ON A SUBCLASS OF ANALYTIC FUNCTIONS INVOLVING CERTAIN FRACTIONAL CALCULUS OPERATORS

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Abstract

Let $\mathcal{A}$ be the class of normalized analytic functions in the open unit disk $\mathbb{U}$. We consider a subclass $\mathcal{A}(\alpha,\beta,\gamma)$ of $\mathcal{A}$ which is defined by using certain fractional calculus operators. The main object of this paper is to investigate subordination theorems, argument theorems and the Fekete-Szegö problem of maximizing $|a_3-\mu a_2^2|$ for functions belonging to the class $\mathcal{A}(\alpha,\beta,\gamma)$, where μ is real. We also obtain certain class-preserving integral operators for the class $\mathcal{A}(\alpha,\beta,\gamma)$.

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z : z\in\mathbb{C} \text{ and } |z|<1\}$. Also let $\mathcal{S}$, $\mathcal{S}^*(\gamma)$ and $\mathcal{K}(\gamma)$ denote, respectively, the subclasses of $\mathcal{A}$ consisting of functions which are univalent, starlike of order γ and convex of order γ in $\mathbb{U}$ (see, e.g., [15]). In particular, the classes $\mathcal{S}^*(0)=\mathcal{S}^*$ and $\mathcal{K}(0)=\mathcal{K}$ are the familiar classes of starlike and convex functions in $\mathbb{U}$, respectively.

Given two functions $f(z)$ and $g(z)$, which are analytic in $\mathbb{U}$ with $f(0)=g(0)$, $f(z)$ is said to be subordinate to $g(z)$ if there exists an analytic function $w(z)$ on $\mathbb{U}$ such that $w(0)=0$, $|w(z)|<1$ and $f(z)=g(w(z))$ for $z\in\mathbb{U}$. We denote this subordination by

$$f(z)\prec g(z) \quad \text{in } \mathbb{U}.$$

Note that if $g(z)$ is univalent in $\mathbb{U}$, then $f(z)\prec g(z)$ if and only if $f(0)=g(0)$ and $f(\mathbb{U})\subset g(\mathbb{U})$.

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Several essentially equivalent definitions of fractional calculus (that is, fractional integral and fractional derivative) have been studied in the literature (*cf.*, *e.g.*, [3], [11] and [12, p.28 *et seq.*]). We state the following definitions due to Owa [8] which have been used rather frequently in the theory of analytic functions (see also [10] and [14]).

Definition 1. The fractional integral of order λ $(\lambda > 0)$ is defined, for a function $f(z)$, by

$$\mathcal{D}_z^{-\lambda} f(z) := \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta, \tag{1.2}$$

and the fractional derivative of order λ $(0 \leq \lambda < 1)$ by

$$\mathcal{D}_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta, \tag{1.3}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ involved in (1.2) (and that of $(z-\zeta)^{-\lambda}$ in (1.3)) is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 2. Under the hypotheses of Definition 1, the fractional derivative of order $n+\lambda$ $(0 \leq \lambda < 1; n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ is defined by

$$\mathcal{D}_z^{n+\lambda} f(z) := \frac{d^n}{dz^n} \mathcal{D}_z^{\lambda} f(z). \tag{1.4}$$

With the aid of the above definitions, Owa and Srivastava [10] defined the fractional calculus operator $\mathcal{J}_z^{\lambda}$ $(\lambda \in \mathbb{R}; \lambda \neq 2, 3, 4, \cdots)$ by

$$\mathcal{J}_z^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} \mathcal{D}_z^{\lambda} f(z) \tag{1.5}$$

for functions (1.1) belonging to the class $\mathcal{A}$.

Recently, Choi *et al.* [2] investigated the subclass $\mathcal{A}(\alpha, \beta, \gamma)$ of $\mathcal{A}$ for $\alpha < 2$, $\beta < 2$ and $\gamma < 1$, which was defined by

$$\mathcal{A}(\alpha, \beta, \gamma) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{\mathcal{J}_z^{\alpha} f(z)}{\mathcal{J}_z^{\beta} f(z)} \right) > \gamma \text{ in } \mathbb{U} \right\}. \tag{1.6}$$

We note that $\mathcal{A}(1, 0, \gamma) = \mathcal{S}^*(\gamma)$ and $\mathcal{A}(\lambda+1, 0, \gamma) = \mathcal{S}^*(\gamma, \lambda)$ $(\lambda < 1; 0 \leq \gamma < 1)$ which was studied by Owa and Shen [9]. Recently, Srivastava *et al.* [13] proved inclusion and subordination properties of the class $\mathcal{A}(\lambda+1, \lambda, (\rho-\lambda)/(1-\lambda)) = \mathcal{S}_{\lambda}(\rho)$ $(0 \leq \lambda < 1; 0 \leq \rho < 1)$.

In this paper, we investigate subordination theorems, argument theorems and the upper bound of the quantity $a_3 - \mu a_2^2$ for functions belonging to the class $\mathcal{A}(\alpha, \beta, \gamma)$, where μ is real. We also consider certain class-preserving integral operators for the class $\mathcal{A}(\alpha, \beta, \gamma)$.

2. Preliminary results

In order to prove our results, we need the following lammas.

Lemma 1. (Choi *et al.* [2]) *Let* $\lambda < 1$ *and* $f(z) \in \mathcal{A}$. *Then*

$$z\left(\mathcal{J}_z^{\lambda} f(z)\right)' = (1-\lambda)\mathcal{J}_z^{\lambda+1} f(z) + \lambda \mathcal{J}_z^{\lambda} f(z) \qquad (z \in \mathbb{U}), \tag{2.1}$$

where the operator $\mathcal{J}_z^{\lambda}$ *is given by* (1.5).

Lemma 2. (Hallenbeck and Ruscheweyh [4]) *Let* $g(z)$ *be convex univalent in* $\mathbb{U}$ *with* $g(0) = 1$. *If* $\operatorname{Re}(\eta) > 0$ *and* $f(z)$ *is analytic in* $\mathcal{D}$ *with* $f(z) \prec g(z)$, *then*

$$\frac{1}{z^{\eta}} \int_0^z f(t) t^{\eta-1} dt \prec \frac{1}{z^{\eta}} \int_0^z g(t) t^{\eta-1} dt. \tag{2.2}$$

Lemma 3. (Jack [5]) *Let* $w(z)$ *be analytic in* $\mathbb{U}$ *with* $w(0) = 0$. *Then if* $|w(z)|$ *attains its maximum value on the circle* $|z| = r$ $(r < 1)$ *at a point* z_0, *we can write*

$$z_0 w'(z_0) = k w(z_0),$$

where k *is real and* $k \geq 1$.

Lemma 4. (Ma and Minda [7]) *Let* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *be analytic in* $\mathbb{U}$ *with* $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{U})$. *Then*

$$|c_2 - \nu c_1^2| \leq \begin{cases} -4\nu + 2 & if\ \nu \leq 0 \\ 2 & if\ 0 \leq \nu \leq 1 \\ 4\nu - 2 & if\ \nu \geq 1. \end{cases} \tag{2.3}$$

3. Subordination and argument theorems

First, by using Lemma 2, we prove

Theorem 1. *Let* $\alpha < 2$, $\beta < 2$ *and* $\gamma < 1$. *If* $f(z) \in \mathcal{A}(\alpha, \beta, \gamma)$, *then*

$$\frac{1}{z} \int_0^z \left(\frac{\mathcal{J}_z^{\alpha} f(t)}{\mathcal{J}_z^{\beta} f(t)} \right) dt \prec 2\gamma - 1 - \frac{2(1-\gamma)}{z} \log(1-z). \tag{3.1}$$

Proof. Let $f(z) \in \mathcal{A}(\alpha, \beta, \gamma)$ and set

$$g(z) = \frac{1 + (1-2\gamma)z}{1-z} \qquad (z \in \mathbb{U})$$

which maps the unit disk $\mathbb{U}$ onto the half domain such that Re $(w) > \gamma$. Then, from the definition of the class $\mathcal{A}(\alpha,\beta,\gamma)$ we have

$$\frac{\mathcal{J}_z^{\alpha} f(z)}{\mathcal{J}_z^{\beta} f(z)} \prec g(z) = \frac{1+(1-2\gamma)z}{1-z}. \tag{3.2}$$

Furthermore, the function $g(z)$ is convex univalent in $\mathbb{U}$ with $g(0) = 1$. Hence, by applying Lemma 2 with $\eta = 1$, we observe that

$$\frac{1}{z}\int_0^z \left(\frac{\mathcal{J}_z^{\alpha} f(t)}{\mathcal{J}_z^{\beta} f(t)}\right) dt \prec \frac{1}{z}\int_0^z \frac{1+(1-2\gamma)t}{1-t} dt$$

which yields (3.1).

Remark 1. If $\alpha = \lambda + 1$ and $\beta = 0$ in Theorem 1, then it would immediately yield the result of Owa and Shen [9, Theorem 2.1].

Corollary 1. *Let $\lambda < 1$ and $\gamma < 1$. If $f(z) \in \mathcal{A}(\lambda+1,\lambda,\gamma)$, then*

$$\frac{1}{z}\int_0^z \left(\frac{t\left(\mathcal{J}_z^{\lambda} f(t)\right)'}{\mathcal{J}_z^{\lambda} f(t)}\right) dt \prec 2\lambda - 1 + 2(1-\lambda)\left(\gamma - \frac{1-\gamma}{z}\log(1-z)\right). \tag{3.3}$$

Proof. Let $f(z) \in \mathcal{A}(\lambda+1,\lambda,\gamma)$. Then, by using Lemma 1, it is easily verified that

$$\text{Re}\left(\frac{z\left(\mathcal{J}_z^{\lambda} f(z)\right)'}{\mathcal{J}_z^{\lambda} f(z)}\right) > (1-\lambda)\gamma + \lambda. \tag{3.4}$$

Hence, by using the same techniques as in the proof of Theorem 1 with

$$g(z) = \frac{1+(2(1-\gamma)(1-\lambda)-1)\,z}{1-z}, \tag{3.5}$$

we conclude that

$$\frac{1}{z}\int_0^z \left(\frac{t\left(\mathcal{J}_z^{\lambda} f(t)\right)'}{\mathcal{J}_z^{\lambda} f(t)}\right) dt \prec \frac{1}{z}\int_0^z \frac{1+(2(1-\gamma)(1-\lambda)-1)\,t}{1-t} dt$$

which evidently implies (3.3).

Putting $\gamma = 0$ in Theorem 1, we obtain

Corollary 2. *Let $\alpha < 2$ and $\beta < 2$. If $f(z) \in \mathcal{A}(\alpha,\beta,0)$, then*

$$\frac{1}{z}\int_0^z \left(\frac{\mathcal{J}_z^{\alpha} f(t)}{\mathcal{J}_z^{\beta} f(t)}\right) dt \prec -1 - \frac{2}{z}\log(1-z).$$

Next, we derive the arguments for functions belonging to the class $\mathcal{A}(\alpha,\beta,\gamma)$.

Theorem 2. *Let $\alpha < 2$, $\beta < 2$ and $\gamma < 1$. If $f(z) \in \mathcal{A}(\alpha,\beta,\gamma)$, then*

$$\left|\arg\left(z^{\alpha-\beta}\frac{\mathcal{D}_z^{\alpha}f(z)}{\mathcal{D}_z^{\beta}f(z)}\right)\right| \leq \sin^{-1}\left(\frac{2(1-\gamma)|z|}{1+(1-2\gamma)|z|^2}\right) \qquad (z \in \mathbb{U}) \tag{3.6}$$

and

$$\frac{1-(1-2\gamma)|z|}{1+|z|} \leq \left|\frac{\mathcal{J}_z^{\alpha}f(z)}{\mathcal{J}_z^{\beta}f(z)}\right| \leq \frac{1+(1-2\gamma)|z|}{1-|z|} \qquad (z \in \mathbb{U}).$$

Proof. Since $f(z) \in \mathcal{A}(\alpha,\beta,\gamma)$, in view of (3.2), we can write

$$\frac{\mathcal{J}_z^{\alpha}f(z)}{\mathcal{J}_z^{\beta}f(z)} = \frac{1+(1-2\gamma)w(z)}{1-w(z)}, \tag{3.7}$$

where $w(z)$ is analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$. We now consider the function

$$h(z) = \frac{1+Aw(z)}{1+Bw(z)} \qquad (-1 \leq B < A; z \in \mathbb{U}).$$

It is well known that $h(z)$, for $-1 \leq B \leq 1$, is the conformal map of the disk $|w(z)| < |z|$ onto the disk

$$\left|h(z) - \frac{1-AB|z|^2}{1-B^2|z|^2}\right| \leq \frac{(A-B)|z|}{1-B^2|z|^2}. \tag{3.8}$$

By virtue of (3.7) and (3.8), we have

$$\left|\frac{\mathcal{J}_z^{\alpha}f(z)}{\mathcal{J}_z^{\beta}f(z)} - \frac{1+(1-2\gamma)|z|^2}{1-|z|^2}\right| \leq \frac{2(1-\gamma)|z|}{1-|z|^2}, \tag{3.9}$$

which immediately yields the assertion (3.6).

Moreover, it follows from (3.9) that

$$\frac{1-(1-2\gamma)|z|}{1+|z|} \leq \left|\frac{\mathcal{J}_z^{\alpha}f(z)}{\mathcal{J}_z^{\beta}f(z)}\right| \leq \frac{1+(1-2\gamma)|z|}{1-|z|}.$$

This completes the proof of Theorem 2.

Corollary 3. *Let $\lambda < 1$ and $\gamma < 1$. If $f(z) \in \mathcal{A}(\lambda+1,\lambda,\gamma)$, then*

$$\left|\arg\left(\frac{z\left(\mathcal{D}_z^{\lambda}f(z)\right)'}{\mathcal{D}_z^{\lambda}f(z)}\right)\right| \leq \sin^{-1}\left(\frac{2(1-\gamma)|z|}{1+(1-2\gamma)|z|^2}\right) \qquad (z \in \mathbb{U})$$

and

$$\frac{(1-\lambda)(1-(1-2\gamma)|z|)}{1+|z|} \leq \left|\frac{z\left(\mathcal{D}_z^{\lambda}f(z)\right)'}{\mathcal{D}_z^{\lambda}f(z)}\right| \leq \frac{(1-\lambda)(1+(1-2\gamma)|z|)}{1-|z|} \qquad (z \in \mathbb{U}).$$

Proof. In view of (3.4) and (3.5), we set

$$\frac{z\left(\mathcal{J}_z^{\lambda} f(z)\right)'}{\mathcal{J}_z^{\lambda} f(z)} = \frac{1+(2(1-\gamma)(1-\lambda)-1)\,w(z)}{1-w(z)} \qquad (z \in U).$$

Here $w(z)$ is analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$. Then, by using same argument of Theorem 2, we can easily verify Corollary 3, and so we omit it.

4. Coefficient bound and class-preserving integral operators

We begin by applying Lemma 4 to prove

Theorem 3. *Let $\beta < \alpha < 2$, $\gamma < 1$ and $\mu \in \mathbb{R}$. If $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{A}(\alpha,\beta,\gamma)$, then*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \left(\dfrac{2(\alpha-\beta+2(1-\gamma)(2-\alpha))}{\alpha-\beta} - \dfrac{6(1-\gamma)(2-\alpha)(2-\beta)(5-\alpha-\beta)}{(\alpha-\beta)(3-\alpha)(3-\beta)}\mu\right)K \\ \qquad if\ \mu \le \dfrac{2(3-\alpha)(3-\beta)}{3(2-\beta)(5-\alpha-\beta)} \\ 2K \qquad if\ \dfrac{2(3-\alpha)(3-\beta)}{3(2-\beta)(5-\alpha-\beta)} \le \mu \le \dfrac{2(3-\alpha)(3-\beta)(2-\beta-\gamma(2-\alpha))}{3(2-\alpha)(2-\beta)(1-\gamma)(5-\alpha-\beta)} \\ \left(\dfrac{6(1-\gamma)(2-\alpha)(2-\beta)(5-\alpha-\beta)}{(\alpha-\beta)(3-\alpha)(3-\beta)}\mu - \dfrac{2(\alpha-\beta+2(1-\gamma)(2-\alpha))}{\alpha-\beta}\right)K \\ \qquad if\ \dfrac{2(3-\alpha)(3-\beta)(2-\beta-\gamma(2-\alpha))}{3(2-\alpha)(2-\beta)(1-\gamma)(5-\alpha-\beta)} \le \mu, \end{cases}$$

where

$$K = \frac{(1-\gamma)(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)}{6(\alpha-\beta)(5-\alpha-\beta)}. \tag{4.1}$$

Proof. If we set

$$p(z) = \frac{\dfrac{\mathcal{J}_z^{\alpha} f(z)}{\mathcal{J}_z^{\beta} f(z)} - \gamma}{1-\gamma} = 1 + c_1z + c_2z^2 + \cdots \qquad (f \in \mathcal{A}), \tag{4.2}$$

then $p(z)$ is analytic with $p(0)=1$ and has a positive real part in $\mathbb{U}$. In view of (4.2), a simple calculation shows

$$a_2 = \frac{(1-\gamma)(2-\alpha)(2-\beta)}{2(\alpha-\beta)}c_1 \tag{4.3}$$

and

$$a_3 = K\left(c_2 + \frac{(1-\gamma)(2-\alpha)}{\alpha-\beta}c_1^2\right), \tag{4.4}$$

where K is given by (4.1). Therefore, using (4.3) and (4.4), we see that

$$\left|a_3 - \mu a_2^2\right| = K\left|c_2 - \nu c_1^2\right|,$$

where

$$\nu = \frac{3(1-\gamma)(2-\alpha)(2-\beta)(5-\alpha-\beta)}{2(\alpha-\beta)(3-\alpha)(3-\beta)}\mu - \frac{(1-\gamma)(2-\alpha)}{\alpha-\beta}.$$

Hence, by applying Lemma 4, we obtain the desired result. We omit further details.

Setting $\alpha = \beta + 1$ in Theorem 3, we have

Corollary 4. *Let $\beta < 1$, $\gamma < 1$ and $\mu \in \mathbb{R}$. If $f(z) \in \mathcal{A}(\beta+1, \beta, \gamma)$, then*

$$\left|a_3 - \mu a_2^2\right|$$

$$\leq \begin{cases} (1-\gamma)(1-\beta)(2-\beta)\left[(3-\beta)\left(\frac{1}{2} - \frac{1}{3}(\beta+\gamma(1-\beta))\right) - (1-\gamma)(1-\beta)(2-\beta)\mu\right] \\ \quad if \ 3(2-\beta)\mu \leq 3-\beta \\ \frac{(1-\gamma)(1-\beta)(2-\beta)(3-\beta)}{6} \qquad if \ \frac{3-\beta}{3(2-\beta)} \leq \mu \leq \frac{(3-\beta)(1+(1-\gamma)(1-\beta))}{3(1-\gamma)(1-\beta)(2-\beta)} \\ (1-\gamma)(1-\beta)(2-\beta)\left[(1-\gamma)(1-\beta)(2-\beta)\mu - (3-\beta)\left(\frac{1}{2} - \frac{1}{3}(\beta+\gamma(1-\beta))\right)\right] \\ \quad if \ (3-\beta)(1+(1-\gamma)(1-\beta)) \leq 3(1-\gamma)(1-\beta)(2-\beta)\mu. \end{cases}$$

Remark 2. If $\gamma = (\rho-\beta)/(1-\beta)$ $(0 \leq \beta < 1; 0 \leq \rho < 1)$ in Corollary 4, then it would immediately yields the result of Srivastava *et al.* [13, Theorem 4].

Next, we consider the generalized Bernardi-Libera-Livingston integral operator I_c $(c > -1)$ defined by (cf. [1], [6] and [15])

$$I_c(f)(z) := \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \qquad (f \in \mathcal{A}; c > -1). \tag{4.5}$$

It follows from (4.5) that

$$\begin{aligned} I_c(f)(z) &= \frac{c+1}{z^c}\int_0^z \left(t^c + \sum_{n=2}^{\infty} a_n t^{n+c-1}\right)dt \\ &= z + \sum_{n=2}^{\infty}\frac{c+1}{c+n}a_n z^n. \end{aligned} \tag{4.6}$$

Theorem 4. *Let $\lambda < 1$, $\gamma < 1$ and $c \geq -\lambda - (1-\lambda)\gamma$. Suppose that $f(z) \in \mathcal{A}(\lambda+1, \lambda, \gamma_0)$, where*

$$\gamma_0 \equiv \gamma_0(c,\gamma,\lambda) = \begin{cases} \gamma - \dfrac{(1-\gamma)(1-\lambda)}{2(c+\lambda+\gamma(1-\lambda))} & \text{if } (1-\lambda)(1-2\gamma)-\lambda \leq c \\ \gamma - \dfrac{c+\lambda+\gamma(1-\lambda)}{2(1-\gamma)(1-\lambda)} & \text{if } (1-\lambda)(1-2\gamma)-\lambda \geq c. \end{cases} \tag{4.7}$$

Then $I_c(f)(z) \in \mathcal{A}(\lambda+1, \lambda, \gamma)$.

Proof. Making use of (1.5) and (4.6), we obtain

$$\begin{aligned} z\left(\mathcal{J}_z^\lambda I_c(f)(z)\right)' &= z + \sum_{n=2}^{\infty} \frac{n(c+1)}{c+n} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+\lambda-1)} a_n z^n \\ &= z + \sum_{n=2}^{\infty} \left(c+1 - \frac{c(c+1)}{c+n}\right) \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+\lambda-1)} a_n z^n \\ &= (c+1)\mathcal{J}_z^\lambda f(z) - c\mathcal{J}_z^\lambda I_c(f)(z). \end{aligned} \tag{4.8}$$

Define the function $w(z)$ by

$$\frac{z\left(\mathcal{J}_z^\lambda I_c(f)(z)\right)'}{\mathcal{J}_z^\lambda I_c(f)(z)} = \frac{1 + (2(1-\gamma)(1-\lambda)-1)\, w(z)}{1-w(z)} \qquad (z \in \mathbb{U}). \tag{4.9}$$

Then $w(z)$ is analytic in $\mathbb{U}$ with $w(0)=0$ and $w(z) \neq -1$. Hence, by applying the method of the aforementioned of [2, Theorem 4] with (4.8) and (4.9), we can easily prove Theorem 4, and so we omit the details.

Finally, we state and prove

Theorem 5. *Let $c \geq 0$, $\alpha < 2$, $\beta < 1$ and $\gamma < 1$. Suppose that $f(z) \in \mathcal{A}(\alpha,\beta,\gamma) \bigcap \mathcal{A}(\beta+1,\beta,\gamma_1)$, where*

$$\gamma_1 \equiv \gamma_1(c,\beta) = \begin{cases} \dfrac{\beta(1-2c)-1}{2c(1-\beta)} & \text{if } 1 \leq c \\ \dfrac{\beta(c-2)-c}{2(1-\beta)} & \text{if } 0 \leq c \leq 1. \end{cases} \tag{4.10}$$

Then $I_c(f)(z) \in \mathcal{A}(\alpha,\beta,\gamma)$.

Proof. This proof is much akin to that of [9, Theorem 6.1], so we shall omit some details here. If we define the function $w(z)$ by

$$\frac{\mathcal{J}_z^\alpha I_c(f)(z)}{\mathcal{J}_z^\beta I_c(f)(z)} = \frac{1+(1-2\gamma)w(z)}{1-w(z)} \qquad (\gamma < 1; z \in \mathbb{U}), \tag{4.11}$$

then $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$ and $w(z) \neq -1$. We need to show that $|w(z)| < 1$ for all $z \in \mathbb{U}$. Thus, by using similar way as in the proof of [9, Theorem 6.1] with Lemma 3, and putting $w(z_0) = e^{i\theta}$, we observe that

$$\operatorname{Re}\left(\frac{\mathcal{J}_z^\alpha f(z_0)}{\mathcal{J}_z^\beta f(z_0)}\right) = \gamma - \frac{k(1-\gamma)}{1-\cos\theta}\operatorname{Re}\left(\frac{1}{\frac{z_0(\mathcal{J}_z^\beta I_c(f)(z_0))'}{\mathcal{J}_z^\beta I_c(f)(z_0)} + c}\right) \qquad (k \geq 1). \tag{4.12}$$

Since $f(z) \in \mathcal{A}(\beta+1, \beta, \gamma_1)$, in view of Theorem 4, we have

$$I_c(f)(z) \in \mathcal{A}\left(\beta+1, \beta, -\frac{\beta}{1-\beta}\right). \tag{4.13}$$

Therefore, it follows from (2.1) and (4.13) that

$$\operatorname{Re}\left(\frac{1}{\frac{z_0(\mathcal{J}_z^\beta I_c(f)(z_0))'}{\mathcal{J}_z^\beta I_c(f)(z_0)} + c}\right)$$
$$= \frac{(1-\beta)\operatorname{Re}\left(\frac{\mathcal{J}_z^{\beta+1} I_c(f)(z_0)}{\mathcal{J}_z^\beta I_c(f)(z_0)}\right) + \beta + c}{\left[\operatorname{Re}\left(\frac{z_0(\mathcal{J}_z^\beta I_c(f)(z_0))'}{\mathcal{J}_z^\beta I_c(f)(z_0)} + c\right)\right]^2 + \left[\operatorname{Im}\left(\frac{z_0(\mathcal{J}_z^\beta I_c(f)(z_0))'}{\mathcal{J}_z^\beta I_c(f)(z_0)} + c\right)\right]^2} > 0. \tag{4.14}$$

Consequently, we obtain that

$$\operatorname{Re}\left(\frac{\mathcal{J}_z^\alpha f(z)}{\mathcal{J}_z^\beta f(z)}\right) \leq \gamma$$

which contradicts the hypothesis $f(z) \in \mathcal{A}(\alpha, \beta, \gamma)$. Hence $|w(z)| < 1$ for all $z \in \mathbb{U}$, and by (4.11), we have the desired result.

Remark 3. Taking $\alpha = \lambda + 1$ and $\beta = 0$ in Theorem 5, we see that

$$f(z) \in \mathcal{S}^*(\gamma, \lambda) \bigcap \mathcal{A}(1, 0, \gamma_1(c, 0)) \quad \text{implies} \quad I_c(f)(z) \in \mathcal{S}^*(\gamma, \lambda),$$

where $\gamma_1(c, 0)$ is given by (4.10). Since $\gamma_1(c, 0) \leq 0$,

$$\mathcal{S}^* = \mathcal{A}(1, 0, 0) \subset \mathcal{A}(1, 0, \gamma_1(c, 0)).$$

Hence Theorem 5 provides a improvement of the result due to Owa and Shen [9, Theorem 6.1].

Corollary 5. *Let* $c \geq 0$, $\beta < 1$ *and* $\gamma_1 \leq \gamma < 1$, *where* γ_1 *is given by* (4.10). *If* $f(z) \in \mathcal{A}(\beta+1, \beta, \gamma)$, *then* $I_c(f)(z) \in \mathcal{A}(\beta+1, \beta, \gamma)$.

Proof. Since $\gamma_1 \leq \gamma < 1$, in view of (1.6), we obtain

$$\mathcal{A}(\beta+1, \beta, \gamma) \bigcap \mathcal{A}(\beta+1, \beta, \gamma_1) = \mathcal{A}(\beta+1, \beta, \gamma).$$

Hence, by virtue of Theorem 5, we conclude that

$$f(z) \in \mathcal{A}(\beta+1,\beta,\gamma) \Longrightarrow I_c(f)(z) \in \mathcal{A}(\beta+1,\beta,\gamma),$$

which completes the proof of Corollary 5.

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