数理解析研究所講究録 1365 巻 2004 年 138-145

THE LENGTH OF CONTRACTIBILITY OF COMPACT CONTRACTIONS ON A HILBERT SPACE

Yuan-Chuan Li
Department of Applied Mathematics, National Chung-Hsing University, Taichung, 402 Taiwan
Chung-Li 32023, Taiwan
E-mail address: ycli@ fish.amath.nchu.edu.tw

and

Mau-Hsiang Shih
Department of Mathematics, National Taiwan Normal University
88 Sec.4, Ting Chou Road, Taipei 116, Taiwan
E-mail address: mhshih@ math.ntnu.edu.tw

1. Introduction

Let A be a bounded linear operator on a complex Banach space, and $r(A)$ the spectral radius. The Gelfand spectral radius asserts that

$$r(A)=\inf_{n\geq 1}||A^n||^{1/n}=\lim_{n\to\infty}||A^n||^{1/n}.$$

In 1960, Rota and Strang[6] introduced the notion of joint spectral radius as follows : Let Σ be a bounded subset of $n\times n$ matrices. Define

$$\Sigma^k=\{A_1A_2\cdots A_k; A_i\in\Sigma, i=1,2,\ldots,k\},\ \text{where } k=1,2,\ldots.$$

The *joint spectral radius* $r(\Sigma)$ of Σ is defined to be

$$\hat{r}(\Sigma)=\limsup_{k\to\infty}\sup_{A\in\Sigma^k}||A||^{1/k}.$$

1991 Mathematics Subject Classification. Primary 46C05, 46B99.

Key words and phrases. Hilbert space, spectral set, spectral radius, point spectrum, point spectral radius, compact operator, contraction, semigroup, C_0-semigroup, infinitesimal generator.

Research supported in part by the National Science Council of Taiwan.

1992, Daubechies and Lagarias [2] introduced the notion of *generalized spectral radius* $r(\Sigma)$ as follows.

$$r(\Sigma) = \limsup_{k\to\infty} r_k(\Sigma)^{1/k},$$

where $r_k(\Sigma) = \sup_{A\in\Sigma^k} r(A)$. It is easy to see that the notion of the joint spectral radius is independent of all equivalent matrix norm. They also conjectured that

$$(1) \qquad r(\Sigma) = \hat{r}(\Sigma).$$

This conjecture is not true whenever Σ is not bounded. For example,

$$\Sigma = \{\begin{bmatrix} \frac{1}{2} & 2^n \\ 0 & \frac{1}{2} \end{bmatrix}; n = 1, 2, \ldots\}.$$

This conjecture was solved by Berger and Wang in 1992 [1].

In 1995, Lagarias and Wang [3] studied the following problem:
For every finite set Σ of $n \times n$ matrices, is there a positive integer k such that

$$(2) \qquad r(\Sigma) = \hat{r}(\Sigma) = r_k(\Sigma)^{1/k} \ ?$$

They also showed that if Σ is a finite set of contraction $n \times n$ matrices, then this problem is true whenever the associated norm is ℓ^p-norms with p rational. They also proved the following result.

Theorem L-W. Let $||\cdot||_2$ be the Euclidean norm on $\mathbf{R}^n$, and Σ a set of m contraction matrices on $\mathbf{R}^n$. Put $r_0 = 1$ and $r_{k+1} = m^{r_k} + r_k$ for $k = 0, 1, 2, \ldots, n-1$. If $r(\Sigma) = 1$, then there exists some finite product $A_{d_k}A_{d_{k-1}}\cdots A_{d_1}$ with $k \le r_{n-1}$, which has spectral radius $r(A_{d_k}A_{d_{k-1}}\cdots A_{d_1}) = 1$.

If Σ consists of a single matrix $\Sigma = \{A\}$, Theorem L-W says that if $||A||_2 = ||A^n||_2 = 1$ then $r(A) = 1$ (since $r_{n-1} = n$). Note that if $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ then $||A||_2 = 1$ and $r(A) = 0$. So, the bound n is the best integer. This result is not true when the space is infinite dimensional. We give a counterexample as follows :

Example. Let $H = \ell^2$ with the coordinate unit vectors $e_1, e_2, \ldots$ as an ordered basis and the 2-norm $||\cdot||_2$. Define the linear operator A determined by $Ae_{k+1} = e_k$ for $k = 1, 2, \ldots, n$ and $Ae_j = 0$ for $j = 1$ or $n+2, n+3, \ldots$. It is easy to see that $||A||_2 = ||A^n||_2 = 1$ and $n(A) = n$. But $r(A) = 0$ because $A^{n+1} = 0$. Therefore the constant n in the result of Lagarias and Wang is not the best constant when H is infinite dimensional.

2. Main Theorems

In this paper, we shall study how to generalize Theorem L-W to the infinite dimensional Hilbert spaces. Let H be a complex Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the associated norm $||\cdot||$, and $B(H)$ the algebra of all bounded linear operators on H. An operator $A \in B(H)$ is called a *contraction* on H if $||Ax|| \le ||x||$ for all $x \in H$; and A is said to be *compact* if it maps the unit ball of H onto a totally bounded set in H. It is a well-known fact that the set of all compact operators on H forms a two sided ideal of $B(H)$ (cf. [7]). It is readily seen from the Cauchy-Schwarz inequality that if A is positive, i.e., $\langle Ay, y\rangle \ge 0$ for all $y \in H$, and $\langle Ax, x\rangle = 0$ for some $x \in H$, then $Ax = 0$. Therefore, if A is a contraction, then the null space of $I - A^*A$ becomes

$$\ker(I - A^*A) = \{x \in H; ||Ax|| = ||x||\}. \tag{3}$$

For a contraction A, we define

$$n(A) \equiv \dim \ker(I - A^*A). \tag{4}$$

In particular, if $n(A) < \infty$ and λ is an eigenvalue of A with $|\lambda| = 1$, then $\dim N(\lambda I - A) \le n(A)$. When A is of finite rank, it is easy to see that $n(A) \le \dim R(A)$, where $R(A)$ is the range of A.

In [4], we showed that (2) holds for a finite set Σ of compact contractions on H except an operator in Σ is not compact. We showed that if A is a compact contraction on H and $||A|| = ||A^{2^{n(A)}}|| = 1$ then $r(A) = 1$. Such constant $2^{n(A)}$ is not optimal. In this paper, we will show that $n(A) + 1$ is the best constant. First, we list the properties of $n(\cdot)$ as follows.

Lemma 1. (see also [4]) *Suppose A, B are two contractions on H.*
(a) $n(A) = n(A^*)$.
(b) $n(AB) \le \min\{n(A), n(B)\}$.
(c) *If A is compact, then $n(A)$ is finite.*
(d) *If A is compact and $n(A) = 0$, then $||A|| < 1$.*

For a finite set Σ of operators in $B(H)$, we define $|\Sigma|$ = the number of all elements in Σ and the semigroup generated by Σ to be the set $\Sigma' = \bigcup_{m=1}^{\infty} \Sigma^m$.

Proposition 2. *Let $A_1, A_2, \ldots, A_m$ be contractions on H and let $B = A_m A_{m-1} \cdots A_1$. Suppose*
(a) *there are nonnegative integers j, k, p with $1 \le j < k \le k + p \le m$ such that*

$$A_{j+p}A_{j+p-1}\cdots A_j = A_{k+p}A_{k+p-1}\cdots A_k;$$

(b) $1 \le \ell = n(B) = n(A_{j+p}A_{j+p-1}\cdots A_j) < \infty$.

Then $A_{k-1}A_{k-2}\cdots A_j$ *has an eigenvalue* λ *with* $|\lambda| = 1$. *In particular, we have* $r(A_{k-1}A_{k-2}\cdots A_j) = 1$.

Proof. Let $K = N(I - B^*B)$. Then we have

$$x \in K \iff ||A_m A_{m-1}\cdots A_1 x|| = ||x||. \tag{5}$$

Since each A_i is contraction, this implies that for every $1 \le i \le m$ $A_i A_{i-1}\cdots A_1$ is an isometry on K and for $i = 1, 2, \ldots, m$

$$A_i A_{i-1}\cdots A_1 K \subset N(I - (A_{i+p}A_{i+p-1}\cdots A_i)^*(A_{i+p}A_{i+p-1}\cdots A_i)).$$

It follows from Lemma 1(b) that for every $i = 1, 2, \ldots, m$,

$$\dim A_{i-1}\cdots A_1 K = \dim K = n(B) \le n(A_{i+p}A_{i+p-1}\cdots A_i). \tag{6}$$

Therefore we obtain from the condition (b) that

$$\begin{aligned} &A_{j-1}A_{j-2}\cdots A_1 K \\ &= N(I - (A_{j+p}A_{j+p-1}\cdots A_j)^*(A_{j+p}A_{j+p-1}\cdots A_j)) \\ &= N(I - (A_{k+p}A_{k+p-1}\cdots A_k)^*(A_{k+p}A_{k+p-1}\cdots A_k)) \\ &= A_{k-1}A_{k-2}\cdots A_1 K. \end{aligned}$$

Since $A_{k-1}A_{k-2}\cdots A_j$ is an isometry on $A_{j-1}A_{j-2}\cdots A_1 K$ by (5), this shows that A_{k-1} $A_{k-2}\cdots A_j$ is an isometry from $A_{j-1}A_{j-2}\cdots A_1 K$ onto itself. It follows from (6) that $A_{k-1}A_{k-2}\cdots A_j$ has an eigenvalue λ with $|\lambda| = 1$. In particular, we have $r(A_{k-1}A_{k-2}\cdots A_j)$ $= 1$. This completes the proof.

If Σ is a finite subset of contractions on H, we define $n(\Sigma) = \max_{A\in\Sigma} n(A)$.

Corollary 3. *Let* A *be a contraction on* H, $m = n(A)$, *and let* $\Sigma = \{A\}$. *If* $n(A^{m+1}) \ge 1$, *then* $r(A) = 1$. *In particular, if* A *is a compact contraction, then* $||A^{m+1}|| =$ 1 *implies* $r(A) = 1$.

Proof. Suppose $r_p(A) < 1$. Then $r(A^k) < 1$ for all $k = 2, 3, \ldots$ by the spectral mapping theorem. Since Σ is a singleton, by Proposition 2 with $r = 1$, we have

$$n(A^{m+1}) < n(A^m) < n(A^{m-1}) < \cdots < n(A) = m.$$

Then $n(A^{m+1}) = 0$. This is impossible. Therefore $r(A) = 1$. The proof is complete.

Lemma 4. *Let Σ be a finite set of contractions on H. Let $r = |\Sigma|$, $s \geq 1$, and $\Sigma_1 = \Sigma^s$. Suppose $n(\Sigma_1) < \infty$ and $r_p(B) < 1$ for all $B \in \bigcup_{j=1}^{r^s+s} \Sigma^j$, where $r_p(B) = \sup\{\lambda; \lambda = 0$ or is an eigenvalue of $B\}$. Then there is a smallest integer q with $s \leq q \leq r^s + s$ such that $n(\Sigma^q) < n(\Sigma_1)$.*

Proof. Let $q = r^s + s$. Suppose $B \in \Sigma^q$ be such that $n(B) = n(\Sigma_1)$. Let us write $B = A_q A_{q-1} \cdots A_1$ for some $A_1, A_2, \ldots, A_q \in \Sigma$. Then

$$A_{i+s-1}A_{i+s-1} \cdots A_i \in \Sigma^s \text{ for } i = 1, 2, \ldots, r^s + 1.$$

Since $r^s \geq |\Sigma^s|$, there are integers i and j with $1 \leq i < j \leq r^s + 1$ such that

$$A_{i+s-1}A_{i+s-2} \cdots A_i = A_{j+s-1}A_{j+s-2} \cdots A_j.$$

Since $n(B) \leq n(A_{i+s-1}A_{i+s-2} \cdots A_i) \leq n(\Sigma_1) = n(B)$ by Lemma 1(b), it follows from Proposition 2 that $A_{j-1} \cdots A_i$ has an eigenvalue λ with $|\lambda| = 1$. This contradicts to $r_p(B) < 1$ for all $B \in \bigcup_{j=1}^{r^s+s} \Sigma^j$.

Remark 5. If A is an $n \times n$ matrix with $||A||_2 = 1$, where $||\cdot||_2$ is the 2-norm on $\mathbf{C}^n$, then $n(A) = n$ implies that A is an isometry from $\mathbf{C}^n$ onto itself. Therefore $r(A) = 1$. From this, we see that if Σ is a finite set of contractions on $\mathbf{C}^n$ and $r(A) < 1$ for every $A \in \Sigma$, then $n(\Sigma) \leq n - 1$.

Corollary 6. *If A is an $n \times n$ matrix with $||A||_2 \leq 1$, then $r(A) < 1$ if and only if $||A^n||_2 < 1$.*

Proof. Suppose $r(A) < 1$. By Remark 5, we have $n(A) \leq n - 1$. It follows from Proposition 2 that

$$n(A^n) < n(A^{n-1}) < \cdots < n(A) \leq n - 1.$$

This implies $n(A^n) = 0$ and hence $||A^n||_2 < 1$. The converse is obvious.

Theorem 7. *Let Σ be the set of finite contractions $A_1, A_2, \cdots, A_r$ on H and $m = n(\Sigma) < \infty$. Put $r_1 = r + 1$ and $r_k = r^{r_{k-1}} + r_{k-1}$ for $k = 2, 3, \ldots, m$. If $n(\Sigma^{r_m}) \geq 1$, then there is some $B \in \bigcup_{j=1}^{r_m} \Sigma^j$ such that $r_p(B) = 1$. In particular, if each A_i is a compact contraction on H and $||A|| = 1$ for some $A \in \Sigma^{r_m}$, then there is some $B \in \bigcup_{j=1}^{r_m} \Sigma^j$ such that $r_p(B) = 1$.*

Proof. Suppose $r_p(B) < 1$ for all $B \in \bigcup_{j=1}^{r_m} \Sigma^j$. It follows from Lemma 4 m-times that

$$n(\Sigma^{r_m}) < n(\Sigma^{r_{m-1}}) < \cdots < n(\Sigma^{r_1} < n(\Sigma) = m.$$

Therefore we must have $n(\Sigma^{r_m}) = 0$. This contradicts our assumption that $n(\Sigma^{r_m}) \geq 1$.

The following result gives an infinite-dimensional version of Theorem L-W.

Theorem 8. *Let Σ be the set of finite contractions $A_1, A_2, \cdots, A_r$ on $\mathbf{C}^n$. Put $r_1 = r + 1$ and $r_k = r^{r_{k-1}} + r_{k-1}$ for $k = 2, 3, \ldots, n-1$. If $n(\Sigma^{r_{n-1}}) \geq 1$, then there is some $B \in \bigcup_{j=1}^{r_{n-1}} \Sigma^j$ such that $r(B) = 1$.*

Proof. Suppose $r(B) < 1$ for all $B \in \bigcup_{j=1}^{r_{n-1}} \Sigma^j$. By Remark 5, we have $n(\Sigma) \leq n - 1$. It follows from Lemma 4 $n-1$-times that

$$n(\Sigma^{r_{n-1}}) < n(\Sigma^{r_{n-2}}) < \cdots < n(\Sigma^{r_1} < n(\Sigma) \leq n - 1.$$

Therefore we must have $n(\Sigma^{r_{n-1}}) = 0$. This contradicts our assumption that $n(\Sigma^{r_{n-1}}) \geq 1$.

Proposition 9. *Let $T(\cdot)$ be a C_0-semsigroup on H with the infinitesimal generator A. Suppose $T(t_0)$ is contraction with $m = n(T(t_0)) < \infty$ for some $t_0 > 0$. If $\|T(t_0)^{m+1}\| = 1$, then A has an eigenvalue μ with $Re\mu = 0$.*

Proof. If $\|T(t_0)^{m+1}\| = 1$, it follows from Proposition 2 that there is some $1 \leq n \leq m+1$ such that $T(t_0)^n = T(nt_0)$ has an eigenvalue λ with $|\lambda| = 1$. By the spectral mapping theorem (cf. [5, Theore A-III.6.3.]) of C_0-semigroup for the point spectrum, we have $\sigma_p(T(t)) \setminus \{0\} = e^{t\sigma_p(A)}$ for $t \geq 0$. Therefore $\lambda \in e^{nt_0\sigma_p(A)}$, that is, $\lambda = e^{t\mu}$ for some $\mu \in \sigma_p(A)$. Since $|\lambda| = 1$, we have $\text{Re}\mu = 0$. This completes the proof.

We denote the ideal of all compact contractions on H by $\mathcal{K}_C(H)$.

Lemma 10. *Let $\{A_m\}$ be a sequence of contractions on H and let $A \in B(H)$. Suppose*

(a) *A is compact;*
(b) *there is a positive integer r such that $n(A_m) \geq r$ for all $m = 1, 2, \ldots$;*
(c) *$\|A_m - A\| \to 0$ as $m \to \infty$. Then $n(A) \geq r$. Moreover, for every positive integer r the set $\{A \in \mathcal{K}_C(H); n(A) < r\}$ is open in $\mathcal{K}_C(H)$.*

Proof. Since each A_m is a contraction, so is A by (c). By (b), for every $m \geq 1$ there is an orthonormal set $\{x_{m1}, x_{m2}, \ldots, x_{mr}\}$ contained in $N(I - A_m^* A_m)$. Since A_m is an isometry on $N(I - A_m^* A_m)$, $\{A_m x_{mk}; 1 \leq k \leq r\}$ is also an orthonormal set.

Since A is compact, there are integers $1 \leq n_1 < n_2 < \cdots$ such that $\{Ax_{n_k j}\}$ converges to some element $y_j \in H$ for $j = 1, 2, \ldots, r$. Since $||A_m - A|| \to 0$ as $m \to \infty$, it is easy to see that $A_{n_k} x_{n_k j} \to y_j$ strongly as $k \to \infty$ for $j = 1, 2, \ldots, r$. Since each $\{A_{n_k} x_{n_k j}; 1 \leq j \leq r\}$ is an orthonormal set, so is $\{y_1, y_2, \ldots, y_r\}$. On the other hand, for every k and j, $x_{n_k j} \in N(I - A^*_{n_k} A_{n_k})$ implies $A^*_{n_k} A_{n_k} x_{n_k j} = x_{n_k j}$. Therefore we have for every $j = 1, 2, \ldots, r$

$$y_j = \lim_{k \to \infty} A_{n_k} A^*_{n_k} A_{n_k} x_{n_k j} = AA^* y_j.$$

This shows that $n(A^*) \geq r$ and hence $n(A) = n(A^*) \geq r$. The proof is complete.

Proposition 11. *Let Σ be a compact set of compact contractions on H. Suppose*
(a) *Σ is a countable set;*
(b) *$r(A) < 1$ for all $A \in \Sigma'$. Then Σ is asymptotically stable (a.s.).*

Proof. Since Σ is compact, so are the Σ^m $(m \geq 1)$. By Lemma 1(b), $\{n(\Sigma^m)\}$ is a decreasing sequence. We show that $n(\Sigma^m) = 0$ for some m. Since Σ^m is compact, by Lemma 1(d), this will imply $||A|| < 1$ for all $A \in \Sigma^m$. Thus the compactness of Σ^m shows that Σ is a.s. Suppose $n(\Sigma^m)$ is never zero. Then there is some integer $m_0 \geq 1$ such that the $n(\Sigma^m)$ $(m \geq m_0)$ are all equal to a positive integer ℓ. Put $\Omega = \Sigma^{m_0}$ and let $S = \{E \subset \Omega; E$ is compact and $n(E^m) = \ell$ for all $m = 1, 2, \ldots\}$. Then $(S, \subset)$ is a partially ordered set. We claim that S has a minimal element.

If $\{E_i\}$ is a decreasing chain in S, then $E = \bigcap_i E_i$ is nonempty compact set. If $n(E^m) < \ell$ for some positive integer m, then, by Lemma 10, there is an open subset V of $\mathcal{K}_C(H)$ such that $E^m \subset V$ and $n(V) < \ell$. Since $\{E_i\}$ is a decreasing chain of compact sets, so is $\{E_i^m\}$. Therefore there must have some i such that $E_i^m \subset V$. This contradicts to $E_i \in S$. Hence E is a lower bound of $\{E_i\}$. By Zorn's Lemma, S has a minimal element, say Ω_0. Clearly, Ω_0 is also countable. So, Ω_0 has an isolated point, say B. Since $\Omega_1 = \Omega_0 \setminus \{B\}$ is not in S, there is some positive integer m_1 such that

$$(*) \qquad n(\Omega_1^{m_1}) < \ell.$$

We claim that $n(\Omega_0^{2m_1}) < \ell$. Thus $\Omega_0 \notin S$; a contradiction. We part the proof into three cases.

Let $A_1, A_2, \ldots, A_{2m_1} \in \Omega_0$.
Case 1. $A_i = A_j$ for some $1 \leq j < k \leq 2m_1$. Then, by Proposition 2 with $p = 0$, we have $r(A_{k-1}A_{k-2}\cdots A_j) = 1$. This contradicts to (b). So, we can assume $A_i \neq A_j$ for all $1 \leq i \neq j \leq 2m_1$.
Case 2. $B \neq A_i$ for all $i = 1, 2, \ldots, 2m_1$. Then $A_{2m_1}A_{2m_1-1}\cdots A_1 \in \Omega_1^{2m_1}$. By (*), $n(A_{2m_1}A_{2m_1-1}\cdots A_1) < \ell$.

Case 3. $B = A_i$ for some $1 \leq i \leq 2m_1$. Then either $1 \leq i \leq m_1$ or $m_1 + 1 \leq i \leq 2m_1$. Anyway, we have either $A_{m_1} A_{m_1-1} \cdots A_1 \in \Omega_1^{m_1}$ or $A_{2m_1} A_{2m_1-1} \cdots A_{m_1+1} \in \Omega_1^{m_1}$. It follows from Lemma 1(b) and (*) that $n(A_{2m_1} A_{2m_1-1} \cdots A_1) < \ell$.

This completes the proof.

References

[1] Berger, M. A. and Wang, Y. Bounded semigroups of matrices, *Linear Algebra Appl.* **166** (1992), 21-27.

[2] Daubechies, I. and Lagarias, J. C., Sets of matrices all infinite products of which converge, *Linear Algebra Appl.* **162** (1992), 227-263.

[3] Lagarias, J. C. and Y. Wang, The finiteness conjecture for the generalized spectral radius of a set of matrices, *Linear Algebra Appl.* **214** 1995, 17-42.

[4] Li, Y.-C. and M.-H. Shih, Contractibility of Compact Contractions in Hilbert Space, *Linear Algebra Appl.* **341** (2002), 369-378.

[5] Nagel, R., *One-parameter Semigroups of Positive Operators*, LNM, **1184**, 1986.

[6] Rota, G.-C. and G. Strang, A note on the joint spectral radius, Indag. Math. **22** (1960), 379-381.

[7] Taylor, A. E. and D. C. Lay, *Introduction to Functional Analysis*, 2nd ed., Wiley, New York, 1980.