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SCATTERING THEORY FOR THE ZAKHAROV EQUATIONS IN THREE SPACE DIMENSIONS

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1. Introduction and Main Results

We study the scattering theory for the Zakharov equation in three space dimensions:

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = uv, \\ \partial_t^2 v - \Delta v = \Delta|u|^2. \end{cases} \tag{1.1}$$

Here u and v are $\mathbb{C}^3$-valued and real valued unknown functions of $(t,x) \in \mathbb{R} \times \mathbb{R}^3$, respectively. The first and the second equations of the system (1.1) are the Schrödinger and the wave components, respectively. In this article, we prove the existence and the uniqueness of an asymptotically free solution for the equation (1.1) without any restrictions on the size of the final data and on the support of the Fourier transform of the Schrödinger data.

Ozawa and Y. Tsutsumi [11] showed the existence and the uniqueness of an asymptotically free solution for the Zakharov equation (1.1). They assumed either a restriction on size of the final data $(u_+, v_+, \dot{v}_+)$ or a restriction on the support of the Fourier transform $\hat{u}_+$ of the Schrödinger component of the final data. More precisely, the Fourier transform of the Schrödinger data vanishes in a neighborhood of the unit sphere so that they could use the difference between the propagation property of the Schrödinger equation and the wave one and obtained additional time decay estimates for the nonlinear term uv. Here we remark that we can not apply the phase correction method, (which is applicable to the long range scattering for the linear and nonlinear Schrödinger equations), to the nonlinear term uv, because all derivatives of the solution for the free wave equation decay t^{-1} in L^∞. Note that, roughly speaking, the phase correction method is applicable if a time-dependent long range potential and its k-th order derivative decay as t^{-1-k} in L^∞. (For details about the long range scattering to the nonlinear Schrödinger equation by the phase correction method, see, e.g., Ginibre and Ozawa [2] and Ozawa [9]).

There are several results on the scattering theory for other coupled systems related with the Schrödinger equations, that is, the existence of the wave operators for the Klein-Gordon-Schrödinger equation in two space dimensions ([10], [14], [16] and [17]) and the existence of the modified wave operators for the wave-Schrödinger and the Maxwell-Schrödinger equations in three space dimensions ([3], [4], [5], [12], [13] and [19]).

Notations. Let $\mathcal{S}$ be the Schwartz class on $\mathbb{R}^3$ and let $\mathcal{S}'$ be the set of tempered distributions on $\mathbb{R}^3$. For $s, m \in \mathbb{R}$, let

$$H^{m,s} \equiv \{\psi \in \mathcal{S}' \colon \|\psi\|_{H^{m,s}} \equiv \|(1+|x|^2)^{s/2}(1-\Delta)^{m/2}\psi\|_{L^2} < \infty\}$$

and $H^m = H^{m,0}$. For $1 \le p \le \infty$ and a positive integer k, we set

$$W_p^k \equiv \left\{ \psi \in L^p \colon \|\psi\|_{W_p^k} \equiv \sum_{|\alpha|\le k} \|\partial^\alpha \psi\|_{L^p} < \infty \right\}.$$

For $s \in \mathbb{R}$, let $\dot{H}^s$ be the homogeneous Sobolev space of order s, and let

$$\|w\|_{\dot{H}^s} \equiv \|(-\Delta)^{s/2} w\|_{L^2}.$$

We set for $t \in \mathbb{R}$,

$$U(t) \equiv e^{\frac{it}{2}\Delta}, \quad \omega \equiv (-\Delta)^{1/2},$$

$$\mathcal{L} \equiv i\partial_t + \frac{1}{2}\Delta, \quad \Box \equiv \partial_t^2 - \Delta.$$

Throughout this article, we assume that the space dimension is three. In this article, we prove the existence and the uniqueness of an asymptotically free solution for the equation (1.1) without any restrictions on the size of the final data and on the support of the Fourier transform of the Schrödinger data. Namely, we remove the size restriction on the final data and the support restriction on the Fourier transform of the Schrödinger component of the final data from the result by Ozawa and Tsutsumi [11].

Let $(u_+, v_+, \dot{v}_+)$ be final data. u_+ and $(v_+, \dot{v}_+)$ are the Schrödinger and the wave components. Let

$$u_0(t,x) = (U(t)u_+)(x), \tag{1.2}$$

$$v_0(t,x) = ((\cos\omega t)v_+)(x) + ((\omega^{-1}\sin\omega t)\dot{v}_+)(x). \tag{1.3}$$

u_0 and v_0 are free solutions for the Schrödinger and the wave equations, respectively.

The main result is the following:

Theorem. *Assume that* $u_+ \in H^{6,9}$, $\omega^{-2}v_+ \in H^{9,2}$ *and* $\omega^{-2}\dot{v}_+ \in H^{8,2} \cap \dot{H}^{-1}$. *Then there exists a constant* $T > 0$ *such that the equation* (1.1) *has a unique solution* (u, v) *satisfying*

$$u \in C([T,\infty); H^3),$$

$$v \in C([T,\infty); H^2),$$

$$\partial_t v \in C([T,\infty); H^1 \cap \dot{H}^{-1}),$$

$$\sup_{t\geq T}(t^{5/4}\|u(t)-u_0(t)\|_{L^2}+t\|u(t)-u_0(t)\|_{\dot{H}^1\cap\dot{H}^3})<\infty,$$
$$\sup_{t\geq T}[t\{\|v(t)-v_0(t)\|_{H^2}+\|\partial_t v(t)-\partial_t v_0(t)\|_{H^1\cap\dot{H}^{-1}}\}]<\infty.$$

A similar result holds for negative time.

Remark 1.1. The assumptions $v_+\in\dot{H}^{-2}$ and $\dot{v}_+\in\dot{H}^{-3}$ in Theorem implies that their Fourier transforms $\hat{\psi}_0$ and $\hat{\psi}_1$ vanish at the origin.

The constant T which appears in Theorem depends only on

$$\eta\equiv\|u_+\|_{H^{6,9}}+\|\omega^{-2}v_+\|_{H^{9,2}}+\|\omega^{-2}\dot{v}_+\|_{H^{8,2}\cap\dot{H}^{-1}}. \tag{1.4}$$

In Theorem, we do not restrict the size of η.

The strategy of the proof is the following:

- solving the Cauchy problem at $t=\infty$ to the equation (1.1) for a given asymptotic profile (A,B) with appropriate time decay estimates of A, B, $\mathcal{L}A-AB$ and $\Box B-\Delta|A|^2$,
- constructing an asymptotic profile (A,B) satisfying the assumptions of above Cauchy problem at $t=\infty$ by the final data $(u_+,v_+,\dot{v}_+)$ which belong to suitable function spaces.

We solve the Cauchy problem at $t=\infty$ for the equation (1.1) by the energy estimates. Note that since $\mathcal{L}A-AB$ is an error of the approximate solution (A,B) for the Schrödinger equation, it is difficult to solve this Cauchy problem if $\mathcal{L}A-AB$ decays slowly in time. In fact, in order to solve the Cauchy problem at $t=\infty$ without any size restrictions on the asymptotic profile (A,B), it is necessary that $\mathcal{L}A-AB$ decays faster than $t^{-9/4}$ in H^3. If we set $(A,B)=(u_0,v_0)$, where u_0 and v_0 are free solutions for the Schrödinger and the wave equations, respectively, then unfortunately $\mathcal{L}A-AB=-u_0v_0$ decays as $t^{-3/2}$ in L^2, since u_0 and v_0 decay as $t^{-3/2}$ and as t^{-1} in $L^\infty(\mathbb{R}^3)$. (This is not sufficient). To overcome this difficulty and to obtain an additional time decay estimate of $\mathcal{L}A-AB$ without assuming the support restriction on the Fourier transform on the Schrödinger data, we construct an asymptotic profile of the form $(A,B)=(u_0+u_1,v_0)$. We find a second correction term u_1 such that u_1 and $\mathcal{L}u_1-u_0v_0$ decay faster than u_0 and u_0v_0, respectively. Actually, we can choose u_1 such that $\mathcal{L}A-AB$ decays as $t^{-5/2}$ in H^3. The similar method is applicable to the other coupled systems of the Schrödinger equation and the wave equations (see [5, 12, 13, 14, 16, 17]) and to the nonlinear Schrödinger equation with non-gauge invariant nonlinearity (see [8, 18]).

The outline of this article is as follows. In Section 2, we solve the Cauchy problem at $t=\infty$ to the equation (1.1) for a given asymptotic profile (A,B) with appropriate time decay estimates of A, B, $\mathcal{L}A-AB$ and $\Box B-\Delta|A|^2$. In Section 3, we construct an asymptotic profile (A,B) satisfying the assumptions of above Cauchy problem at $t=\infty$ by the final data $(u_+,v_+,\dot{v}_+)$ which satisfy the assumptions in Theorem.

2. The Cauchy Problem at Infinity

In this section, we solve the Cauchy problem at infinity for the equation (1.1) of general form. Namely, for an asymptotic profile (A, B) satisfying suitable assumptions, we construct a unique solution (u, v) for the equation (1.1) which approaches (A, B) as $t \to \infty$.

Let (A, B) be an asymptotic profile. Here A and B are $\mathbb{C}^3$ and real valued, respectively. We introduce the following functions:

$$R_1[A, B] = \mathcal{L}A - AB, \tag{2.1}$$

$$R_2[A, B] = \Box B - \Delta|A|^2. \tag{2.2}$$

The functions R_1 and R_2 are the errors of the approximation (A, B) for the system (1.1), since they are the differences between the left hand sides and the right hand ones of the first and the second equalities in that system.

Proposition 2.1. *Assume that there exist constants $\delta > 0$ and $\varepsilon > 0$ such that for $t \geq 1$,*

$$\begin{gathered} \|A(t)\|_{W^3_\infty} \leq \delta t^{-3/2}, \\ \|B(t)\|_{W^2_\infty} \leq \delta t^{-1}, \\ \|R_1[A, B](t)\|_{H^3} + \|\partial_t R_1[A, B](t)\|_{H^1} \leq \delta t^{-9/4-\varepsilon}, \\ \|R_2[A, B](t)\|_{H^2 \cap \dot{H}^{-1}} + \|\partial_t^2 R_2[A, B](t)\|_{L^2} \leq \delta t^{-2-\varepsilon}. \end{gathered}$$

Then there exists a constant $T \geq 1$, depending only on δ, such that the equation (1.1) has a unique solution (u, v) satisfying

$$\begin{gathered} u \in C([T, \infty); H^3), \\ v \in C([T, \infty); H^2), \quad \partial_t v \in C([T, \infty); H^1 \cap \dot{H}^{-1}), \\ \sup_{t \geq T}(t^{5/4}\|u(t) - A(t)\|_{L^2} + t\|u(t) - A(t)\|_{\dot{H}^1 \cap \dot{H}^3}) < \infty, \\ \sup_{t \geq T}[t\{\|v(t) - B(t)\|_{H^2} + \|\partial_t v(t) - \partial_t B(t)\|_{H^1 \cap \dot{H}^{-1}}\}] < \infty. \end{gathered}$$

We can prove this proposition by the standard energy estimates for the functions $(u-A, v-B, \partial_t(v-B))$ in the space $H^3 \oplus H^2 \oplus (H^1 \cap \dot{H}^{-1})$. For the detailed proof, see Section 3 in [15].

Remark 2.1. In Proposition 2.1, the asymptotic profile (A, B) is not determined explicitly. In Section 3, we construct the asymptotic profile satisfying the assumptions of Proposition 2.1.

Remark 2.2. Note that in the assumptions of Proposition 2.1, the asymptotic profile (A, B) decays as fast as the free solution (u_0, v_0) as $t \to \infty$.

3. Asymptotics and Proof of Theorem

In this section, we construct an asymptotic profile (A, B) satisfying the assumptions of Proposition 2.1.

First we recall time decay estimates of the solutions for the free Schrödinger and wave equations, which are the principal part of the asymptotic profile. (see, e.g., Section 2 in Ozawa and Tsutsumi [11]):

Lemma 3.1. *Let k be a non-negative integer. There exists a constant $C > 0$ such that for $t \geq 1$,*

$$\sum_{|\alpha|+2j\leq k} \|\partial_x^\alpha \partial_t^j u_0(t)\|_{L^2} \leq C\|u_+\|_{H^k},$$

$$\sum_{|\alpha|+2j\leq k} \|\partial_x^\alpha \partial_t^j u_0(t)\|_{L^\infty} \leq C\|u_+\|_{W_1^k} t^{-3/2},$$

$$\sum_{|\alpha|+2j\leq k} \|\partial_x^\alpha \partial_t^j u_0(t)\|_{L^\infty} \leq C\|u_+\|_{H^{k,2}} t^{-3/2},$$

$$\sum_{|\alpha|+j\leq k} \|\partial_x^\alpha \partial_t^j v_0(t)\|_{L^2} \leq C(\|v_+\|_{H^k} + \|\dot{v}_+\|_{H^{k-1}} + \|\dot{v}_+\|_{\dot{H}^{-1}}),$$

$$\sum_{|\alpha|+j\leq k} \|\partial_x^\alpha \partial_t^j v_0(t)\|_{L^\infty} \leq C(\|v_+\|_{W_1^{k+2}} + \|\dot{v}_+\|_{W_1^{k+1}}) t^{-1}.$$

According to Lemma 3.1, we see that if we put $(A, B) = (u_0, v_0)$, then $\|R_1[A, B](t)\|_{L^2} = \|u_0(t)v_0(t)\|_{L^2} = O(t^{-3/2})$. This time decay estimate does not satisfy the assumptions of Proposition 2.1. To overcome this difficulty, we find an asymptotic profile of the form $(A, B) = (u_0 + u_1, v_0)$, where u_1 is a second correction term which will be determined below. We see

$$R_1[A, B] = (\mathcal{L}u_1 - u_0 v_0) - u_1 v_0, \tag{3.1}$$

$$R_1[A, B] = \Delta|u_0 + u_1|^2. \tag{3.2}$$

We construct a second correction term u_1 of the Schrödinger component such that u_1 and $\mathcal{L}u_1 - u_0 v_0$ decays faster than u_0 and $u_0 v_0$ as $t \to \infty$, respectively, and so that $R_1[A, B]$ satisfies the assumption of Proposition 2.1.

We construct a second correction u_1 of the form

$$u_1(t, x) = u_0(t, x)V(t, x), \tag{3.3}$$

where

$$V(t, x) = ((\cos \omega t)Q_0)(x) + ((\omega^{-1} \sin \omega t)Q_1)(x). \tag{3.4}$$

We determine functions Q_0 and Q_1 of $x \in \mathbb{R}^3$. We first note the following identity:

$$\mathcal{L}(wz) = w\frac{1}{2}\Delta z + z\mathcal{L}w + \frac{1}{t}\left(-i\sum_{k=1}^{3}(J_k w)(\partial_k z) + iwPz\right) \tag{3.5}$$

for a $\mathbb{C}^3$-valued function w and a real valued function z, where

$$J_k \equiv x_k + it\partial_k \ (k = 1, 2, 3), \quad J \equiv (J_1, J_2, J_3),$$
$$P \equiv t\partial_t + x \cdot \nabla.$$

It is well-known that if w and z solve the free Schrödinger and wave equations, then so do $J_k w$ and Pz because $J\mathcal{L} - \mathcal{L}J = 0$ and $\Box P = (P + 2)\Box$. Noting this fact and putting $w = u_0$ and $z = V$, we expect that the most slowly decaying part of $\mathcal{L}u_1$ is $(1/2)u_0\Delta V$. Now we set

$$Q_0(x) \equiv -2(-\Delta)^{-1}v_+(x) = -2\omega^{-2}v_+(x), \tag{3.6}$$
$$Q_1(x) \equiv -2(-\Delta)^{-1}\dot{v}_+(x) = -2\omega^{-2}\dot{v}_+(x), \tag{3.7}$$

so that the equality

$$\frac{1}{2}u_0\Delta V = u_0 v_0$$

holds. Then it is expected that $\mathcal{L}u_1 - u_0v_0$ decays faster than u_0v_0 as $t \to \infty$.

From the equality (3.5), we have

$$\mathcal{L}u_1 - u_0v_0 = \frac{1}{t}\left(-i\sum_{k=1}^{3}(J_k u_0)(\partial_k V) + iu_0PV\right). \tag{3.8}$$

Remark 3.1. It is well known that

$$J_k u_0(t, \cdot) = J_k(t)U(t)u_+ = U(t)(\mathcal{M}_{x_k}u_+) \quad (k = 1, 2, 3),$$
$$PV(t, \cdot) = (\cos\omega t)(\mathcal{M}_x \cdot \nabla Q_0) + (\omega^{-1}\sin\omega t)((1 + \mathcal{M}_x \cdot \nabla)Q_1),$$

where $\mathcal{M}_{x_k}$ and $\mathcal{M}_x$ are the multiplication operators by the function x_k and x, respectively.

The time decay estimates of u_1 and $\mathcal{L}u_1 - u_0v_0$ are as follows.

Lemma 3.2. *There exists a constant $C > 0$ such that for $t \geq 1$,*

$$\sum_{j=0}^{2}\|\partial_t^j u_1(t)\|_{H^{4-j}} \leq C\eta^2 t^{-3/2},$$
$$\sum_{j=0}^{2}\|\partial_t^j u_1(t)\|_{W^{4-j}_\infty} \leq C\eta^2 t^{-5/2},$$
$$\|\mathcal{L}u_1(t) - u_0(t)v_0(t)\|_{H^3} + \|\partial_t(\mathcal{L}u_1(t) - u_0(t)v_0(t))\|_{H^1} \leq C\eta^2 t^{-5/2},$$

where $\eta > 0$ is defined in (1.4).

Noting Lemmas 3.1, Remark 3.1 and the equality (3.8), we can prove this lemma exactly in the same way as in the proof of Lemma 3.3 in [12].

We set $(A, B) = (u_0 + u_1, v_0)$. Recalling the equalities (3.1) and (3.2) and using the Hölder inequality, we have the time decay estimates of the asymptotic profile (A,B) and the functions $R_1[A,B]$ and $R_2[A,B]$ by Lemmas 3.1 and 3.2.

Lemma 3.3. *There exists a constant $C>0$ such that for $t\geq 1$,*

$$\|A(t)\|_{W^3_\infty} \leq C(\eta+\eta^2)t^{-3/2},$$

$$\|B(t)\|_{W^2_\infty} \leq C\eta t^{-1},$$

$$\|R_1[A,B](t)\|_{H^3} + \|\partial_t R_1[A,B](t)\|_{H^1} \leq C(\eta^2+\eta^3)t^{-5/2},$$

$$\sum_{j=0}^{2} \|\partial_t^j R_2[A,B](t)\|_{H^{2-j}} \leq C(\eta^2+\eta^4)t^{-7/2},$$

$$\|R_2[A,B](t)\|_{\dot{H}^{-1}} \leq C(\eta^2+\eta^4)t^{-5/2},$$

where $\eta>0$ is defined in (1.4).

Proof of Theorem. From Lemma 3.3, we see that the asymptotic profile (A,B) and the functions $R_1[A,B]$ and $R_2[A,B]$ satisfy the assumptions of Proposition 2.1 for $\delta=\eta+\eta^4$ and $\varepsilon=1/4$. Theorem immediately follows from Proposition 2.1. □

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