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Some recent results on Schrödinger equations with time-periodic potentials

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1 Introduction. Statement of Results

We report on some recent results on Schrödinger equations with time-periodic potentials. The full report on the results will be published in [1]. We consider the Schrödinger equation

$$i\partial_t u = (-\Delta + V(t,x))\,u, \quad (t,x) \in \mathbf{R} \times \mathbf{R}^3. \tag{1.1}$$

Note that the results presented here depend on the configuration space being of dimension three.

We make the following assumption on the potential $V(t,x)$. We write $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ for the unit circle and let $\langle x\rangle = (1+x^2)^{1/2}$.

Assumption 1.1. *The function $V(t,x)$ is real-valued and is 2π-periodic with respect to t: $V(t,x) = V(t+2\pi, x)$. For $\beta > 2$ we assume that*

$$\sum_{j=0}^{2} \sup_{x\in\mathbf{R}^3} \langle x\rangle^{\beta} \left(\int_0^{2\pi} |\partial_t^j V(t,x)|^2 dt\right)^{\frac{1}{2}} < \infty. \tag{1.2}$$

Associated with the equation (1.1) is a unitary propagator $U(t,s)$, which is a family of unitary operators on $\mathcal{H} = L^2(\mathbf{R}^3)$ with the following properties. We let $H^2(\mathbf{R}^3)$ denote the usual Sobolev space of order 2.

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1. $(t,s) \mapsto U(t,s)$ is strongly continuous.
2. $U(t,r) = U(t,s)U(s,r)$ for all $t,s,r \in \mathbf{R}$.
3. $U(t+2\pi, s+2\pi) = U(t,s)$ for all $t,s \in \mathbf{R}$.
4. $U(t,s)H^2(\mathbf{R}^3) = H^2(\mathbf{R}^3)$. For $u_0 \in H^2(\mathbf{R}^3)$, $U(t,s)u_0$ is an $\mathcal{H}$-valued C^1-function of (t,s), which satisfies

$$i\partial_t U(t,s)u_0 = H(t)U(t,s)u_0,$$
$$i\partial_s U(t,s)u_0 = -U(t,s)H(s)u_0.$$

To study the properties of $U(t,s)$ in detail one introduces the extended phase space $\mathcal{K} = L^2(\mathbf{T} \times \mathbf{R}^3) \equiv L^2(\mathbf{T}; \mathcal{H})$. Define

$$K_0 = -i\partial_t - \Delta,$$
$$K = -i\partial_t - \Delta + V(t,x).$$

These operators are self-adjoint on $\mathcal{K}$, on the natural domain. The relation to the propagator is as follows. Let $\mathcal{U}(\sigma) = e^{-i\sigma K}$. Then

$$(\mathcal{U}(\sigma)u)(t) = U(t, t-\sigma)u(t-\sigma)$$

for $u = u(t,\cdot) \in \mathcal{K}$. The extended phase space formalism was introduced by Howland in [2], and implemented for the time-periodic case by Yajima in [6].

One of the problems considered in [1] is the large time behavior of a solution $u(t) = U(t,0)u_0$. The usual approach is to use scattering theory. Let $\lambda_j \in [0,1)$ be eigenvalues of K with eigenfunctions φ_j (the spectrum of K is invariant under integer translations). Under the above conditions the wave operators exist and are complete:

$$W_\pm = \underset{t\to\pm\infty}{s\text{-lim}}\, U(0,t)e^{-itH_0}, \quad H_0 = -\Delta.$$

Completeness means that $\operatorname{Ran} W_\pm = \mathcal{H}_{\mathrm{ac}}(\mathcal{U}_0)$ and $\mathcal{H}_{\mathrm{sc}}(\mathcal{U}_0) = \{0\}$. Here $\mathcal{U}_0 = U(2\pi, 0)$ denotes the monodromy (Floquet) operator. Consequently $u(t) = U(t,0)u_0$ can be written as

$$u(t,x) = \sum a_j e^{-it\lambda_j}\varphi_j(t,x) + u_{\mathrm{scat}}(t,x),$$

where

$$\|u_{\mathrm{scat}}(t,x) - e^{-itH_0}\psi(x)\| \to 0 \quad \text{for } t \to \infty$$

for some $\psi \in \mathcal{H}$. Actually, these results hold under the short range assumption $\beta > 1$ in (1.2), see [6], [3], [7].

Here we consider a different approach to the large time behavior of a solution. Let for $\delta \in \mathbf{R}$

$$\mathcal{H}_\delta = L^2_\delta(\mathbf{R}^3) = \{f \in L^2_{\mathrm{loc}}(\mathbf{R}^3) \mid \langle x\rangle^\delta f(x) \in L^2(\mathbf{R}^3)\}$$

denote the weighted spaces. Then we take $u_0 \in \mathcal{H}_\delta$ for a sufficiently large $\delta > 0$, and look at the solution $U(t,0)u_0$ in the space $\mathcal{H}_{-\delta}$.

To state our results we introduce the weighted Sobolev spaces

$$\mathcal{K}^s_\delta = H^s(\mathbf{T}, \mathcal{H}_\delta),$$

where s is a nonnegative integer, and $\delta \in \mathbf{R}$. We introduce the following definition:

Definition 1.2. *$n \in \mathbf{Z}$ is said to be a threshold resonance of K, if there exists a solution $u(t,x)$ of the equation*

$$-i\partial_t u - \Delta u + Vu = nu,$$

such that, with a constant $C \neq 0$,

$$u(t,x) = \frac{Ce^{int}}{|x|} + u_1(t,x), \quad u_1 \in \mathcal{K}.$$

Such a solution is called an n-resonant solution.

The main results on the large time behavior of a solution can be stated as follows:

Theorem 1.3. *Let $\beta_k = \max\{2k+1, 4\}$. Let V satisfy the Assumption 1.1 for some $\beta > \beta_k$, $k \in \mathbf{N}$, and let $\{\phi_j\}$ be an orthonormal basis of eigenfunctions of K corresponding to the eigenvalues $0 \leq \lambda_j < 1$. Set $\delta = \beta/2$ and $\varepsilon_0 = \min\{1, \frac{\beta-\beta_k}{2}\}$. We have the following results.*

(1) *Suppose K has neither threshold resonances nor integer eigenvalues. Then there exist finite rank operators $B_1, \ldots, B_k$ from $\mathcal{H}_\delta$ to $\mathcal{K}^1_{-\delta}$, such that $B_j = 0$, unless j is odd, and such that, for any $u_0 \in \mathcal{H}_\delta$, and for any ε, $0 < \varepsilon < \varepsilon_0$, as $t \to \infty$,*

$$U(t,0)u_0 = \sum_j c_j e^{-it\lambda_j}\phi_j(t,x) + t^{-\frac{3}{2}}B_1u_0(t,x) + \cdots$$
$$\cdots + t^{-\frac{k}{2}-1}B_ku_0(t,x) + O(t^{-\frac{k+\varepsilon}{2}-1}),$$

where $c_j = 2\pi(\phi_j(0), u_0)_{\mathcal{H}}$, and $O(t^{-\frac{k+\varepsilon}{2}-1})$ stands for an $\mathcal{H}_{-\delta}$-valued function of t such that its norm in $\mathcal{H}_{-\delta}$ is bounded by $Ct^{-\frac{k+\varepsilon}{2}-1}\|u_0\|_{\mathcal{H}_\delta}$, when $t \geq 1$.

(2) *Suppose K has either threshold resonances, integer eigenvalues, or both, and assume that $\beta > \beta_k$, $k \geq 2$. Furthermore, $\{\phi_{01}, \ldots, \phi_{0m}\} \subset \{\phi_j\}$ is an orthonormal basis of eigenfunctions of K with eigenvalue 0. Then there exist a 0-resonant solution $\psi(t,x)$, finite rank operators $B_1, \ldots, B_{k-2}$ from $\mathcal{H}_\delta$ to $\mathcal{K}^1_{-\delta}$, such that $B_j = 0$, unless j is odd, and such that, for any $u_0 \in \mathcal{H}_\delta$ and for any $0 < \varepsilon < \varepsilon_0$, as $t \to \infty$,*

$$U(t,0)u_0 = \sum_j c_j e^{-it\lambda_j}\phi_j(t,x) + t^{-\frac{1}{2}}\Big(d_0\psi(t,x) + \sum_{\ell=1}^{m} d_\ell \phi_{0\ell}(t,x)\Big)$$
$$+ t^{-\frac{3}{2}}B_1 u_0(t,x) + \cdots + t^{-\frac{k-2}{2}-1}B_{k-2}u_0(t,x) + O(t^{-\frac{k-2+\varepsilon}{2}-1}),$$

where c_j and $O(t^{-\frac{k-2+\varepsilon}{2}-1})$ are as in Part (1), $d_0 = 2\pi(u_0, \psi(0))_{\mathcal{H}}$, and d_ℓ are linear functionals of u_0 of the form

$$d_\ell(u_0) = a_{\ell 1}(u_0, \phi_{01}(0))_{\mathcal{H}} + \cdots + a_{\ell n}(u_0, \phi_{0m}(0))_{\mathcal{H}}, \quad \ell = 1, \ldots, m.$$

In particular, all d_ℓ vanish on the orthogonal complement of the 1-eigenspace of the monodromy operator $\mathcal{U}_0 = U(2\pi, 0)$.

The statement in Part (2) is written for the case that we have both a zero threshold resonance and zero eigenvalues. In the other cases obvious modifications are needed.

2 Remarks on the Proof

The proof of Theorem 1.3 is quite long and rather involved. The starting point is a detailed study of the resolvent $R_0(z) = (K_0 - z)^{-1}$, in the form of asymptotic expansions, in the norm topology of the bounded operators $B(\mathcal{K}^s_\delta, \mathcal{K}^s_{-\delta})$ for $\delta > 1/2$ and $s \geq 0$ an integer. This study is based on the explicit integral kernel of $(-\Delta - z)^{-1}$ on $L^2(\mathbf{R}^3)$. This fact explains why the results depend on the dimension being equal to three. The expansions are needed around every $z \in \mathbf{R}$, the integers being particularly important, since they are the thresholds for our operators K_0 and K.

Let us briefly state some of the results on $R_0(z)$. Using Taylor's theorem the integral kernel of $r_0(z) = (-\Delta - z)^{-1}$ can be expanded as

$$\frac{e^{i\sqrt{z}|x-y|}}{4\pi|x-y|} = \sum_{j=0}^{k} \frac{1}{4\pi j!}(i\sqrt{z})^j |x-y|^{j-1} + d_k(z;x,y),$$

with the remainder given by

$$d_k(z;x,y) = \frac{(i\sqrt{z})^k |x-y|^{k-1}}{4\pi(k-1)!}\int_0^1 (1-s)^{k-1}(e^{is\sqrt{z}|x-y|} - 1)ds.$$

Using the terms above to define operators we get an expansion

$$r_0(z) = g_0 + \sqrt{z}g_1 + \cdots + z^{k/2}g_k + d_k(z), \quad d_k(z) = \mathcal{O}(z^{\frac{k+\varepsilon}{2}}),$$

in the topology of $B(\mathcal{H}_\gamma, \mathcal{H}_{-\gamma})$, $\gamma > \beta_k = \max\{2k+1, 4\}$, and $0 \le \varepsilon < \varepsilon_0 = \min\{1, \gamma - \frac{\beta_k}{2}\}$.

Now use $\mathcal{K} = L^2(\mathbf{T}) \otimes \mathcal{H}$, and let p_n denote the projection onto the subspace of $L^2(\mathbf{T})$ spanned by e^{int}. We can then write

$$R_0(z) = (K_0 - z)^{-1} = \sum_{m\in\mathbf{Z}}{}^{\oplus} p_m \otimes r_0(z-m).$$

Inserting the expansion for $r_0(z)$ we get an expansion

$$R_0(z+n) = R_0^+(n) + \sqrt{z}D_1(n) + \cdots + z^{k/2}D_k(n) + \widetilde{R}_{0k}(n,z),$$

where the coefficients can be found explicitly in terms of the g_j's introduced above.

The next step is the study of $1 + R_0(z)V$. Invertibility of the boundary values of $1 + R_0(\lambda + i0)V$ in $B(\mathcal{K}^s_\delta, \mathcal{K}^s_{-\delta})$ for $\delta > 1/2$, and $s \ge 0$ an integer, is first studied. The relation to eigenvalues and threshold resonances is also established. Based on the asymptotic expansion of $R_0(z)$ one obtains an asymptotic expansion of $(1 + R_0(z)V)^{-1}$, which then together with the ssecond resolvent equation $R(z) = (1 + R_0(z)V)^{-1}R_0(z)$ yields asymptotic expansions for $R(z)$. We use the technique developed by Murata [5] to obtain these results. One can also use the techniques developed in [4].

The main results can be stated as follows. Here E_n denotes multiplication by e^{int}.

Theorem 2.1. *Let V satisfy the Assumption* 1.1 *for $\beta > \beta_k \equiv \max\{2k + 1, 4\}$, $k \in \mathbf{N}$. Let $\delta = \beta/2$ and $\varepsilon_0 = \min\{1, \frac{\beta-\beta_k}{2}\}$.*

(1) *Suppose that there are no integer eigenvalues or resonances. Then, as a $B(\mathcal{K}^s_\delta, \mathcal{K}^s_{-\delta})$-valued function of $z \in \overline{\mathbf{C}^+}$, $s = 0, 1$, for any $0 < \varepsilon < \varepsilon_0$, we have*

$$R(z+n) = F_0(n) + \sqrt{z}F_1(n) + zF_2(n) + \cdots + z^{k/2}F_k(n) + \mathcal{O}(z^{(k+\varepsilon)/2})$$

in a neighborhood of $z = 0$.
(i) *$F_j(n) = E_nF_j(0)E_n^*$ for all $n \in \mathbf{Z}$ and $j = 0, 1, \ldots$.*
(ii) *If j is odd, $F_j(0)$ are operators of finite rank and may be written as a finite sum $\sum a_{j\nu} \otimes b_{j\nu}$, where $a_{j\nu}, b_{j\nu} \in \mathcal{K}^1_{-\delta}$.*
(iii) *The first few terms are given as*

$$F_0(n) = G^+(n)R_0^+(n)(= R^+(n)),$$

$$F_1(n) = G^+(n)D_1(n)G^-(n)^*,$$
$$F_2(n) = G^+(n)\left[D_2(n) - D_1(n)VG^+(n)D_1(n)\right]G^-(n)^*,$$

where $G^\pm(n) = (1 + R_0^\pm(n)V)^{-1}$, *and where* $D_j(n)$ *are the operators defined in result on the free resolvent.*

(2) *Suppose that we have either integer eigenvalues, threshold resonances, or both. Then, as a* $B(\mathcal{K}_\delta^s, \mathcal{K}_{-\delta}^s)$*-valued function of* $z \in \overline{\mathbf{C}^+}$, $s = 0, 1$,

$$R(z+n) = -\frac{1}{z}F_{-2}(n) + \frac{1}{\sqrt{z}}F_{-1}(n) + F_0(n) + \cdots$$
$$\cdots + z^{(k-2)/2}F_{k-2}(n) + \mathcal{O}(z^{(k-2+\varepsilon)/2})$$

in a neighborhood of $z = 0$. *Here*
(i) $F_j(n) = E_nF_j(0)E_n^*$ *for* $n \in \mathbf{Z}$ *and* $j = -2, -1, \ldots$.
(ii) $F_j(n)$ *is of finite rank, when* j *is odd, and may be written as a finite sum* $\sum a_{j\nu} \otimes b_{j\nu}$, *where* $a_{j\nu}, b_{j\nu} \in \mathcal{K}_{-\delta}^1$.
(iii) $F_{-2}(n) = E_K(\{n\})$.
(iv) $F_{-1}(n) = E_K(\{n\})VD_3(n)VE_K(\{n\}) - 4\pi i\bar{Q}_n$, *where* $\bar{Q}_n = \langle\cdot, \psi^{(n)}\rangle\psi^{(n)}$, *and* $\psi^{(n)}$ *is a suitably normalized n-resonant function.*

Now we use the relation

$$e^{-i\sigma K}u = \lim_{\varepsilon\downarrow 0}\lim_{N\to\infty}\frac{1}{2\pi i}\int_{-N}^{N} e^{-i\sigma\lambda}R(\lambda + i\varepsilon)u\,d\lambda,$$

where the right hand side should be understood as a weak integral. It is combined with the asymptotic expansion results, to obtain the following result.

Let $J\colon \mathcal{H} \to \mathcal{K}$ be given by $(Ju)(t,x) = u(x)$. We then have the following asymptotic expansion for $e^{-i\sigma K}Ju_0$, $u_0 \in \mathcal{H}_\delta$ (δ sufficiently large).

$$e^{-i\sigma K}Ju_0 = \sum_{j=1}^{k}\sigma^{-(j+2)/2}\Big(\sum_{n\in\mathbf{Z}} e^{-i\sigma n}\varepsilon_jF_j(n)Ju_0\Big) + O(\sigma^{-\frac{k+2+\varepsilon}{2}})$$

Here $\varepsilon_j = 0$ for j even, and $\varepsilon_j = 1$ for j odd. The expansion coefficients satisfy $F_j(n) = E_nF_j(0)E_n^*$, and

$$F_j(0) = \sum_{\substack{\nu\\ \text{finite}}} f_{j\nu} \otimes g_{j\nu}.$$

By the Sobolev embedding theorem $\mathcal{K}_{-\delta}^1$ is continuously embedded in $\mathcal{H}_{-\delta}$ such that

$$\sup_{t\in\mathbf{T}}\|u(t)\|_{\mathcal{H}_{-\delta}} \le C\|u\|_{\mathcal{K}_{-\delta}^1}.$$

Using the expansion for $e^{-i\sigma K}$ we get

$$\sup_{t\in\mathbf{T}} \|U(t,t-\sigma)u_0 - \sum_{j=1}^{k} \varepsilon_j \sigma^{-(j+2)/2} Z_j(\sigma) J u_0(t)\|_{\mathcal{H}_{-\delta}} \leq C\sigma^{-\frac{k+2+\varepsilon}{2}} \|u_0\|_{\mathcal{H}_\delta}.$$

Now let $t = \sigma$, and then replace σ by t to get

$$\|U(t,0)u_0 - \sum_{j=1}^{k} \varepsilon_j t^{-(j+2)/2} B_j(t) u_0\|_{\mathcal{H}_{-\delta}} \leq Ct^{-\frac{k+2+\varepsilon}{2}} \|u_0\|_{\mathcal{H}_\delta}.$$

These computations are for the Part (1) of Theorem 1.3. For Part (2) a somewhat more involved argument is needed.

To get the properties of the coefficients stated above we use the following Lemma.

Lemma 2.2. *Let $B = f \otimes g$, $f, g \in \mathcal{K}^1_{-\delta}$. Let $u_0 \in \mathcal{H}_\delta$, and let*

$$Z(\sigma)u_0 = \sum_{n=-\infty}^{\infty} e^{-in\sigma} E_n B E_n^* J u_0.$$

Then $Z(\sigma) \in B(\mathcal{H}_\delta, \mathcal{K}^1_{-\delta})$ has the integral kernel

$$2\pi f(t,x) g(t-\sigma, y).$$

Let us briefly outline the proof. We compute as follows, where we use the Fourier inversion theorem in the last step.

$$\begin{aligned} Z(\sigma)u_0 &= \sum_{n=-\infty}^{\infty} e^{-in\sigma} e^{int} f(t,x) \int_{\mathbf{T}} \int_{\mathbf{R}^3} g(s,y) e^{-ins} u_0(y)\, dy\, ds \\ &= f(t,x) \sum_{n=-\infty}^{\infty} e^{in(t-\sigma)} \int_{\mathbf{T}} e^{-ins} \Big(\int_{\mathbf{R}^3} g(s,y) u_0(y) dy \Big) ds \\ &= 2\pi f(t,x) \int_{\mathbf{R}^3} g(t-\sigma, y) u_0(y) dy. \end{aligned}$$

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