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Nonsmooth Fractional Programming with Generalized Ratio Invexity

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Abstract: In this paper, we consider nonsmooth fractional programming problems with generalized ratio invexity. We present necessary and sufficient optimality theorems and establish duality theorems for nonsmooth fractional programming under suitable ρ-invexity assumptions.

1 Intorduction

We consider the following nonsmooth fractional programming problem :

$$\text{(NFP)} \quad \text{Minimize} \quad \frac{f(x)}{g(x)}$$

$$\text{subject to} \quad x \in X = \{x \in \mathbb{R}^n | h_j(x) \leqq 0, \ j = 1, \cdots, m\},$$

where $f, g : \mathbb{R}^n \to \mathbb{R}$, and $h_j : \mathbb{R} \to \mathbb{R}$, $j = l, \cdots, m$, are locally Lipschitz functions. We assume in the sequel that $f(x) \geqq 0$ and $g(x) > 0$ on $\mathbb{R}^n$.

Jeyakumar [3] defined ρ-invexity for nonsmooth optimization problems, and Kuk *et al.* [6] defined the concept of V-ρ-invexity for vector valued functions, which is a generalization of the V-invex function [4]. Khan and Hanson [5]and Reddy and Mukherjee [8] applied the (generalized) ratio invexity concept for single objective fractional programming problems.

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Recently, Liang *et al.* [7] introduced the concept of (F, α, ρ, d)-convexity and presented optimality and duality results for a class of nonlinear fractional programming problems under generalized convexity and the properties of sublinear functional. In this paper, we present a result about the fractional objective function based on ρ-invexity assumptions. By using ρ-invexity of fractional function, we obtain necessary and sufficient optimality conditions and duality theorems for nonsmooth fractional programming problems.

2 Definitions and Generalized Invexity of Fractional Function

The real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be locally Lipschitz if for any $z \in \mathbb{R}^n$ there exists a positive constant K and a neighborhood N of z such that, for each x, $y \in N$,

$$|f(x) - f(y)| \leqq K\|x - y\|,$$

where $\|\cdot\|$ denotes any norm in $\mathbb{R}^n$. The Clarke generalized directional derivative of a locally Lipschitz function f at x in the direction $d \in \mathbb{R}^n$, denote by $f^0(x; d)$, is defined as follows:

$$f^0(x; d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} t^{-1}(f(y + td) - f(y)),$$

where y is a vector in $\mathbb{R}^n$.

The Clarke generalized subgradient of f at x is denoted by

$$\partial f(x) = \{\xi : f^0(x; d) \geqq \xi d, \quad \forall d \in \mathbb{R}^n\}.$$

Definition 2.1 *f is said to be regular at x if for all $d \in \mathbb{R}^n$ the one-sided directional derivative $f'(x; d)$ exists and $f'(x; d) = f^0(x; d)$.*

Definition 2.2 *A locally Lipschitz function $f : X_0 \to \mathbb{R}$ is said to be ρ-invex at $x_0 \in X_0$ with respect to functions η and $\theta : X_0 \times X_0 \to \mathbb{R}^n$ if there exists $\rho \in \mathbb{R}$ such that for any $x \in X_0$, and any $\xi \in \partial f(x_0)$,*

$$f(x) - f(x_0) \geqq \xi\eta(x, x_0) + \rho\|\theta(x, x_0)\|^2,$$

where $\theta(x, x_0) \neq 0$ if $x \neq x_0$.

When $\rho = 0$, the definition of ρ-invexity reduces to the notion of invexity in the sense of Hanson [2].

Remark. *When f is of class C^1 in Definition 2.2, then the above inequality reduces to*

$$f(x) - f(x_0) \geqq f'_{x_0}\eta + \rho\|\theta(x, x_0)\|^2$$

where f'_{x_0} is the Frechet derivative of f at x_0.

Theorem 2.1 *If f and $-g$ are ρ-invex with respect to η and θ, and f and $-g$ are regular at x_0, then the fractional objective function $f(x)/g(x)$ is ρ-invex with respect to $\bar{\eta}$ and $\bar{\theta}$, where*

$$\bar{\eta}(x, x_0) = (g(x_0)/g(x))\eta(x, x_0), \quad \bar{\theta}(x, x_0) = (1/g(x))^{1/2}\theta(x, x_0).$$

Proof. Let $x, x_0 \in X_0$. By the ρ-invexity of f and $-g$, we have

$$\begin{aligned}
&f(x)/g(x) - f(x_0)/g(x_0)\\
&= (f(x) - f(x_0))/g(x) - f(x_0)(g(x) - g(x_0))/g(x)g(x_0)\\
&\geqq (1/g(x))\xi\eta(x, x_0) + \rho\|(1/g(x))^{1/2}\theta(x, x_0)\|^2\\
&\quad +(f(x_0)/(g(x)g(x_0))(-\zeta\eta(x, x_0) + \rho\|\theta(x, x_0)\|^2),
\end{aligned}$$

for any $x \in X_0$, any $\xi \in \partial f(x_0)$ and any $\zeta \in \partial g(x_0)$. Since $f(x) \geq 0$ and $g(x) > 0$,

$$\begin{aligned}
&f(x)/g(x) - f(x_0)/g(x_0)\\
&\geqq (g(x_0)/g(x))((\xi/g(x_0))\eta(x, x_0) + (-f(x_0)\zeta/(g^2(x_0))\eta(x, x_0))\\
&\quad +\rho\|(1/g(x))^{1/2}(1 + (f(x_0)/g(x_0)))^{1/2}\theta(x, x_0)\|^2.
\end{aligned}$$

Since f and $-g$ are regular at x_0, we obtain, for any $\delta \in \partial(f(x_0)/g(x_0))$,

$$\begin{aligned}
&f(x)/g(x) - f(x_0)/g(x_0)\\
&\geqq (g(x_0)/g(x))\delta\eta(x, x_0) + \rho\|(1/g(x))^{1/2}(1 + (f(x_0)/g(x_0)))^{1/2}\theta(x, x_0)\|^2.
\end{aligned}$$

Considering that

$$1 + f(x_0)/g(x_0) \geqq 1,$$

we have

$$f(x)/g(x) - f(x_0)/g(x_0)$$
$$\geqq (g(x_0)/g(x))\delta\eta(x, x_0) + \rho\|(1/g(x))^{1/2}\theta(x, x_0)\|^2.$$

Therefore, the function $f(x)/g(x)$ is ρ-invex, where

$$\bar{\eta}(x, x_0) = (g(x_0)/g(x))\eta(x, x_0),$$
$$\bar{\theta}(x, x_0) = (1/g(x))^{1/2}\theta(x, x_0).$$

3 Optimality Conditions

The Cottle constraint qualification

The Cottle constraint qualification is satisfied at x_0 if either $h_j(x_0) < 0$ for all $j = 1, \cdots, m$ or $0 \notin conv\{\partial h_j(x_0) : h_j(x_0) = 0\}$, where $convS$ denotes the convex hull of a set S.

By Theorem 6.1.1 in [1], we can present the following Fritz John necessary conditions.

Theorem 3.1 (Fritz John Necessary Conditions). *If $x_0 \in X$ is an optimal solution of (NFP), then there exist λ and r_j, $j = 1, 2, \cdots, m$, such that*

$$0 \in \lambda\partial\Big(\frac{f(x_0)}{g(x_0)}\Big) + \sum_{j=1}^{m} r_j \partial h_j(x_0),$$

$$\sum_{j=1}^{m} r_j h_j(x_0) = 0,$$

$$(\lambda, r_1, \cdots, r_m) \geqq 0 \text{ and } (\lambda, r_1, \cdots, r_m) \neq 0.$$

Assuming the Cottle constraint qualification, we obtain the Karush-Kuhn-Tucker necessary conditions.

Theorem 3.2 (Karush-Kuhn-Tucker Necessary Conditions). *Assume that $x_0 \in X$ is an optimal solution for (NFP) at which the Cottle constraint qualification is satisfied. Then there exist $\mu_j \geqq 0, j = 1, 2, \cdots, m$, such that*

$$0 \in \partial\Big(\frac{f(x_0)}{g(x_0)}\Big) + \sum_{j=1}^{m} \mu_j \partial h_j(x_0),$$

$$\sum_{j=1}^{m} \mu_j h_j(x_0) = 0,$$

$$(\mu_1, \cdots, \mu_m) \geqq 0.$$

Theorem 3.3 (Karush-Kuhn-Tucker Sufficient Conditions). *Let (x_0, μ) satisfy the Karush-Kuhn-Tucker conditions as follows:*

$$0 \in \partial\Big(\frac{f(x_0)}{g(x_0)}\Big) + \sum_{j=1}^{m} \mu_j \partial h_j(x_0),$$

$$\sum_{j=1}^{m} \mu_j h_j(x_0) = 0,$$

$$(\mu_1, \cdots, \mu_m) \geqq 0.$$

Assume that f and $-g$ are ρ-invex at x_0 with respect to η and θ, and f and $-g$ are regular at x_0, and h_j is ρ'_j-invex at x_0 with respect to $\bar{\eta}$ and $\bar{\theta}$ with $\rho + \sum_{j=1}^{m} \mu_j \rho'_j \geqq 0$.
Then x_0 is an optimal solution of (NFP).

Proof. Let x_0, $x \in X$ and (x_0, μ) satisfy the Karush-Kuhn-Tucker conditions. Then there exist $\delta \in \partial(f(x_0)/g(x_0))$ and $\gamma_j \in \partial h_j(x_0)$ such that $\delta + \sum_{j=1}^{m} \mu_j \gamma_j = 0$ and $\sum_{j=1}^{m} \mu_j h_j(x_0) = 0$. Since f and $-g$ are ρ-invex at x_0 with respect to η and θ and regular at x_0, then by Theroem 2.1 we have

$$f(x)/g(x) - f(x_0)/g(x_0)$$

$$\geqq (-g(x)/g(x_0)) \sum_{j=1}^{m} \mu_j \gamma_j \eta(x, x_0) + \rho \|(1/g(x))^{1/2} \theta(x, x_0)\|^2$$

$$-\sum_{j=1}^{m} \mu_j h_j(x_0) + \sum_{j=1}^{m} \mu_j h_j(x).$$

Since h_j is ρ'_j-invex at x_0 with respect to $\bar{\eta}$ and $\bar{\theta}$, we obtain

$$f(x)/g(x) - f(x_0)/g(x_0)$$

$$\geqq (\rho + \sum_{j=1}^{m} \mu_j \rho'_j) \|\bar{\theta}(x, x_0)\|^2$$

$$\geqq 0.$$

Therefore, x_0 is an optimal solution of (NFP).

4 Duality Theorems

We consider the following Mond-Weir dual problem to (NFP):

$$(\mathrm{NFD})_M \qquad \text{Maximize} \qquad \frac{f(u)}{g(u)}$$

$$\text{subject to} \qquad 0 \in \partial\Big(f(u)/g(u)\Big) + \sum_{j=1}^{m} \mu_j \partial h_j(u),$$

$$\sum_{j=1}^{m} \mu_j h_j(u) \geqq 0,$$

$$(\mu_1, \cdots, \mu_m) \geqq 0.$$

Theorem 4.1 (Weak Duality). *Let x be feasible for (NFP) and (u, μ) feasible for $(NFD)_M$. Assume that f and $-g$ are ρ-invex with respect to η and θ, and f and $-g$ are regular functions, and h_j is ρ'_j-invex with respect to $\bar{\eta}$ and $\bar{\theta}$ with $\rho + \sum_{j=1}^{m} \mu_j \rho'_j \geqq 0$.*

Then

$$\frac{f(x)}{g(x)} \geqq \frac{f(u)}{g(u)}.$$

Proof. Since f and $-g$ are ρ-invex with respect to η and θ, and regular, and (u, μ) is feasible for $(\mathrm{NFD})_M$, then by Theorem 2.1 we have

$$\begin{aligned} & f(x)/g(x) - f(u)/g(u) \\ & \geqq (-g(u)/g(x)) \sum_{j=1}^{m} \mu_j \gamma_j \eta(x,u) + \rho \|(1/g(x))^{1/2}\theta(x,u)\|^2, \end{aligned}$$

for some $\gamma_j \in \partial h_j(u)$. Since h_j is ρ_j'-invex with respect to $\bar{\eta}$ and $\bar{\theta}$, we obtain

$$\begin{aligned} & f(x)/g(x) - f(u)/g(u) \\ & \geqq (\rho + \sum_{j=1}^{m} \mu_j \rho_j') \|\bar{\theta}(x,u)\|^2 \\ & \geqq 0. \end{aligned}$$

Theorem 4.2 (Strong Duality). *Let $\bar{x}$ be an optimal solution for (NFP) at which the Cottle constraint qualification is satisfied. Then there exists $\bar{\mu}$ such that $(\bar{x}, \bar{\mu})$ is feasible for $(NFD)_M$. Moreover, if f, g and h satisfy the conditions of Theorem 4.1, then $(\bar{x}, \bar{\mu})$ is an optimal solution of $(NFD)_M$ and the optimal values of (NFP) and $(NFD)_M$ are equal.*

Proof. From the Karush-Kuhn-Tucker necessary conditions, there exists $\bar{\mu}_j \geqq 0$, $j = 1, 2, \cdots, m$ such that

$$0 \in \partial\Big(\frac{f(\bar{x})}{g(\bar{x})}\Big) + \sum_{j=1}^{m} \bar{\mu}_j \partial h_j(\bar{x}),$$

$$\sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}) = 0.$$

Thus $(\bar{x}, \bar{\mu})$ is feasible for $(\mathrm{NFD})_M$. So, by Theorem 4.1, $(\bar{x}, \bar{\mu})$ is an optimal solution of $(\mathrm{NFD})_M$.

Theorem 4.3 (Strict Converse Duality). *Let $\bar{x}$ be feasible for (NFP) and $(\bar{u}, \bar{\mu})$ be feasible for (NFD)$_M$ such that $f(\bar{x})/g(\bar{x}) \leqq f(\bar{u})/g(\bar{u})$. Assume that f and $-g$ are ρ-invex at $\bar{u}$ with respect to η and θ, and f and $-g$ are regular at $\bar{u}$, and h_j is ρ'_j-invex with respect to $\bar{\eta}$ and $\bar{\theta}$ with $\rho + \sum_{j=1}^m \bar{\mu}_j \rho'_j \geqq 0$.*

Then

$$\bar{x} = \bar{u}.$$

Proof. Since f and $-g$ are ρ-invex at $\bar{u}$ with respect to η and θ, and regular at $\bar{u}$ and $(\bar{u}, \bar{\mu})$ is feasible for (NFD)$_M$, then by Theorem 2.1 we have

$$\begin{aligned} &f(\bar{u})/g(\bar{u}) - f(\bar{x})/g(\bar{x}) \\ &\leqq (g(\bar{u})/g(\bar{x})) \sum_{j=1}^m \bar{\mu}_j \gamma_j \eta(\bar{x}, \bar{u}) - \rho \|(1/g(\bar{x}))^{1/2} \theta(\bar{x}, \bar{u})\|^2 \\ &+ \sum_{j=1}^m \bar{\mu}_j h_j(\bar{u}) - \sum_{j=1}^m \bar{\mu}_j h_j(\bar{x}), \end{aligned}$$

for some $\gamma_j \in \partial h_j(\bar{u})$. Since h_j is ρ'_j-invex with respect to $\bar{\eta}$ and $\bar{\theta}$, we obtain

$$\begin{aligned} &f(\bar{u})/g(\bar{u}) - f(\bar{x})/g(\bar{x}) \\ &\leqq -(\rho + \sum_{j=1}^m \bar{\mu}_j \rho'_j) \|\bar{\theta}(\bar{x}, \bar{u})\|^2 \\ &\leqq 0. \end{aligned}$$

Thus $\bar{x} = \bar{u}$.

We propose the following Wolfe dual problem to (NFP):

$$\begin{aligned} \text{(NFD)}_W \quad \text{Maximize} \quad & \frac{f(u)}{g(u)} + \sum_{j=1}^m \mu_j h_j(u) \\ \text{subject to} \quad & 0 \in \partial (f(u)/g(u)) + \sum_{j=1}^m \mu_j \partial h_j(u), \\ & (\mu_1, \cdots, \mu_m) \geqq 0. \end{aligned}$$

Theorem 4.4 (Weak Duality). *Let x be feasible for (NFP) and (u, μ) feasible for $(NFD)_W$. Assume that f and $-g$ are ρ-invex with respect to η and θ, and f and $-g$ are regular functions, and h_j is $\rho_j{}'$-invex with respect to $\bar{\eta}$ and $\bar{\theta}$ with $\rho + \sum_{j=1}^{m} \mu_j \rho_j{}' \geqq 0$.*

Then

$$\frac{f(x)}{g(x)} \geqq \frac{f(u)}{g(u)} + \sum_{j=1}^{m} \mu_j h_j(u).$$

Proof. Since f and $-g$ are ρ-invex with respect to η and θ, regular and (u, μ) is feasible for $(\text{NFD})_W$, then by Theorem 2.1 we have

$$f(x)/g(x) - ((f(u)/g(u)) + \sum_{j=1}^{m} \mu_j h_j(u))$$

$$\geqq (-g(u)/g(x)) \sum_{j=1}^{m} \mu_j \gamma_j \eta(x, y) + \rho \|(1/g(x)^{1/2}\theta(x, u)\|^2 - \sum_{j=1}^{m} \mu_j h_j(u)$$

for some $\gamma_j \in \partial h_j(u)$. Since h_j is ρ_j'-invex with respect to $\bar{\eta}$ and $\bar{\theta}$, we obtain

$$f(x)/g(x) - ((f(u)/g(u)) + \sum_{j=1}^{m} \mu_j h_j(u))$$

$$\geqq -\sum_{j=1}^{m} \mu_j h_j(x) + (\rho + \sum_{j=1}^{m} \mu_j \rho_j')\|\bar{\theta}(x, u)\|^2$$

$$\geqq 0.$$

Theorem 4.5 (Strong Duality). *Let $\bar{x}$ be an optimal solution for (NFP) at which the Cottle constraint qualification is satisfied. Then there exists $\bar{\mu}$ such that $(\bar{x}, \bar{\mu})$ is feasible for $(NFD)_W$. Moreover, if f, g and h satisfy the conditions of Theorem 4.4, then $(\bar{x}, \bar{\mu})$ is an optimal solution of $(NFD)_W$ and the optimal values of (NFP) and $(NFD)_W$ are equal.*

Proof. From the Karush-Kuhn-Tucker necessary conditions, there exists $\bar{\mu}_j \geqq 0$, $j = 1, 2, \cdots, m$ such that

$$0 \in \partial\Big(\frac{f(\bar{x})}{g(\bar{x})}\Big) + \sum_{j=1}^{m} \bar{\mu}_j \partial h_j(\bar{x}),$$

$$\sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}) = 0.$$

Thus $(\bar{x}, \bar{\mu})$ is feasible for $(\text{NFD})_W$. So, by Theorem 4.4, $(\bar{x}, \bar{\mu})$ is an optimal solution of $(\text{NFD})_W$.

Theorem 4.6 (Strict Converse Duality). *Let $\bar{x}$ be an optimal solution for (NFP) at which the Cottle constraint qualification is satisfied. Assume that f and $-g$ are ρ-invex at $\hat{x}$ with respect to η and θ, and f and $-g$ are regular at $\hat{x}$, and h_j is ${\rho_j}'$-invex with respect to $\bar{\eta}$ and $\bar{\theta}$ with $\rho + \sum_{j=1}^{m} \hat{\mu}_j \rho_j' > 0$. If $(\hat{x}, \hat{\mu})$ is an optimal solution of $(NFD)_W$, then $\hat{x} = \bar{x}$ and the optimal values of (NFP) and $(NFD)_W$ are equal.*

Proof. Assume that $\hat{x} \neq \bar{x}$. Since $\bar{x}$ is an optimal solution of (NFP), there exists $\bar{\mu} \geqq 0$ such that $(\bar{x}, \bar{\mu})$ is an optimal solution of $(\text{NFD})_W$. Then

$$f(\bar{x})/g(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}) = f(\hat{x})/g(\hat{x}) + \sum_{j=1}^{m} \hat{\mu}_j h_j(\hat{x}) = \max_{(x,\mu)\in Y} (f(x)/g(x) + \sum_{j=1}^{m} \mu_j h_j(x)$$

where Y is a feasible set of $(\text{NFD})_W$. Because $(\hat{x}, \hat{\mu}) \in Y$, we have

$$0 \in \partial(f(\hat{x})/g(\hat{x})) + \sum_{j=1}^{m} \hat{\mu}_j \partial h_j(\hat{x}).$$

Since f and $-g$ are ρ-invex at $\hat{x}$ with respect to η and θ, and regular at $\hat{x}$, then by Theorem 2.1 we have

$$\begin{aligned} & f(\bar{x})/g(\bar{x}) - f(\hat{x})/g(\hat{x}) \\ & \geqq (-g(\hat{x})/g(\bar{x})) \sum_{j=1}^{m} \hat{\mu}_j \gamma_j \eta(\bar{x}, \hat{x}) + \rho \|(1/g(\bar{x}))^{1/2} \theta(\bar{x}, \hat{x})\|^2 \end{aligned}$$

for some $\gamma_j \in \partial h_j(\hat{x})$. Since h_j is ρ'_j-invex with respect to $\bar{\eta}$ and $\bar{\theta}$, we obtain

$$f(\bar{x})/g(\bar{x}) + \sum_{j=1}^{m} \hat{\mu}_j h_j(\bar{x}) - (f(\hat{x})/g(\hat{x}) + \sum_{j=1}^{m} \hat{\mu}_j h_j(\hat{x}))$$

$$\geqq (\rho + \sum_{j=1}^{m} \hat{\mu}_j \rho'_j)\|\bar{\theta}(\bar{x}, \hat{x})\|^2 > 0.$$

It follows then that

$$f(\bar{x})/g(\bar{x}) + \sum_{j=1}^{m} \hat{\mu}_j h_j(\bar{x})$$

$$> f(\hat{x})/g(\hat{x}) + \sum_{j=1}^{m} \hat{\mu}_j h_j(\hat{x}) = f(\bar{x})/g(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x})$$

or that

$$\sum_{j=1}^{m} \hat{\mu}_j h_j(\bar{x}) > \sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}).$$

But from Theorem 3.2, we have that $\sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}) = 0$, hence $\sum_{j=1}^{m} \hat{\mu}_j h_j(\bar{x}) > 0$ which contradicts the facts that $\hat{\mu}_j \geqq 0$ and $h_j(\bar{x}) \leqq 0$. Hence $\hat{x} = \bar{x}$.

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