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Some Results on Optimality Conditions for Nonsmooth Vector Optimization Problems*

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Abstract

In this paper, we summarize our recent results ([14]-[17]) about optimality conditions for nonsmooth vector optimization problems. Firstly, we give optimality conditions for a (properly, weakly) efficient solution of a nonsmooth convex vector optimization, which are expressed in terms of vector variational inequalities with subdifferentials. Secondly, we present sequential optimality conditions for an efficient solutions of a nonsmooth convex vector optimization, which hold without any constraint qualifications. Thirdly, we give a necessary optimality condition for a weakly efficient solution of a non-Lipschitzian vector optimization problem. Finally, we present necessary optimality condition for a properly efficient solution of a Lipschitzian vector optimization problem.

1 Introduction

In this paper, we consider the following vector optimization problem:

$$(\mathrm{VP}) \quad \begin{array}{ll} \text{Minimize} & f(x) := (f_1(x), \cdots, f_p(x)) \\ \text{subject to} & x \in D, \end{array}$$

where $f_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, \cdots, p$, are functions and D is a subset of $\mathbb{R}^n$.

Solving (VP) means to find the (properly, weakly) efficient solutions which are defined as follows;

Definition 1.1 (1) $\bar{x} \in D$ is said to be an efficient solution of (VP) if for any $x \in D$,

$$(f_1(x) - f_1(\bar{x}), \cdots, f_p(x) - f_p(\bar{x})) \notin -\mathbb{R}^p_+ \backslash \{0\},$$

where $\mathbb{R}^p_+$ is the nonnegative orthant of $\mathbb{R}^p$.

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(2) $\bar{x} \in D$ is called a properly efficient solution of (VP) if $\bar{x} \in D$ is an efficient solution of (VP) and there exists a constant $M > 0$ such that for each $i = 1, \cdots, p$, we have

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leqq M$$

for some j such that $f_j(x) > f_j(\bar{x})$ whenever $x \in D$ and $f_i(x) < f_i(\bar{x})$.

(3) $\bar{x} \in D$ is said to be a weakly efficient solution of (VP) if for any $x \in D$,

$$(f_1(x) - f_1(\bar{x}), \cdots, f_p(x) - f_p(\bar{x})) \notin -int\mathbb{R}^p_+,$$

where $int\mathbb{R}^p_+$ is the interior of $\mathbb{R}^p_+$.

We denote the set of all the efficient solution of (VP), the set of all the weakly efficient solution of (VP), the set of all the properly efficient solution of (VP) by Eff(VP), $WEff$(VP) and $PrEff$(VP), respectively.

It is clear that $PrEff(\text{VP}) \subset Eff(\text{VP}) \subset WEff(\text{VP})$. For basic meanings and properties of such solution sets, see [25].

Recently many authors have tried to obtain optimality conditions to nonsmooth (nondifferentiable) vector optimization problems involving nonsmooth objective or constraint functions ([1], [2], [4], [6], [7], [10], [13], [18]-[20], [23], [24], [26]-[31]). In particular, Giannessi [3] gave elegant optimality conditions for differentiable vector convex optimization problem, which are expressed by vector variational inequalities with gradients. Many authors ([8]-[13], [27], [28]) have tried to extend the Giannessi's results to (nonsmooth) vector optimization problems. Very recently, Jeyakumar, Lee and Dinh ([5]) gave sequential optimality conditions characterizing the solution without any constraint qualification for a scalar convex optimization problem.

In this paper, we summarize our recent results ([14]-[17]) about optimality conditions for nonsmooth vector optimization problems. Firstly, we give optimality conditions for a (properly, weakly) efficient solution of a nonsmooth convex vector optimization, which are expressed in terms of vector variational inequalities with subdifferentials. Secondly, we present sequential optimality conditions for an efficient solution of a nonsmooth convex vector optimization, which hold without any constraint qualifications, and which are given in sequential forms using subdifferentials and ϵ-subdifferentials. Thirdly, we give a necessary optimality condition for a weakly efficient solution of a non-Lipschitzian vector optimization problem involving lower semicontinuous or continuous functions (not necessarily, locally Lipschitz functions). Finally, we present a necessary optimality condition for a properly efficient solution of a Lipschitzian vector optimization problem involving locally Lipschitz functions.

2 Vector Variational Inequalities for Nonsmooth Convex Vector Optimization Problems

Throughout this section, we will assume that the objective functions of (VP), $f_i, i = 1, \cdots, p$, are convex and the constraint set of (VP), D, is a closed convex subset of $\mathbb{R}^n$.

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a convex function. The subdifferential of φ at $a \in \mathbb{R}^n$ is defined as the non-empty compact convex set

$$\partial\varphi(a) = \{v \in \mathbb{R}^n \mid \varphi(x) - \varphi(a) \geqq \langle v, x - a \rangle, \ \forall x \in \mathbb{R}^n\},$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $\mathbb{R}^n$.

Recently, Giannessi [3] considered the following vector variational inequalities for a differentiable convex vector optimization (VP) (when f_i, $i = 1, \cdots, p$, are differentiable):

(VVI)$_\nabla$ Find $\bar{x} \in D$ such that
$(\langle \nabla f_1(\bar{x}), x - \bar{x} \rangle, \cdots, \langle \nabla f_p(\bar{x}), x - \bar{x} \rangle) \notin -\mathbb{R}^p_+ \backslash \{0\}, \ \forall x \in D,$

where $\nabla f_i(x)$ is the gradient of f_i at x and $\langle \cdot, \cdot \rangle$ is the scalar product on $\mathbb{R}^n$.

(MVVI)$_\nabla$ Find $\bar{x} \in D$ such that
$(\langle \nabla f_1(x), x - \bar{x} \rangle, \cdots, \langle \nabla f_p(x), x - \bar{x} \rangle) \notin -\mathbb{R}^p_+ \backslash \{0\}, \ \forall x \in D.$

(WVVI)$_\nabla$ Find $\bar{x} \in D$ such that
$(\langle \nabla f_1(\bar{x}), x - \bar{x} \rangle, \cdots, \langle \nabla f_p(\bar{x}), x - \bar{x} \rangle) \notin -int\mathbb{R}^p_+, \ \forall x \in D,$

where $int\mathbb{R}^p_+$ is the interior of $\mathbb{R}^p_+$.

He proved that if f_i, $i = 1, \cdots, p$, are differentiable, then

$$sol(\text{VVI})_\nabla \subset sol(\text{MVVI})_\nabla = Eff(\text{VP}) \subset WEff(\text{VP}) = sol(\text{WVVI})_\nabla.$$

In this section, we will consider scalar or vector variational inequalities for the nonsmooth convex vector optimization problem (VP), which are formulated as below, to give theorems which extends the above Giannessi's results to (VP).

(VI)$_\lambda$ Find $\bar{x} \in D$ such that $\exists \xi_i \in \partial f_i(\bar{x}), \ i = 1, \cdots, p$, such that
$\langle \sum_{i=1}^p \lambda_i \xi_i, x - \bar{x} \rangle \geqq 0 \ \forall x \in D,$

where $\lambda = (\lambda_1, \cdots, \lambda_p) \in \mathbb{R}^p_+ \backslash \{0\}$.

(MVI)$_\lambda$ Find $\bar{x} \in D$ such that $\forall x \in D$, $\exists \xi_i \in \partial f_i(x)$, $i = 1, \cdots, p$, $\langle \sum_{i=1}^{p} \lambda_i \xi_i, x - \bar{x} \rangle \geqq 0$.

(VVI)$_1$ Find $\bar{x} \in D$ such that $\forall x \in D$, $\forall \xi_i \in \partial f_i(\bar{x})$, $i = 1, \cdots, p$, $(\langle \xi_1, x - \bar{x} \rangle, \cdots, \langle \xi_p, x - \bar{x} \rangle) \notin -\mathbb{R}^p_+ \backslash \{0\}$.

(VVI)$_2$ Find $\bar{x} \in D$ such that $\exists \xi_i \in \partial f_i(\bar{x})$, $i = 1, \cdots, p$, such that $(\langle \xi_1, x - \bar{x} \rangle, \cdots, \langle \xi_p, x - \bar{x} \rangle) \notin -\mathbb{R}^p_+ \backslash \{0\} \quad \forall x \in D$.

(VVI)$_3$ Find $\bar{x} \in D$ such that $\forall x \in D$, $\exists \xi_i \in \partial f_i(\bar{x})$, $i = 1, \cdots, p$, such that $(\langle \xi_1, x - \bar{x} \rangle, \cdots, \langle \xi_p, x - \bar{x} \rangle) \notin -\mathbb{R}^p_+ \backslash \{0\}$.

(MVVI) Find $\bar{x} \in D$ such that $\forall x \in D$, $\forall \xi_i \in \partial f_i(x)$, $i = 1, \cdots, p$, such that $(\langle \xi_1, x - \bar{x} \rangle, \cdots, \langle \xi_p, x - \bar{x} \rangle) \notin -\mathbb{R}^p_+ \backslash \{0\}$.

(WVVI)$_1$ Find $\bar{x} \in D$ such that $\forall x \in D$, $\forall \xi_i \in \partial f_i(\bar{x})$, $i = 1, \cdots, p$, $(\langle \xi_1, x - \bar{x} \rangle, \cdots, \langle \xi_p, x - \bar{x} \rangle) \notin -int\mathbb{R}^p_+$.

(WVVI)$_2$ Find $\bar{x} \in D$ such that $\exists \xi_i \in \partial f_i(\bar{x})$, $i = 1, \cdots, p$, such that $(\langle \xi_1, x - \bar{x} \rangle, \cdots, \langle \xi_p, x - \bar{x} \rangle) \notin -int\mathbb{R}^p_+ \quad \forall x \in D$.

(WVVI)$_3$ Find $\bar{x} \in D$ such that $\forall x \in D$, $\exists \xi_i \in \partial f_i(\bar{x})$, $i = 1, \cdots, p$, such that $(\langle \xi_1, x - \bar{x} \rangle, \cdots, \langle \xi_p, x - \bar{x} \rangle) \notin -int\mathbb{R}^p_+$.

(WMVVI) Find $\bar{x} \in D$ such that $\forall x \in D$, $\forall \xi_i \in \partial f_i(x)$, $i = 1, \cdots, p$, such that $(\langle \xi_1, x - \bar{x} \rangle, \cdots, \langle \xi_p, x - \bar{x} \rangle) \notin -int\mathbb{R}^p_+$.

We denote the solution sets of the above inequality problems by

$sol(\text{VI})_\lambda$, $sol(\text{MVI})_\lambda$, $sol(\text{VVI})$, $\cdots$, $sol(\text{WMVVI})$, respectively.

Now we give three theorems which show relations among solution sets of (VP) and the vector variational inequality problems, and present optimality conditions for (properly, weakly) efficient solutions of (VP). The following Theorem 2.1, 2.2 and 2.3 are found in [15].

Theorem 2.1 *The following are true:*

(1) $sol(VVI)_1 \subset sol(VVI)_2$.
(2) $PrEff(VP) = \bigcup_{\lambda \in int\mathbb{R}^p_+} sol(VI)_\lambda \subset sol(VVI)_2 \subset sol(VVI)_3 \subset sol(MVVI) = Eff(VP)$.

Theorem 2.2 *The following relations hold:*

$$sol(WVVI)_1 \subset WEff(VP) = \bigcup_{\lambda \in \mathbb{R}^p_+ \backslash \{0\}} sol(VI)_\lambda = \bigcup_{\lambda \in \mathbb{R}^p_+ \backslash \{0\}} sol(MVI)_\lambda$$

$$= sol(WVVI)_2 = sol(WVVI)_3 = sol(WMVVI).$$

Theorem 2.3 *If D is a polyhedral convex set in $\mathbb{R}^n$, then*

$$sol(VVI)_2 = PrEff(VP).$$

3 Sequential Optimality Conditions for Convex Vector Optimization Problems

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a convex function. For $\epsilon \geqq 0$, the ϵ-subdifferential of φ at $a \in \mathbb{R}^n$ is defined as the non-empty closed convex set

$$\partial_\epsilon \varphi(a) = \{v \in \mathbb{R}^n \mid \varphi(x) - \varphi(a) \geqq \langle v, x - a\rangle - \epsilon \quad \forall x \in \mathbb{R}^n\}.$$

In this section, we assume that $D = \{x \in \mathbb{R}^n \mid g_j(x) \leqq 0,\ j = 1, \cdots, m\}$, where $g_j : \mathbb{R}^n \to \mathbb{R},\ j = 1, \cdots, m$ are convex functions, and the objective functions of (VP), $f_i,\ i = 1, \cdots, p$, are convex functions.

Now we give two theorems about sequential optimality conditions for an efficient solution of (VP). The following Theorems 3.1 and 3.2 can be obtained from results in [17].

Theorem 3.1 *Let $(\theta_1, \cdots, \theta_p) \in int\mathbb{R}^p_+$ and $\bar{x} \in D$. Then the following are equivalent:*

(i) $\bar{x} \in Eff(VP)$.

(ii) there exist $u \in \partial(\sum_{i=1}^p \theta_i f_i)(\bar{x})$, $\lambda^n := (\lambda_1^n, \cdots, \lambda_m^n) \in \mathbb{R}^m_+$, $\delta^n \geqq 0$, $v^n \in \partial_{\delta^n}(\sum_{j=1}^m \lambda_j^n g_j)(\bar{x})$, $\mu^n := (\mu_1^n, \cdots, \mu_p^n) \in \mathbb{R}^p_+$, $\epsilon^n \geqq 0$, $w^n \in \partial_{\epsilon^n}(\sum_{i=1}^p \mu_i^n f_i)(\bar{x})$ such that

$$u + \lim_{n\to\infty}(v^n + w^n) = 0,$$
$$\lim_{n\to\infty}\delta^n = \lim_{n\to\infty}\epsilon^n = 0 \quad and$$
$$\lim_{n\to\infty}(\sum_{j=1}^m \lambda_j^n g_j)(\bar{x}) = 0.$$

Theorem 3.2 *Let $(\theta_1, \cdots, \theta_p) \in int\mathbb{R}^p_+$ and $\bar{x} \in D$. Then $\bar{x} \in Eff(VP)$ if and only if there exist $u \in \partial(\sum_{i=1}^p \theta_i f_i)(\bar{x})$, $\lambda^n \in \mathbb{R}^m_+$, $\mu^n := (\mu_1^n, \cdots, \mu_p^n) \in \mathbb{R}^p_+$, $x^n \in X$, $s^n \in \partial(\sum_{j=1}^m \lambda_j^n g_j + \sum_{i=1}^p \mu_i^n f_i)(x^n)$ such that*

$$u + \lim_{n\to\infty} s^n = 0,$$
$$\lim_{n\to\infty}[(\sum_{j=1}^m \lambda_j^n g_j + \sum_{i=1}^p \mu_i^n f_i)(x^n) - (\sum_{i=1}^p \mu_i^n f_i)(\bar{x})] = 0 \quad and$$
$$\lim_{n\to\infty}\|x^n - \bar{x}\| = 0.$$

4 Necessary Optimality Conditions for non-Lipschitzian Vector Optimization Problems

We introduce the normal cone and the (singular) approximate subdifferential studied by Mordukhovich ([21], [22]).

Let C be a nonempty subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. Define

$$P(C,x) := \{w \in clC \mid \|x-w\| = inf_{z\in C}\|x-z\|\},$$

where clC is the closure of the set C. Let $\bar{x} \in clC$. The normal cone to C at $\bar{x}$ is defined by

$$N(C,\bar{x}) := \{y \in \mathbb{R}^n \mid \exists y_k \to y, \ \ x_k \to \bar{x}, t_k \in (0,\infty), \\ c_k \in \mathbb{R}^n \ \text{ with } c_k \in P(C,x_k) \text{ and } y_k = t_k(x_k - c_k)\}.$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and $x \in \mathbb{R}^n$. The approximate subdifferential of f at x is defined by

$$\partial^A f(x) := \{x^* \in \mathbb{R}^n \mid (x^*,-1) \in N(epif,(x,f(x)))\},$$

and the singular approximate subdifferential of f at x is defined by

$$\partial^\infty f(x) := \{x^* \in \mathbb{R}^n \mid (x^*,0) \in N(epif,(x,f(x)))\}.$$

In this section, we assume that $D = \{x \in C \mid g_j(x) \leq 0, \ j = 1,\cdots,m\}$, where $g_j : \mathbb{R}^n \to \mathbb{R}$ is a function and C is a closed subset of $\mathbb{R}^n$.

Now we give a theorem about a necessary optimality condition for a weakly efficient solution of a non-Lipschitzian vector optimization problem (VP) involvolving lower semicontinuous or continuous functions. The following Theorems 4.1 and 4.2 are found in [16].

Theorem 4.1 *Let $\bar{x} \in D$. Assume that $f_i, i = 1,\cdots,p$ and g_j, $j \in I(\bar{x}) := \{j \mid g_j(\bar{x}) = 0\}$, are lower semicontinuous at $\bar{x}$ and g_j, $j \in \{1,\cdots,m\} \setminus I(\bar{x})$ are continuous at $\bar{x}$. Suppose that*

$$\sum_{j\in I(\bar{x})}(r_j a_j + \tilde{z}_j) + \sum_{i=1}^{p} z_i + \eta = 0, \ r_j \geq 0, \ a_j \in \partial^A g_j(\bar{x}), \\ \tilde{z}_j \in \partial^\infty g_j(\bar{x}), \ j \in I(\bar{x}) \text{ and } z_i \in \partial^\infty f_i(\bar{x}), \ i = 1,\cdots,p,$$

$$\begin{aligned}
&\eta \in N(C, \bar{x}) \\
&\textit{imply } r_j = 0,\ \ \tilde{z}_j = 0, j \in I(\bar{x}),\ \ z_i = 0,\ \ i = 1, \cdots, p, \\
&\textit{and } \eta = 0.
\end{aligned}$$

If $\bar{x} \in D$ is a weakly efficient solution of (VP), then there exist $\bar{\lambda}_i \geqq 0,\ i = 1, \cdots, p$, not all zero, $\bar{a}_i \in \mathbb{R}^n,\ i = 0, 1, \cdots, p,$ such that

$$\begin{aligned}
&(\bar{a}_i, -\bar{\lambda}_i) \in N(epi f_i, (\bar{x}, f_i(\bar{x}))),\ i = 1, \cdots, p \textit{ and} \\
&0 \in \sum_{i=1}^{p} \bar{a}_i + \sum_{j \in I(\bar{x})} \Big[\bigcup_{r_j > 0} r_j \partial^A g_j(\bar{x})\Big] \cup \partial^{\infty} g_i(\bar{x}) + N(C, \bar{x}).
\end{aligned}$$

Theorem 4.2 *Let $\bar{x} \in D$. Suppose that $f_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, \cdots, p,$ are locally Lipschitz at $\bar{x}$. Assume that*

$$\begin{aligned}
&\sum_{j \in I(\bar{x})} (\alpha_j a_j + \tilde{z}_j) + \eta = 0,\ \ \alpha_j \geqq 0,\ \ a_j \in \partial^A g_j(\bar{x}), \\
&\tilde{z}_j \in \partial^{\infty} g_j(\bar{x}), j \in I(\bar{x}),\ \ \eta \in N(C, \bar{x}) \\
&\textit{imply } \alpha_j = 0,\ \ \tilde{z}_j = 0, j \in I(\bar{x}),\ \ \eta = 0.
\end{aligned}$$

If $\bar{x} \in D$ is a weakly efficient solution of (VP), then there exist $\bar{\lambda}_i \geqq 0,\ i = 1, \cdots, p$, not all zero, such that

$$0 \in \sum_{i=1}^{p} \bar{\lambda}_i \partial^A f_i(\bar{x}) + \sum_{j \in I(\bar{x})} \Big[\bigcup_{r_j > 0} r_j \partial^A g_j(\bar{x})\Big] \cup \partial^{\infty} g_j(\bar{x}) + N(C, \bar{x}).$$

5 Necessary Optimality Conditions for Lipschitzian Vector Optimization Problems

We first recall some notions of Nonsmooth Analysis ([1]). Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The Clarke subdifferential of ψ at $x_0 \in \mathbb{R}^n$ is the set

$$\partial \psi(x_0) = \{\xi \in \mathbb{R}^n \mid \langle x, \xi \rangle \leqq \psi^0(x_0; x) \quad \forall x \in \mathbb{R}^n\}$$

where

$$\psi^0(x_0; x) = \limsup_{\substack{x' \to x_0 \\ t \downarrow 0}} \frac{1}{t}\Big[\psi(x' + tx) - \psi(x')\Big].$$

The Clarke tangent cone and the Clarke normal cone of a subset $C \subset \mathbb{R}^n$ at $x_0 \in C$ are denoted by $T_C(x_0)$ and $N_C(x_0)$, respectively. Recall that

$$T_C(x_0) = \{\eta \in \mathbb{R}^n \mid \rho_C^0(x_0;\eta) = 0\},$$
$$N_C(x_0) = \{\xi \in \mathbb{R}^n \mid \langle \xi, \eta \rangle \leq 0, \quad \forall \eta \in T_C(x_0)\}$$

where $\rho_C(x) = \rho(x, C)$ i.e. $\rho_C(x)$ is the distance from $x \in \mathbb{R}^n$ to C.

In this section, we assume that

$$D = \{x \in C \mid g_j(x) \leqq 0, \ j = 1, 2, \dots, m, \ h_l(x) = 0, \ l = 1, 2, \dots, q\}$$

where $C \subset \mathbb{R}^n$ is a closed subset, and g_j and h_l are given functions. Let $x_0 \in D$ and let

$$I(x_0) = \{j : g_j(x_0) = 0\}.$$

We say that condition (CQ) holds at $x_0 \in D$ if there do not exist $\mu_j \geqq 0, \ j \in I(x_0)$, and $r_l \in \mathbb{R}, \ l = 1, 2, \dots, q$, such that $\sum_{j \in I(x_0)} \mu_j + \sum_{l=1}^{q} |r_l| \neq 0$ and

$$0 \in \sum_{j \in I(x_0)} \mu_j \partial g_j(x_0) + \sum_{l=1}^{q} r_l \partial h_l(x_0) + N_C(x_0)$$

where $\partial g_j(x_0)$ and $\partial h_l(x_0)$ are the Clarke subdifferentials of g_j and h_l at x_0, and $N_C(x_0)$ stands for the Clarke normal cone to C at x_0.

Now we give a necessary optimality condition for a properly efficient solution of (VP). The following Theorem 5.1 is found in [14].

Theorem 5.1 *Assume that all functions f_i, g_j and h_l of (VP) are locally Lipschitz. If $\bar{x} \in D$ is a properly efficient solution of (VP) and if condition (CQ) holds at $\bar{x}$, then there exist $\lambda_i > 0, \ i = 1, \dots, p, \ \mu_j \geq 0, \ j \in I(\bar{x}), \ r_l \in \mathbb{R}, \ l = 1, 2, \dots, q$, such that*

$$0 \in \sum_{i=1}^{p} \lambda_i \partial f_i(\bar{x}) + \sum_{j \in I(\bar{x})} \mu_j \partial g_j(\bar{x}) + \sum_{l=1}^{q} r_l \partial h_l(\bar{x}) + N_C(\bar{x}).$$

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