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Coexistence problems for the Hill equations with 3-step potentials

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Abstract We study the coexistence of two linearly independent, periodic solutions of the Hill equation with a 3-step potential. We give a simple, necessary and sufficient condition for the coexistence.

Keywords Hill's equation, 3-step potential, Coexistence, Monodromy matrix

AMS Subject Classification 34L15, 34L40

1 Introduction

The purpose of this talk is to give a simple, necessary and sufficient condition for the Hill equation with a three-step potential to admit two linearly independent, periodic solutions.

Given a subdivision

$$0 = t_0 < t_1 < t_2 < t_3 = 2\pi$$

of the interval $[0, 2\pi]$, we put

$$t = (t_1, t_2) \quad \text{and} \quad s_i = t_i - t_{i-1} \quad \text{for} \quad i = 1, 2, 3.$$

For $a = (a_1, a_2, a_3) \in \mathbb{R}^3$, let $Q(a, t, \cdot) : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic step function such that

$$Q(a, t, \cdot) = a_i \quad \text{on} \quad [t_{i-1}, t_i) \quad \text{for} \quad i = 1, 2, 3.$$

We are concerned with the Hill equation of the form

$$-y''(x) + Q(a, t, x)y(x) = \lambda y(x) \quad \text{on} \quad \mathbb{R}, \quad y, y' \in AC_{loc}(\mathbb{R}), \tag{1}$$

where λ is a real parameter.

In order to formulate our claims, we recall from [5] some fundamental results and terminologies in the general theory of Hill's equations. Let $y_1(a, t, \lambda, x)$ and $y_2(a, t, \lambda, x)$ be the solutions of the equation (1) subject to the initial conditions

$$y_1(a, t, \lambda, 0) - 1 = y_1'(a, t, \lambda, 0) = 0$$

and

$$y_2(a, t, \lambda, 0) = y_2'(a, t, \lambda, 0) - 1 = 0,$$

respectively. We introduce the discriminant of the equation (1):

$$D(a,t,\lambda) := y_1(a,t,\lambda,2\pi) + y_2'(a,t,\lambda,2\pi),$$

which is analytic in λ. Denoting by $\lambda_j(a,t)$ the jth root of the equation $D(a,t,\cdot)^2 - 4 = 0$ counted with multiplicity for each $j \in \mathbb{N} = \{1,2,3,\dots\}$, we have by the Liapounoff oscillation theorem (see [5, Theorem 2.1])

$$\lambda_1(a,t) < \lambda_2(a,t) \le \lambda_3(a,t) < \dots < \lambda_{2k}(a,t) \le \lambda_{2k+1}(a,t) < \dots. \qquad (2)$$

This sequence also gives all the eigenvalues of (1) with the 4π-periodicity condition $y(\cdot + 4\pi) = y(\cdot)$ on $\mathbb{R}$ repeated according to multiplicity, while the subsequence

$$\lambda_1(a,t) < \lambda_4(a,t) \le \lambda_5(a,t) < \dots < \lambda_{4k}(a,t) \le \lambda_{4k+1}(a,t) < \dots$$

provides all the eigenvalues of (1) with the 2π-periodicity condition repeated according to multiplicity. If the equation (1) admits two linearly independent, periodic solutions of period 2π or 4π, we say that two such solutions coexist. Such coexistence is equivalent to the condition

$$\lambda = \lambda_{2k}(a,t) = \lambda_{2k+1}(a,t) \quad \text{for some} \quad k \in \mathbb{N}.$$

The sequence (2) also characterizes the stability of the solutions of (1). Whenever all solutions of (1) are bounded on $\mathbb{R}$ we say that they are stable; otherwise we say that they are unstable. By the Liapounoff theorem, we see that the solutions of (1) are stable if and only if $\{\lambda\}$ is an interior point of the set

$$\bigcup_{k=1}^{\infty} [\lambda_{2k-1}(a,t), \lambda_{2k}(a,t)].$$

We call $(\lambda_{2k}(a,t), \lambda_{2k+1}(a,t))$ the kth instability interval for $k \in \mathbb{N}$. So the coexistence is also equivalent to the absence of the instability interval.

We define

$$p_i = p_i(a_i,\lambda) = \sqrt{\lambda - a_i}, \qquad \arg p_i \in \{0, \frac{\pi}{2}\} \quad \text{for} \quad i = 1,2,3.$$

Our main result is the following claim.

Theorem 1.1. *Let $k \in \mathbb{N}$. Assume that $a_m \ne a_n$ for $m \ne n$. Then the statements* (i) *and* (ii) *below are equivalent.*

(i) $\lambda = \lambda_{2k}(a,t) = \lambda_{2k+1}(a,t)$.

(ii) $s_1p_1(a_1,\lambda) + s_2p_2(a_2,\lambda) + s_3p_3(a_3,\lambda) = k\pi$ *and* $s_ip_i(a_i,\lambda) \in \pi\mathbb{N}$ *for* $i = 1,2,3$.

As a byproduct of Theorem 1.1, we have the following assertions.

Corollary 1.2. *Assume that $a_m \ne a_n$ for $m \ne n$. Then the following statements* (a), (b), *and* (c) *are equivalent for $k \in \mathbb{N}$.*

(a) *The kth instability interval is absent.*

(b) *There exists $\lambda \in \mathbb{R}$ satisfying the statement* (ii).

(c) *There exists $(n_1,n_2,n_3) \in \mathbb{N}^3$ for which*

$$a_1 + \frac{\pi^2}{s_1^2}n_1^2 = a_2 + \frac{\pi^2}{s_2^2}n_2^2 = a_3 + \frac{\pi^2}{s_3^2}n_3^2 \quad \text{and} \quad n_1 + n_2 + n_3 = k.$$

Corollary 1.3. *The first instability interval and the second instability interval are always present, provided $a_m \neq a_n$ for $m \neq n$.*

The coexistence problems for Hill's equations with 2-step potentials have been studied in [2], [3], [4], and [6]. In order to review those results, we introduce needed notations. Given $0 < \kappa < 2\pi$ and $b = (b_1, b_2) \in \mathbb{R}^2$ with $b_1 \neq b_2$, let $W(b, \kappa, \cdot) : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function such that $W(b, \kappa, \cdot) = b_1$ on $[0, \kappa)$ and that $W(b, \kappa, \cdot) = b_2$ on $[\kappa, 2\pi)$. Meissner [6] was the first to study the characteristic value problem

$$-z''(x) = \nu^2 W(b, \kappa, x) z(x) \quad \text{on} \quad \mathbb{R}, \quad \nu > 0,$$

where $b_1, b_2 > 0$. He solved the coexistence problem for this equation in the case when $\kappa = \pi$. Furthermore, Hochstadt [2] investigated this problem for general κ. He proved that two linearly independent, periodic solutions to this equation can coexist for some ν if and only if $\sqrt{b_2/b_1}\,(2\pi - \kappa)/\kappa$ is a rational number. His method is based on a factorization of the discriminant. Recently, Gan and Zhang [3], [4] studied the eigenvalue problem

$$-z''(x) + W(b, \kappa, x) z(x) = \nu z(x) \quad \text{on} \quad \mathbb{R}, \quad \nu \in \mathbb{R},$$

where $b_1, b_2 \in \mathbb{R}$. They obtained a necessary and sufficient condition for the coexistence (see Theorem 2.3 in [3] and Proposition 3.1 in [4]). Their method is based on a characterization of the eigenvalue by the rotation number of the Prüfer transform of the solution.

Our idea to prove Theorem 1.1 is entirely different from the ones in [2], [3], [4], and [6]; we make effective use of the full components of the monodromy matrix. This enables us to reduce the problem to a simple arithmetic.

2 Proof of theorem

By $M(a, t, \lambda)$ we denote the monodromy matrix of (1):

$$M(a, t, \lambda) = \begin{pmatrix} y_1(a, t, \lambda, 2\pi) & y_2(a, t, \lambda, 2\pi) \\ y_1'(a, t, \lambda, 2\pi) & y_2'(a, t, \lambda, 2\pi) \end{pmatrix}.$$

Using $-y_j''(x) = (\lambda - a_i) y_j(x)$ on (t_{i-1}, t_i) for $i = 1, 2, 3$ and $j = 1, 2$, we have the following formulae in the case when $p_1(a_1, \lambda) p_2(a_2, \lambda) p_3(a_3, \lambda) \neq 0$.

$$\begin{aligned} y_1(a, t, \lambda, 2\pi) &= \cos s_1 p_1 \cos s_2 p_2 \cos s_3 p_3 - \frac{p_1}{p_2} \sin s_1 p_1 \sin s_2 p_2 \cos s_3 p_3 \\ &\quad - \frac{p_1}{p_3} \sin s_1 p_1 \cos s_2 p_2 \sin s_3 p_3 - \frac{p_2}{p_3} \cos s_1 p_1 \sin s_2 p_2 \sin s_3 p_3. \end{aligned} \tag{3}$$

$$\begin{aligned} y_1'(a, t, \lambda, 2\pi) &= -p_1 \sin s_1 p_1 \cos s_2 p_2 \cos s_3 p_3 - p_2 \cos s_1 p_1 \sin s_2 p_2 \cos s_3 p_3 \\ &\quad - p_3 \cos s_1 p_1 \cos s_2 p_2 \sin s_3 p_3 + \frac{p_1 p_3}{p_2} \sin s_1 p_1 \sin s_2 p_2 \sin s_3 p_3. \end{aligned} \tag{4}$$

$$y_2(a,t,\lambda,2\pi) = \frac{1}{p_1}\sin s_1p_1\cos s_2p_2\cos s_3p_3 + \frac{1}{p_2}\cos s_1p_1\sin s_2p_2\cos s_3p_3 + \frac{1}{p_3}\cos s_1p_1\cos s_2p_2\sin s_3p_3 - \frac{p_2}{p_1p_3}\sin s_1p_1\sin s_2p_2\sin s_3p_3. \quad (5)$$

$$y_2'(a,t,\lambda,2\pi) = \cos s_1p_1\cos s_2p_2\cos s_3p_3 - \frac{p_2}{p_1}\sin s_1p_1\sin s_2p_2\cos s_3p_3 - \frac{p_3}{p_1}\sin s_1p_1\cos s_2p_2\sin s_3p_3 - \frac{p_3}{p_2}\cos s_1p_1\sin s_2p_2\sin s_3p_3. \quad (6)$$

Notice that the statement (**i**) in Theorem 1.1 is equivalent to the condition

$$M(a,t,\lambda) = (-1)^k\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \text{and} \quad \lambda \in \{\lambda_{2k}(a,t),\ \lambda_{2k+1}(a,t)\} \quad (7)$$

(see the proof of Lemma 2.4 in [5]). Let us demonstrate Theorem 1.1.

Proof of Theorem 1.1. It suffices to show that (**ii**) in Theorem 1.1 and (7) are equivalent.

Let us prove that (7) yields (**ii**). Assume that (7) holds. Our first task is to deduce that $\sin s_1p_1\sin s_2p_2\sin s_3p_3 = 0$ by contradiction. Suppose $\sin s_1p_1\sin s_2p_2\sin s_3p_3 \neq 0$. We put $x_i = \cot s_ip_i$ for $i = 1,2,3$. Inserting (3) $\sim$ (6) into three equalities

$$y_1'(a,t,\lambda,2\pi) = 0, \quad y_2(a,t,\lambda,2\pi) = 0, \quad y_2'(a,t,\lambda,2\pi) - y_1(a,t,\lambda,2\pi) = 0,$$

and dividing those by $\sin s_1p_1\sin s_2p_2\sin s_3p_3$, we obtain

$$\frac{p_1p_3}{p_2} - p_1x_2x_3 - p_2x_1x_3 - p_3x_1x_2 = 0, \quad (8)$$

$$-\frac{p_2}{p_1p_3} + \frac{1}{p_1}x_2x_3 + \frac{1}{p_2}x_1x_3 + \frac{1}{p_3}x_1x_2 = 0, \quad (9)$$

$$x_3 = -\frac{(p_1^2-p_3^2)p_2}{(p_1^2-p_2^2)p_3}x_2 - \frac{(p_2^2-p_3^2)p_1}{(p_1^2-p_2^2)p_3}x_1. \quad (10)$$

We deduce from (8) and (9) that

$$(-p_1p_2^2+p_1p_3^2)x_2x_3 + (-p_2^3+\frac{p_1^2p_3^2}{p_2})x_1x_3 + (-p_3p_2^2+p_1^2p_3)x_1x_2 = 0. \quad (11)$$

Plugging (10) into (11), we have

$$(p_2^2-p_3^2)(p_1^2-p_3^2)p_1p_2x_2^2 + 2p_1^2(p_2^2-p_3^2)^2x_1x_2 - \frac{p_1(p_1^2p_3^2-p_2^4)(p_2^2-p_3^2)}{p_2}x_1^2 = 0$$

and hence

$$x_2 = \left\{-\frac{p_1(p_2^2-p_3^2)}{p_2(p_1^2-p_3^2)} \pm \frac{p_3(p_1^2-p_2^2)}{p_2(p_1^2-p_3^2)}\right\}x_1. \quad (12)$$

This together with (10) implies that

$$x_3 = \mp x_1. \tag{13}$$

Combining (8) with (12) and (13), we conclude that

$$x_1^2 = -1.$$

This violates the fact that $\cot z \neq \pm\sqrt{-1}$ for $z \in \mathbb{C}$. Thus we obtain

$$\sin s_1 p_1 \sin s_2 p_2 \sin s_3 p_3 = 0.$$

Next we shall show that $p_1 p_2 p_3 \neq 0$. Let us first prove that $p_1 \neq 0$ by contradiction. Suppose that $p_1 = 0$. Noting $y_j''(x) = 0$ on (t_0, t_1) for $j = 1, 2$, we have

$$\begin{aligned}
y_1(a,t,\lambda,2\pi) &= \cos s_2 p_2 \cos s_3 p_3 - \frac{p_2}{p_3} \sin s_2 p_2 \sin s_3 p_3, && (14)\\
y_2'(a,t,\lambda,2\pi) &= \cos s_2 p_2 \cos s_3 p_3 - \frac{p_3}{p_2} \sin s_2 p_2 \sin s_3 p_3 \\
&\quad - s_1(p_2 \sin s_2 p_2 \cos s_3 p_3 + p_3 \cos s_2 p_2 \sin s_3 p_3), && (15)\\
y_1'(a,t,\lambda,2\pi) &= -p_2 \sin s_2 p_2 \cos s_3 p_3 - p_3 \cos s_2 p_2 \sin s_3 p_3, && (16)\\
y_2(a,t,\lambda,2\pi) &= s_1 \cos s_2 p_2 \cos s_3 p_3 - \frac{s_1 p_2}{p_3} \sin s_2 p_2 \sin s_3 p_3 \\
&\quad + \frac{1}{p_2} \sin s_2 p_2 \cos s_3 p_3 + \frac{1}{p_3} \cos s_2 p_2 \sin s_3 p_3. && (17)
\end{aligned}$$

Inserting (14) and (15) into $y_1(a,t,\lambda,2\pi) - y_2'(a,t,\lambda,2\pi) = 0$, and combining that with (16) and $y_1'(a,t,\lambda,2\pi) = 0$, we obtain

$$\frac{p_2^2 - p_3^2}{p_2 p_3} \sin s_2 p_2 \sin s_3 p_3 = 0$$

and hence $\sin s_2 p_2 \sin s_3 p_3 = 0$. This together with $y_1(a,t,\lambda,2\pi) = (-1)^k$ and (14) implies that $\cos s_2 p_2 \cos s_3 p_3 = (-1)^k$ and thus $\sin s_2 p_2 = \sin s_3 p_3 = 0$. Therefore, we infer by (17) that $y_2(a,t,\lambda,2\pi) = s_1(-1)^k \neq 0$ which is a contradiction. Hence we have $p_1 \neq 0$. Similarly we get $p_2 \neq 0$ and $p_3 \neq 0$.

Our next task is to demonstrate that $\sin s_1 p_1 = \sin s_2 p_2 = \sin s_3 p_3 = 0$. Because $\sin s_1 p_1 \sin s_2 p_2 \sin s_3 p_3 = 0$, we have $\sin s_1 p_1 = 0$ or $\sin s_2 p_2 = 0$ or $\sin s_3 p_3 = 0$. We first consider the case that $\sin s_1 p_1 = 0$. By (3), (6), and $y_1(a,t,\lambda,2\pi) = y_2'(a,t,\lambda,2\pi) = \pm 1$, we obtain

$$\begin{aligned}
\pm 1 &= \cos s_2 p_2 \cos s_3 p_3 - \frac{p_2}{p_3} \sin s_2 p_2 \sin s_3 p_3 \\
&= \cos s_2 p_2 \cos s_3 p_3 - \frac{p_3}{p_2} \sin s_2 p_2 \sin s_3 p_3.
\end{aligned}$$

Thus we have $\sin s_1 p_1 = \sin s_2 p_2 = \sin s_3 p_3 = 0$. This conclusion also follows from $\sin s_2 p_2 = 0$ or $\sin s_3 p_3 = 0$ in a similar manner.

Because $\sin s_1p_1 = \sin s_2p_2 = \sin s_3p_3 = 0$ and $p_1p_2p_3 \neq 0$, we have $s_ip_i \in \pi\mathbb{N}$ for $i = 1, 2, 3$. So we get

$$y_1(a,t,\lambda,x) = \begin{cases} \cos xp_1 & \text{for} \quad x \in [0, t_1), \\ \cos(x-t_1)p_2 \cos s_1p_1 & \text{for} \quad x \in [t_1, t_2), \\ \cos(x-t_2)p_3 \cos s_2p_2 \cos s_1p_1 & \text{for} \quad x \in [t_2, 2\pi). \end{cases}$$

Therefore we see that the number of zeros of $y_1(a,t,\lambda,\cdot)$ inside $[0, 2\pi)$ is equal to

$$(s_1p_1 + s_2p_2 + s_3p_3)/\pi.$$

Since

$$M(a,t,\lambda) = (-1)^k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we infer that $y_1(a,t,\lambda,x)$ is a periodic solution of (1) of period 2π or 4π. Because $\lambda \in \{\lambda_{2k}(a,t),\ \lambda_{2k+1}(a,t)\}$, the Haupt Theorem (see Theorem 3.1 in Chapter 8 of [1]) implies that $y_1(a,t,\lambda,\cdot)$ has exactly k zeros in $[0, 2\pi)$. Thus it follows that

$$(s_1p_1 + s_2p_2 + s_3p_3)/\pi = k.$$

Hence we obtain (ii).

Finally we shall prove that (ii) implies (7). We assume (ii). By (3) $\sim$ (6) we have

$$M(a,t,\lambda) = (-1)^k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

As in the above observation, we see that $y_1(a,t,\lambda,x)$ is a periodic solution of (1) of period 2π or 4π and that the number of zeros of $y_1(a,t,\lambda,\cdot)$ inside $[0,2\pi)$ is k. Thus the Haupt theorem again implies $\lambda \in \{\lambda_{2k}(a,t),\ \lambda_{2k+1}(a,t)\}$. □

Remark 2.1. *We consider the case that the potential Q is complex-valued, namely, $(a_1, a_2, a_3) \in \mathbb{C}^3$. Suppose that $a_m \neq a_n$ for $m \neq n$. We claim that the following statements* (**d**) *and* (**e**) *are equivalent.*

(**d**) *The equation* (1) *admits two linearly independent, periodic solution of period 2π or 4π.*

(**e**) $s_i^2(\lambda - a_i) \in \{\pi^2 j^2 |\ \ j \in \mathbb{N}\}$ *for $i = 1, 2, 3$.*

In particular, if there exist p and q for which $\operatorname{Im} a_p \neq \operatorname{Im} a_q$, then all the eigenvalues of (1) *are simple.*

Remark 2.2. *For the Hill equation with 4-step potential, there is no analogy to Theorem 1.1. To see this we give a counterexample. We put*

$$t_0 = 0, \quad t_1 = \frac{\pi}{2}, \quad t_2 = \frac{9-\sqrt{17}}{8}\pi, \quad t_3 = \frac{11-\sqrt{17}}{8}\pi, \quad t_4 = 2\pi,$$

$$s_j = t_j - t_{j-1}, \qquad a_j = \frac{\pi^2}{4s_j^2} \quad \textit{for} \quad j = 1, 2, 3, 4.$$

Let $V : \mathbb{R} \to \mathbb{R}$ *be a* 2π*-periodic function such that*

$$V(\cdot) = a_j \quad on \quad [t_{j-1}, t_j) \quad for \quad j = 1, 2, 3, 4.$$

Then the equation

$$-y''(x) + V(x)y(x) = 0 \quad on \quad \mathbb{R}$$

admits two linearly independent, periodic solutions of period 2π*, because the monodromy matrix of this equation equals the identity matrix. However, we have*

$$s_j\sqrt{a_j} = \frac{\pi}{2} \notin \pi\mathrm{N} \quad for \quad j = 1, 2, 3, 4.$$

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