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# On The Order of Starlikeness and Strongly Starlikeness of Convex Functions of Order $\alpha$ and Strongly Convex of Order $\beta$

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Let $\mathcal{A}$ denote the set of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic in the unit disc $\mathbb{E} = \{z : |z| < 1\}$. It is said to be starlike of order $\alpha, 0 \leqq \alpha < 1$, if $f(z) \in \mathcal{A}$ and

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad \text{in } \mathbb{E}.$$

We denote by $\mathcal{S}_t(\alpha)$ this family of functions. It is said to be convex of order $\alpha, 0 \leqq \alpha < 1$, if $f(z) \in \mathcal{A}$ and

$$1 + \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) > \alpha \qquad \text{in } \mathbb{E}.$$

We also denote by $\mathcal{C}(\alpha)$ this family of functions.

A function $f(z) \in \mathcal{A}$ is said to be strongly starlike of order $\beta, 0 < \beta \leqq 1$, if

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\beta \qquad \text{in } \mathbb{E}.$$

We denote this family of functions by $\mathcal{SS}_t(\beta)$. A function $f(z) \in \mathcal{A}$ is said to be strongly convex of order $\beta, 0 < \beta \leqq 1$, if

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}\beta \qquad \text{in } \mathbb{E}.$$

This family of functions is denoted by $\mathcal{SC}(\beta)$.

A.Marx [3] and E.Strohhäcker [5] showed that if $f(z) \in \mathcal{C}(0)$ then $f(z) \in \mathcal{S}_t\left(\frac{1}{2}\right)$. It is well known that the number $\frac{1}{2}$ is the largest value of $\beta$ for which the assertion $\mathcal{C}(0) \subset \mathcal{S}_t(\beta)$ holds, as is seen by the function $f(z) = \frac{z}{1-z}$.

I.S.Jack [1] posed the more general problem:

What is the largest number $\beta(\alpha)$ so that $\mathcal{C}(\alpha) \subset \mathcal{S}_t(\beta(\alpha))$?

Now, we introduce the new classes of starlike and convex functions. It is said to be strongly starlike of order $\beta, 0 < \beta \leqq 1$, and starlike of order $\alpha, 0 \leqq \alpha < 1$, if $f(z) \in \mathcal{A}$ and

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta \quad \text{in } \mathbb{E}.$$

We denote by $\mathcal{SS}_t(\alpha,\beta)$ this family of functions.

On the other hand, it is said to be strongly convex of order $\beta, 0 < \beta \leqq 1$, and convex of order $\alpha, 0 \leqq \alpha < 1$, if $f(z) \in \mathcal{A}$ and

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta \quad \text{in } \mathbb{E}.$$

This family of functions is also denoted by $\mathcal{SC}(\alpha,\beta)$. In [1], I.S.Jack obtained the following rsult.

**Theorem A.** *If $f(z) \in \mathcal{C}(\alpha)$, then $f(z) \in \mathcal{S}_t(\beta(\alpha))$, where*

$$\beta(\alpha) \geqq \frac{2\alpha - 1 + \sqrt{9 - 4\alpha + 4\alpha^2}}{4}.$$

In [2], T.H.MacGregor claimed and conjectured the sharp result of $\beta(\alpha)$ which improved Theorem A as the following.

**Theorem B.** *If $f(z) \in \mathcal{C}(\alpha)$, then $f(z) \in \mathcal{S}_t(\beta(\alpha))$, where*

$$\beta(\alpha) = \begin{cases} \dfrac{1-2\alpha}{2^{2-2\alpha}(1-2^{2\alpha-1})} & if\ \alpha \neq \dfrac{1}{2} \\ \dfrac{1}{2\log 2} & if\ \alpha = \dfrac{1}{2}. \end{cases}$$

In [6], D.R.Wilken and J.Feng completed the proof of Theorem B.

**Theorem 1.** *If $f(z) \in \mathcal{SC}(\alpha, n(\beta))$, then $f(z) \in \mathcal{SS}_t(\beta(\alpha),\beta)$, where $0 \leqq \alpha < 1, 0 < \beta \leqq 1$,*

$$n(\beta) = \beta + \frac{2}{\pi}\mathrm{Tan}^{-1}F(a_0),$$

$$F(a_0) = \mathrm{Min}_{0<a<\infty} F(a) = \mathrm{Min}_{0<a<\infty} \frac{G(a)}{H(a)},$$

$$G(a) = \frac{(a+a^{-1})}{2(a,\beta,l)}\left(a^{\beta}\beta\sin\left(\frac{\pi}{2}(1-\beta)\right) + \beta l\right) - \frac{(\beta(\alpha)-\alpha)}{a^{\beta}}\sin\left(\frac{\pi}{2}\beta\right),$$

$$H(a) = (1-\beta(\alpha)) + \frac{(a+a^{-1})}{2(a,\beta,l)}a^{\beta}\beta\cos\left(\frac{\pi}{2}(1-\beta)\right) + \frac{(\beta(\alpha)-\alpha)}{a^{\beta}}\cos\left(\frac{\pi}{2}\beta\right),$$

$$l = \frac{\beta(\alpha)}{1-\beta(\alpha)},$$

$$(a,\beta,l) = a^{2\beta} + 2a^{\beta}l\cos\left(\frac{\pi}{2}\beta\right) + l^2$$

*and*

$$\beta(\alpha) = \begin{cases} \dfrac{1-2\alpha}{2^{2-2\alpha}(1-2^{2\alpha-1})} & if\ \alpha \neq \dfrac{1}{2} \\ \dfrac{1}{2\log 2} & if\ \alpha = \dfrac{1}{2}. \end{cases}$$

*Proof* Let us put

$$p(z)=\frac{zf'(z)}{f(z)},\qquad p(0)=1,$$

and

$$q(z)=\frac{p(z)-\beta(\alpha)}{1-\beta(\alpha)},\qquad q(0)=1.$$

Then we have

$$p(z)=(1-\beta(\alpha))q(z)+\beta(\alpha),$$

and

$$\frac{zp'(z)}{p(z)}=\frac{(1-\beta(\alpha))zq'(z)}{(1-\beta(\alpha))q(z)+\beta(\alpha)}=\left(\frac{zq'(z)}{q(z)}\right)\frac{q(z)}{q(z)+\frac{\beta(\alpha)}{1-\beta(\alpha)}}.$$

Then it follows that

$$1+\frac{zf''(z)}{f'(z)}-\alpha=p(z)+\frac{zp'(z)}{p(z)}-\alpha$$

$$=(1-\beta(\alpha))q(z)+\beta(\alpha)+\left(\frac{zq'(z)}{q(z)}\right)\frac{q(z)}{q(z)+\frac{\beta(\alpha)}{1-\beta(\alpha)}}+\beta(\alpha)-\alpha$$

$$=q(z)\left\{(1-\beta(\alpha))+\left(\frac{zq'(z)}{q(z)}\right)\frac{q(z)}{q(z)+\frac{\beta(\alpha)}{1-\beta(\alpha)}}+\frac{\beta(\alpha)-\alpha}{q(z)}\right\}.$$

If there exists a point $z_0\in\mathbb{E}$ such that

$$|\arg q(z)|<\frac{\pi}{2}\beta\qquad for\ |z|<|z_0|$$

and

$$|\arg q(z_0)|=\frac{\pi}{2}\beta,$$

then from [4], we have

$$\frac{z_0q'(z_0)}{q(z_0)}=ik\beta$$

where

$$k\geqq\frac{1}{2}\left(a+\frac{1}{a}\right)\qquad\text{when }\arg q(z_0)=\frac{\pi}{2}\beta$$

and

$$k\leqq-\frac{1}{2}\left(a+\frac{1}{a}\right)\qquad\text{when }\arg q(z_0)=-\frac{\pi}{2}\beta$$

where

$$q(z_0)^{\frac{1}{\beta}} = \pm ia, \qquad \text{and} \qquad a > 0.$$

At first, let us suppose

$$\arg q(z_0) = \frac{\pi}{2}\beta, \qquad q(z_0) = (ia)^{\beta},$$

and $a > 0$, then we have

$$\arg\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right)$$

$$= \arg q(z_0) + \arg\left\{(1-\beta(\alpha)) + \frac{i\beta k}{(ia)^{\beta} + \frac{\beta(\alpha)}{1-\beta(\alpha)}} + \left(\frac{\beta(\alpha)-\alpha}{a^{\beta}}\right)e^{-i\frac{\pi}{2}\beta}\right\}$$

$$= \frac{\pi}{2}\beta + \arg\left\{(1-\beta(\alpha)) + \frac{\beta k e^{i\frac{\pi}{2}}\left(a^{\beta}e^{-i\frac{\pi}{2}\beta} + l\right)}{a^{2\beta} + 2a^{\beta}l\cos\left(\frac{\pi}{2}\beta\right) + l^2} + \left(\frac{\beta(\alpha)-\alpha}{a^{\beta}}\right)e^{-i\frac{\pi}{2}\beta}\right\}$$

$$= \frac{\pi}{2}\beta + J \qquad \text{say.}$$

Then it follows that

$$J \geqq \arg\left\{(1-\beta(\alpha)) + \left(\frac{a+a^{-1}}{2(a,\beta,l)}\right)\left(a^{\beta}\beta e^{i\frac{\pi}{2}(1-\beta)} + i\beta l\right) + \left(\frac{\beta(\alpha)-\alpha}{a^{\beta}}\right)e^{-i\frac{\pi}{2}\beta}\right\}$$

$$= \mathrm{Tan}^{-1}\left\{\frac{\left(\frac{a+a^{-1}}{2(a,\beta,l)}\right)\left(a^{\beta}\beta\sin\left(\frac{\pi}{2}(1-\beta)\right) + \beta l\right) - \left(\frac{\beta(\alpha)-\alpha}{a^{\beta}}\right)\sin\left(\frac{\pi}{2}\beta\right)}{(1-\beta(\alpha)) + \left(\frac{a+a^{-1}}{2(a,\beta,l)}\right)a^{\beta}\beta\cos\left(\frac{\pi}{2}(1-\beta)\right) + \left(\frac{\beta(\alpha)-\alpha}{a^{\beta}}\right)\cos\left(\frac{\pi}{2}\beta\right)}\right\}$$

$$= \mathrm{Tan}^{-1}_{0<a<\infty}\left(\frac{G(a)}{H(a)}\right) = \mathrm{Tan}^{-1}_{0<a<\infty} F(a) \geqq \mathrm{Tan}^{-1} F(a_0).$$

Therefore, we have

$$\arg\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \alpha\right) \geqq \frac{\pi}{2}\beta + \mathrm{Tan}^{-1} F(a_0)$$

$$= \frac{\pi}{2}n(\beta).$$

This contradicts the hypothesis of Theorem 1.

For the case $\arg q(z_0) = -\frac{\pi}{2}\beta$, applying the same method as the above, we can complete the proof of Theorem 1.

Putting $\beta = 1$ in Theorem 1, we obtain T.H.MacGregor [3] and D.R.Wilken and J.Feng's result [6].

**Corollary 1.** *If $f(z) \in \mathcal{SC}(\alpha, n(1)) = \mathcal{SC}(\alpha, 1) = \mathcal{C}(\alpha)$, then $f(z) \in \mathcal{SS}_t(\beta(\alpha), 1) = \mathcal{S}_t(\beta(\alpha))$.*

*Proof* In the proof of Theorem 1, let us suppose that if there exists a point $z_0 \in \mathbb{E}$ such that

$$|\arg q(z)| < \frac{\pi}{2} \qquad \text{for } |z| < |z_0|$$

and

$$|\arg q(z_0)| = \frac{\pi}{2},$$

then we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik,$$

where

$$k \geqq \frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \text{when } \arg q(z_0) = \frac{\pi}{2}$$

and

$$k \leqq -\frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \text{when } \arg q(z_0) = -\frac{\pi}{2}$$

where $q(z_0) = \pm ia$ and $a > 0$.
At first, let us suppose

$$\arg q(z_0) = \frac{\pi}{2}, \qquad q(z_0) = ia$$

and $a > 0$, then we easily have $H(a) > 0$.
On the other hand, from A,Marx [3] and E.Strohhäcker's result [5], we have $\beta \geqq \frac{1}{2}$ for $0 \leqq \alpha < 1$, therefore we have

$$\frac{\beta(\alpha)}{1 - \beta(\alpha)} = l \geqq 1.$$

Then it follows that

$$G(a) = \frac{(a + a^{-1})l}{2(a^2 + l^2)} - \frac{\beta(\alpha)}{a}$$

$$= \frac{1}{2a}\left\{\frac{(a^2 + l^2)l + l - l^3}{a^2 + l^2} - \beta(\alpha) + \alpha\right\}$$

$$= \frac{1}{2a}\left\{\frac{l(1 - l^2)}{a^2 + l^2} + l - \beta(\alpha) + \alpha\right\}$$

$$> \frac{1}{2a}\left(\frac{l - l^3}{l^2} + l - \beta(\alpha) + \alpha\right)$$

$$= \frac{1}{2a\beta(\alpha)}\left(1 - (1 - \alpha)\beta(\alpha) - \beta(\alpha)^2\right)$$

$$= \frac{1}{2a\beta(\alpha)} Q(\alpha) \qquad \text{say.}$$

Now then, $Q(\alpha)$ is a quadratic expression of $\beta(\alpha)$, the axis of parabolic curve is $\frac{\alpha - 1}{2} < 0$, this parabola opens downwards,

$$Q(0) = 1 - (1 - 0)\beta(0) - \beta(0)^2 = \frac{1}{4},$$

and

$$Q(1) = 1 - (1-0)\beta(1) - \beta(1)^2 = 1.$$

This shows that $Q(\alpha) \geqq 0$ for $0 \leqq \alpha < 1$, therefore we have $G(a) \geqq 0$ for $0 < a < \infty$ and it follows that

$$\lim_{a\to\infty} G(a) = 0.$$

Therefore, we have

$$\mathrm{Min}_{0<a<\infty} F(a) = \lim_{a\to\infty}\left(\frac{G(a)}{H(a)}\right) = 0,$$

and

$$n(1) = 1 + \frac{2}{\pi}\mathrm{Tan}^{-1}F(a_0) = 1.$$

For the case,

$$\arg q(z_0) = -\frac{\pi}{2}, \qquad q(z_0) = -ia, \qquad a > 0,$$

applying the same method as the above, we can complete the proof of Corollary 1.

# References


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