generalization properties of integral means of $f(z)$ given by H.Silverman. We also show some examples of our theorems.

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

(1.1)
$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

that are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\mathbb{U}$. Also let $\mathcal{S}^*$ and $\mathcal{K}$ denote the subclasses of $\mathcal{S}$ consisting of functions $f(z)$ which are starlike and convex in $\mathbb{U}$, respectively.

The class $\mathcal{T}$ is defined as the subclass of $\mathcal{S}$ consisting of all functions $f(z)$ which are given by

(1.2)
$$f(z)=z-\sum_{n=2}^{\infty}a_nz_n \quad (a_n\geq 0).$$

Further, we denote by $\mathcal{T}^*=\mathcal{S}^*\cap\mathcal{T}$ and $\mathcal{C}=\mathcal{K}\cap\mathcal{T}$. It is well-known by Silverman [3] that

2000 Mathematics Subject Classification: Primary 30C45.
Key Words and Phrases: Integral means, univalent function, starlike function, convex function.

Remark 1.1. A function $f(z) \in \mathcal{T}^*$ if and only if

$$\sum_{n=2}^{\infty} n a_n \leq 1. \tag{1.3}$$

A function $f(z) \in \mathcal{C}$ if and only if

$$\sum_{n=2}^{\infty} n^2 a_n \leq 1. \tag{1.4}$$

For $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{A}$, $f(z)$ is said to be subordinate to $g(z)$ in $\mathbb{U}$ if there exists an analytic function $\omega(z)$ in $\mathcal{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ $(z \in \mathbb{U})$, and $f(z) = g(\omega(z))$. We denote this subordination by

$$f(z) \prec g(z). \quad (cf. Duren[1]) \tag{1.5}$$

For subordinations, Littlewood [2] has give the following integral mean.

Theorem A. *If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$ with $f(z) \prec g(z)$, then, for $\lambda > 0$ and $|z| = r$ $(0 < r < 1)$,*

$$\int_0^{2\pi} |f(re^{i\theta})|^\lambda d\theta \leq \int_0^{2\pi} |g(re^{i\theta})|^\lambda d\theta. \tag{1.6}$$

Furthermore, Silverman [3] has shown that

Remark 1.2. $f_1(z) = z$ and $f_n(z) = z - \dfrac{z^n}{n}$ $(n \geq 2)$ are extreme points of the class $\mathcal{T}^*$(or $\mathcal{T}$). $f_1(z) = z$ and $f_n(z) = z - \dfrac{z^n}{n^2}$ $(n \geq 2)$ are extreme points of the class $\mathcal{C}$.

Applying Theorem A with extreme points of $\mathcal{T}$, Silverman [4] has proved the follwing results.

Theorem B. *Suppose that $f(z) \in \mathcal{T}$, $\lambda > 0$ and $f_2(z) = z - \dfrac{z^2}{2}$. Then, for $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\int_0^{2\pi} |f(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_2(z)|^\lambda d\theta. \tag{1.7}$$

Theorem C. *If $f(z) \in \mathcal{T}$, $\lambda > 0$, and $f_2(z) = z - \dfrac{z^2}{2}$, then, for $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\int_0^{2\pi} |f'(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_2'(z)|^\lambda d\theta. \tag{1.8}$$

In the present paper, we consider the generalization properties for Theorem B and Theorem C with $f(z) \in \mathcal{T}^*$ and $f(z) \in \mathcal{C}$.

2 Generalization properties

Our first result for the generalization properties is contained in

Theorem 2.1. *Let $f(z) \in \mathcal{T}^*$, $\lambda > 0$, and $f_k(z) = z - \frac{z^k}{k}$ $(k \geq 2)$. If $f(z)$ satisfies*

$$\sum_{j=0}^{k-3} \frac{j+1}{k}(a_{2k+j-1} + a_{k+j+1} - a_{k-j-1}) \geq 0 \tag{2.1}$$

for $k \geq 3$, then, for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_k(z)|^\lambda d\theta. \tag{2.2}$$

Proof. For $f(z) \in \mathcal{T}^*$, we have to show that

$$\int_0^{2\pi} |1 - \sum_{n=2}^{\infty} a_n z^{n-1}|^\lambda d\theta \leq \int_0^{2\pi} |1 - \frac{z^{k-1}}{k}|^\lambda d\theta.$$

By TheoremA, it sufficies to prove that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{z^{k-1}}{k}.$$

Let us define the function $\omega(z)$ by

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1}{k}\omega(z)^{k-1}. \tag{2.3}$$

It follws from (2.3) that

$$|\omega(z)|^{k-1} = |k \sum_{n=2}^{\infty} a_n z^{n-1}| \leq |z| \left(\sum_{n=2}^{\infty} k a_n \right).$$

Thus, we only show that

$$\sum_{n=2}^{\infty} k a_n \leq \sum_{n=2}^{\infty} n a_n,$$

or

$$\sum_{n=2}^{\infty} a_n \leq \frac{1}{k} \left(\sum_{n=2}^{\infty} n a_n \right).$$

Indeed, we see that

$$\begin{aligned}\frac{1}{k}\sum_{n=2}^{\infty} na_n =& \left(1-\frac{k-2}{k}\right)a_2+\left(1-\frac{k-3}{k}\right)a_3+\cdots \\ &+\left(1-\frac{2}{k}\right)a_{k-2}+\left(1-\frac{1}{k}\right)a_{k-1}+a_k+\left(1+\frac{1}{k}\right)a_{k+1} \\ &+\left(1+\frac{2}{k}\right)a_{k+2}+\cdots+\left(1+\frac{k+1}{k}\right)a_{2k+1}+\left(1+\frac{k+2}{k}\right)a_{2k+2} \\ &+\cdots \\ =&\frac{k-2}{k}(a_{2k-2}-a_2)+\frac{k-3}{k}(a_{2k-3}-a_3)+\cdots \\ &+\frac{2}{k}(a_{k+2}-a_{k-2})+\frac{1}{k}(a_{k+1}-a_{k-1}) \\ &+\left(1+\frac{k-1}{k}\right)a_{2k-1}+\left(1+\frac{k}{k}\right)a_{2k}+\left(1+\frac{k+1}{k}\right)a_{2k+1} \\ &+\cdots+\sum_{n=2}^{2k-2} a_n.\end{aligned}$$

Nothing that

$$1+\frac{k+j}{k}\geq 1+\frac{2+j}{k} \quad (j=-1,0,1,\cdots),$$

we obtain

$$\begin{aligned}\frac{1}{k}\left(\sum_{n=2}^{\infty} na_n\right) \geq& \frac{k-2}{k}(a_{2k-2}-a_2)+\frac{k-3}{k}(a_{2k-3}-a_3) \\ &+\cdots+\frac{2}{k}(a_{k+2}-a_{k-2})+\frac{1}{k}(a_{k+1}-a_{k-1}) \\ &+\left(1+\frac{1}{k}\right)a_{2k-1}+\left(1+\frac{2}{k}\right)a_{2k}+\cdots \\ &+\left(1+\frac{k-3}{k}\right)a_{3k-5}+\left(1+\frac{k-2}{k}\right)a_{3k-4}+\cdots \\ &+\sum_{n=2}^{2k-2} a_n \\ \geq& \frac{1}{k}(a_{2k-1}+a_{k+1}-a_{k-1})+\frac{2}{k}(a_{2k}+a_{k+2}-a_{k-2}) \\ &+\cdots+\frac{k-2}{k}(a_{3k-4}+a_{2k-2}-a_2)+\sum_{n=2}^{\infty} a_n \\ =&\sum_{j=0}^{k-3}\frac{j+1}{k}(a_{2k+j-1}+a_{k+j+1}-a_{k-j-1})+\sum_{n=2}^{\infty} a_n \\ \geq& \sum_{n=2}^{\infty} a_n\end{aligned} \tag{2.4}$$

with the follwing condition

$$\sum_{j=0}^{k-3} \frac{j+1}{k}(a_{2k+j-1} + a_{k+j+1} - a_{k-j-1}) \geq 0.$$

Thus, we observe that the function $\omega(z)$ defined by (2.3) satisfies $\omega(z)$ is analytic in $\mathbb{U}$ with $\omega(0) = 0$, $|\omega(z)| < 1$ $(z \in \mathbb{U})$. This completes the proof of the theorem. □

Remark 2.1. Taking $k = 2$ in Theorem 2.1, we have Theorem B by Silverman [4].

Example 2.1. *Let us define*

$$f(z) = z - \frac{37}{1200}z^2 - \frac{1}{18}z^3 - \frac{1}{48}z^4 - \frac{1}{100}z^5 \tag{2.5}$$

and

$$f_3(z) = z - \frac{1}{3}z^3 \tag{2.6}$$

with $k = 3$ in Theorem 2.1. Since $f(z)$ satisfies

$$\sum_{n=2}^{\infty} na_n = \frac{217}{600} < 1,$$

we have $f(z) \in \mathcal{T}^*$. Furthermore, $f(z)$ satisfies,

$$\frac{1}{3}(a_5 + a_4 - a_2) = \frac{1}{3}\left(\frac{1}{100} + \frac{1}{48} - \frac{37}{1200}\right) = 0.$$

Thus, $f(z)$ satisfies the coditions in Thorem 2.1 with $k = 3$.
If we take $\lambda = 2$, then we have

$$\int_0^{2\pi} |f(z)|^2 d\theta \leq 2\pi r^2 \left(1 + \frac{1}{9}r^4\right) < \frac{20}{9}\pi = 6.9813\cdots.$$

Corollary 2.1. *Let $f(z) \in \mathcal{T}^*$, $0 < \lambda \leq 2$, and $f_k(z) = z - \frac{z^k}{k}$ $(k \geq 2)$. If $f(z)$ satisfies (2.1) for $k \geq 3$, then, for $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\int_0^{2\pi} |f(z)|^\lambda d\theta \leq 2\pi r^\lambda \left(1 + \frac{1}{k^2}r^{2(k-1)}\right)^{\frac{\lambda}{2}} < 2\pi\left(1 + \frac{1}{k^2}\right)^{\frac{\lambda}{2}}. \tag{2.6}$$

Proof. It follows that

$$\int_0^{2\pi} |f_k(z)|^\lambda d\theta = \int_0^{2\pi} |z|^\lambda |1 - \frac{z^{k-1}}{k}|^\lambda d\theta.$$

Applying Hölder inequality for $0 < \lambda < 2$, we obtain that

$$\begin{aligned}\int_0^{2\pi} |z|^\lambda |1 - \frac{z^{k-1}}{k}|^\lambda d\theta &\leq \left(\int_0^{2\pi} (|z|^\lambda)^{\frac{2}{2-\lambda}} d\theta\right)^{\frac{2-\lambda}{2}} \left(\int_0^{2\pi} \left(|1 - \frac{z^{k-1}}{k}|^\lambda\right)^{\frac{2}{\lambda}} d\theta\right)^{\frac{\lambda}{2}} \\ &= \left(\int_0^{2\pi} |z|^{\frac{2\lambda}{2-\lambda}} d\theta\right)^{\frac{2-\lambda}{2}} \left(\int_0^{2\pi} |1 - \frac{z^{k-1}}{k}|^2 d\theta\right)^{\frac{\lambda}{2}} \\ &= \left(2\pi r^{\frac{2\lambda}{2-\lambda}}\right)^{\frac{2-\lambda}{2}} \left(2\pi \left(1 + \frac{1}{k^2} r^{2(k-1)}\right)\right)^{\frac{\lambda}{2}} \\ &= 2\pi r^\lambda \left(1 + \frac{1}{k^2} r^{2(k-1)}\right)^{\frac{\lambda}{2}} \\ &< 2\pi \left(1 + \frac{1}{k^2}\right)^{\frac{\lambda}{2}}.\end{aligned}$$

Further, it is clear for $\lambda = 2$. □

For the generalization of Theorem C by Silverman [4], we have

Theorem 2.2. *Let $f(z) \in \mathcal{T}^*$, $\lambda > 0$, and $f_k(z) = z - \frac{z^k}{k}$ $(k \geq 2)$. Then, for $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\int_0^{2\pi} |f'(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_k'(z)|^\lambda d\theta. \tag{2.7}$$

Proof. For $f(z) \in \mathcal{T}^*$, it is sufficient to show that

$$1 - \sum_{n=2}^{\infty} n a_n z^{n-1} \prec 1 - z^{k-1}. \tag{2.8}$$

Let us define the function $\omega(z)$ by

$$1 - \sum_{n=2}^{\infty} n a_n z^{n-1} = 1 - \omega(z)^{k-1}, \tag{2.9}$$

or, by

$$\omega(z)^{k-1} = \sum_{n=2}^{\infty} n a_n z^{n-1}.$$

Since $f(z)$ satisfies

$$\sum_{n=2}^{\infty} n a_n \leq 1,$$

the function $\omega(z)$ is analytic in $\mathbb{U}$, $\omega(0) = 0$, and $|\omega(z)| < 1$ $(z \in \mathbb{U})$. □

Remark 2.2. If we take $k = 2$ in Theorem 2.2, then we have Theorem C by Silverman [4].

Using Hölder inequality for Theorem 2.2, we have

Corollary 2.2. *Let $f(z) \in \mathcal{T}^*$, $0 < \lambda \leq 2$, and $f_k(z) = z - \frac{z^k}{k}$ $(k \geq 2)$. Then, for $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\int_0^{2\pi} |f'(z)|^\lambda d\theta \leq 2\pi(1 + r^{2(k-1)})^{\frac{\lambda}{2}} < 2^{\frac{2+\lambda}{2}}\pi.$$

3 Integral means for functions in the class $\mathcal{C}$

In this section, we discuss the integral means for functions in the class $\mathcal{C}$.

Theorem 3.1. *Let $f(z) \in \mathcal{C}$, $\lambda > 0$, and $f_k(z) = z - \frac{z^k}{k^2}$ $(k \geq 2)$. If $f(z)$ satisfies*

$$\sum_{j=2}^{k-1} \frac{(k+j)(k-j)}{k^2}(a_{2k-j} - a_j) \geq 0 \tag{3.1}$$

for $k \geq 3$, then, for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_k(z)|^\lambda d\theta. \tag{3.2}$$

Proof. For the proof, we need to show that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{z^{k-1}}{k^2} \tag{3.3}$$

by Theorem A. Define the function $\omega(z)$ by

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1}{k^2}\omega(z)^{k-1}, \tag{3.4}$$

or by

$$\omega(z)^{k-1} = k^2 \left(\sum_{n=2}^{\infty} a_n z^{n-1} \right). \tag{3.5}$$

Therefore, we have to show that

$$\sum_{n=2}^{\infty} a_n \leq \frac{1}{k^2} \left(\sum_{n=2}^{\infty} n^2 a_n \right).$$

Using the same technique as in the proof of Theorem 2.1, we see that

$$\frac{1}{k^2}\left(\sum_{n=2}^{\infty} n^2 a_n\right) \geq \sum_{j=2}^{k-1} \frac{(k+j)(k-j)}{k^2}(a_{2k-j} - a_j) + \sum_{n=2}^{\infty} a_n$$
$$\geq \sum_{n=2}^{\infty} a_n.$$

□

Example 3.1. *Consider the functions*

$$f(z) = z - \frac{1}{40}z^2 - \frac{1}{18}z^3 - \frac{1}{40}z^4 \tag{3.6}$$

and

$$f_3(z) = z - \frac{1}{9}z^3 \tag{3.7}$$

with $k = 3$ in Theorem 3.1. Then we have that

$$\sum_{n=2}^{\infty} n^2 a_n = \frac{4}{40} + \frac{9}{18} + \frac{16}{40} = 1$$

which implies $f(z) \in C$, and that

$$\frac{5}{9}(a_4 - a_2) = 0.$$

Thus $f(z)$ satisfies the conditions of Theorem 3.1. If we make $\lambda = 2$, then we see that

$$\int_0^{2\pi} |f(z)|^2 d\theta \leq 2\pi r^2\left(1 + \frac{1}{81}r^4\right) < \frac{164}{81}\pi = 6.3607\cdots.$$

Corollary 3.1. *Let $f(z) \in C$, $0 < \lambda \leq 2$, and $f_k(z) = z - \frac{z^k}{k^2}$ $(k \geq 2)$. If $f(z)$ satisfies the condition (3.1) for $k \geq 3$, then, for $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\int_0^{2\pi} |f(z)|^\lambda d\theta \leq 2\pi r^\lambda \left(1 + \frac{1}{k^4}r^{2(k-1)}\right)^{\frac{\lambda}{2}} \tag{3.8}$$
$$< 2\pi\left(1 + \frac{1}{k^4}\right)^{\frac{\lambda}{2}}.$$

Further, we may have

Theorem 3.2. *Let $f(z) \in C$, $\lambda > 0$, and $f_k(z) = z - \frac{z^k}{k^2}$ $(k \geq 2)$. If $f(z)$ satisfies*

(3.9)
$$\sum_{j=2}^{2k-2} j(k-j)a_j \leq 0,$$

then, for $z = re^{i\theta}$ $(0 < r < 1)$,

(3.10)
$$\int_0^{2\pi} |f'(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_k'(z)|^\lambda d\theta.$$

Example 3.2. *Take the functions*

(3.11)
$$f(z) = z - \frac{1}{24}z^2 - \frac{1}{18}z^3 - \frac{1}{48}z^4$$

and

(3.12)
$$f_3(z) = z - \frac{1}{9}z^3$$

with $k = 3$ *in Theorem 3.2. Since*

$$\sum_{n=2}^{\infty} n^2 a_n = \frac{4}{24} + \frac{9}{18} + \frac{16}{48} = \frac{5}{6} < 1$$

and

$$2(3-2)a_2 + 3(3-3)a_3 + 4(3-4)a_4 = \frac{1}{12} - \frac{1}{12} = 0,$$

$f(z)$ *satisfies the conditions in Theorem 3.2. If we take* $\lambda = 2$, *then we have*

$$\int_0^{2\pi} |f'(z)|^2 d\theta \leq 2\pi\left(1 + \frac{1}{9}r^4\right) < \frac{20}{9}\pi.$$

Corollary 3.2. *Let* $f(z) \in \mathcal{C}$, $0 < \lambda \leq 2$, *and* $f_k(z) = z - \frac{z^k}{k^2}$ $(k \geq 2)$. *If* $f(z)$ *satisfies the condition (3.9) for* $k \geq 2$, *then, for* $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f'(z)|^\lambda d\theta \leq 2\pi\left(1 + \frac{1}{k}r^{2(k-1)}\right)^{\frac{\lambda}{2}} < 2\pi\left(1 + \frac{1}{k}\right)^{\frac{\lambda}{2}}.$$

4 Appendix

For analytic functions $h(z)$ and $g(z)$, Hölder inequality gives that, for $z = re^{i\theta}$ $(0 < r < 1)$,

(4.1)
$$\int_0^{2\pi} |h(z)g(z)| d\theta \leq \left(\int_0^{2\pi} |h(z)|^p d\theta\right)^{\frac{1}{p}} \left(\int_0^{2\pi} |g(z)|^q d\theta\right)^{\frac{1}{q}}$$

with $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. It follows from (4.1) that

$$(4.2)\qquad \int_0^{2\pi} |h(z)|^p d\theta \geq \frac{\left(\int_0^{2\pi} |h(z)g(z)|d\theta\right)^p}{\left(\int_0^{2\pi} |g(z)|^q d\theta\right)^{\frac{p}{q}}}.$$

Letting $p = \frac{\lambda}{2}$, $q = \frac{\lambda}{\lambda - 2}$, $\lambda > 2$, we observe that the function $f(z)$ in the class $\mathcal{T}$ satisfies

$$(4.3)\qquad \int_0^{2\pi} |f(z)|^\lambda d\theta = \int_0^{2\pi} (|f(z)|^2)^{\frac{\lambda}{2}} d\theta$$

$$\geq \frac{\left(\int_0^{2\pi} |f(z)|^2 d\theta\right)^{\frac{\lambda}{2}}}{\left(\int_0^{2\pi} d\theta\right)^{\frac{\lambda-2}{2}}}$$

$$= (2\pi)^{\frac{2-\lambda}{2}} \left\{ 2\pi \left(r^2 + \sum_{n=2}^{\infty} a_n{}^2 r^{2n} \right) \right\}^{\frac{\lambda}{2}}$$

$$= 2\pi r^\lambda \left(1 + \sum_{n=2}^{\infty} a_n{}^2 r^{2(n-1)} \right)^{\frac{\lambda}{2}}.$$

Further, when $\lambda = 2$, we see that, for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^2 d\theta = 2\pi r^2 \left(1 + \sum_{n=2}^{\infty} a_n{}^2 r^{2(n-1)} \right)$$

$$< 2\pi \left(1 + \sum_{n=2}^{\infty} a_n{}^2 \right).$$

Therefore, we conclude that

Theorem 4.1. *Let $f(z) \in \mathcal{T}$ and $\lambda \geq 2$. Then, for $z = re^{i\theta}$ $(0 < r < 1)$,*

$$(4.4)\qquad \int_0^{2\pi} |f(z)|^\lambda d\theta \geq 2\pi r^\lambda \left(1 + \sum_{n=2}^{\infty} a_n{}^2 r^{2(n-1)} \right)^{\frac{\lambda}{2}}.$$

References

[1] P.L.Duren, Univalent Functions, Springer-Verlag, New York, 1983.

[2] J.E.Littlewood, *On inequalities in the theory of functions*, Proc. London Math. Soc. **23**(1925), 481 - 519.

[3] H.Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51**(1975), 109 - 116.

[4] H.Silverman, *Integral means for univalent functions with negative coefficients*, Houston J. Math. **23**(1997), 169 - 174.

Shigeyoshi Owa:
Department of mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan
e-mail: owa@math.kindai.ac.jp

Daisuke Yagi:
Department of mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan

Nicolae N. Pascu:
Department of Mathematics
Transilvania University of Brasov
R-2200 Brasov
Romania
e-mail: n.pascu@unitbv.ro