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On Some Closure Properties of Semilinear Sets

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Abstract. We show that closure under the operations of minimum and minimum iteration holds for special cases of linear sets of dimension 2.

1 Introduction

Multiset languages are important in the fields of *Petri nets*, *molecular computing* and *membrane computing* [6, 7, 8]. Multisets can be represented by m-dimensional vectors over the natural numbers. Some research has been done recently [5, 2, 3, 4], especially on characterization and complexity of multiset languages, and the closure properties of various classes of multiset languages. Some of them still remain open problems.

Among the operations for which some closure properties are still open problems are the operations of *minimum* $\sqcap$ and *maximum* $\sqcup$ of multiset languages, as well as their iteration. The mimimum (maximum) of two multisets is defined componentwise, and extended to multiset languages.

An important class of multiset languages is the class of *semilinear sets*, introduced already by S. Ginsburg [1]. It corresponds to the classes of regular, linear, and context-free word languages which coincide for multiset languages since the underlying operation $+$ is commutative.

Here we concentrate on the operation *minimum* and $\sqcap$-*iteration* for the special case of dimension 2, and for *linear sets*. We show that for a linear set its $\sqcap$-iteration is semilinear (Theorem 18), and that the minimum of two "1-dimensional" linear sets (lines) is semilinear (Theorem 21), too.

For all remaining cases as well as for the dual operations of *maximum* $\sqcup$ and *maximum iteration* the closure properties are open problems. We conjecture that closure holds under these operations in general for all semilinear sets and any dimension m.

2 Preliminaries

Definition 1. A set $F \subseteq \mathbb{N}^m$ where $\mathbb{N} = \{0, 1, \dots\}$ and $m \geq 1$ is called a *linear set* iff either $F = \emptyset$ or there exist $r \geq 0$ and $v_0, \dots, v_r \in \mathbb{N}^m$ such that

(S_1) $\quad F = F(v_0; v_1, ..., v_r) = \{\, v_0 + \sum_{i=1}^{r} k_i v_i \mid k_i \in \mathbb{N}\}$

The vector v_0 and the vector set $P = \{v_i \mid i = 1, ..., r\}$ appearing in (S_1) are often called *preperiod* and the set of *periods* of F, respectively. □

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The set E is *semilinear* if it is a finite union of linear sets. We refer to these linear sets as the *components* of E.

Definition 2. Let $v_0 \in \mathbb{N}^m$, and $P = \{v_1, ..., v_r\} \subset \mathbb{N}^m$. The *grid* is defined by

$$G(v_0; P) = \{v_0 + \sum_{i=1}^{r} \lambda_i v_i \mid \lambda_i \in \mathbf{Z}, i = 1, ..., r\} \tag{1}$$

where $\mathbf{Z}$ denotes the set of integers. □

Note that

$$F(v_0; P) \subseteq G(v_0; P) \tag{2}$$

Definition 3. The grid $G(v_0; D)$ with $D = \{d_1, ..., d_m\} \subset \mathbb{N}^m$ is a *basic grid* if the d_i are multiples of basis vectors, i.e. of the form $d_i = (\delta_{i,1}\Delta_1, ..., \delta_{i,r}\Delta_r)$, with $\Delta_i \neq 0$. (Here $\delta_{i,j}$ is the Kronecker symbol: $\delta_{i,j} = 0$ if $i \neq j$ and $\delta_{i,i} = 1$). If $\Delta_i = 1$ for $i \in [m] = \{1, ..., m\}$ then the basic grid $U(v_0) = G(v_0; E)$ with $E = \{e_1, ..., e_r\}$ is called a *unit grid.* □

Obviously, for any unit grid $U(v_0)$ holds

$$U(v_0) = U(0), \tag{3}$$

(thus there is only one unit grid $U = U(0)$), and for every grid G holds

$$G \subseteq U \tag{4}$$

In the sequel we will investigate some properties of subsets of the unit grid U. Let A be a subset of U. As usual, we define a partial order $\leq$ as follows:

Definition 4. For $u, v \in A$ and $u = (u_1, ..., u_m)$, $v = (v_1, ..., v_m)$ let $u \leq v$ if and only if $u_i \leq v_i$ holds for all $i \in [m] = \{1, ..., m\}$.

In some of the applications it is convenient to add the elements "∞" and "$-\infty$" to $\mathbf{Z}$, let $\overline{\mathbf{Z}} = \mathbf{Z} \cup \{\infty, -\infty\}$. We extend the partial order "$\leq$", operations "+" and "$\cdot$" to $\overline{\mathbf{Z}}$ as usual: for any $u \in \overline{\mathbf{Z}}$ let $u \leq \infty$ and $-\infty \leq u$, for $u \in \mathbf{Z}$, let $u + \infty = \infty$, and $u + (-\infty) = u - \infty = -\infty$, for $u > 0$ let $u \cdot (\pm\infty) = \pm\infty$, for $u < 0$ let $u \cdot (\pm\infty) = \mp\infty$, and for $u = 0$ define $0 \cdot (\pm\infty) = 0$. We extend the notions of linear set, grid, basic grid and unit grid to vector sets whose components are elements of $\overline{\mathbf{Z}}$. □

Definition 5. Let A be a subset of the unit grid U. An element $a \in A$ is a *minimal (maximal) element* of A if for any $v \in A$: $v \leq a$ ($a \leq v$) implies $a = v$. □

Definition 6. Let A be a subset of U: $A \subseteq U$.
For $u = (u_1, ..., u_m) \in A$ and $i \in [m]$ the i-section $S(A; u, i)$ of A and u is defined by

$$S(A; u, i) = \{v \in A \mid v = (v_1, ..., v_m), v_j = u_j, j \neq i\} \ . \tag{5}$$

□

Definition 7. Let $A \subseteq U$. An element $v = (v_1, ..., v_m) \in A$ is said to be *i-section maximal* (*i-section minimal*) if it is maximal (minimal) in the i-section $S(A; v, i)$. □

Definition 8. Let A be a subset of U. The set A is *closed* under maximality (minimality) if all sections have maximal (minimal) elements. □

Definition 9. Let A be a subset of U. For the i-section $S(A;v,i)$ we define the *maximum closure (minimum closure)* $[S(A;v,i)]_{max}$ ($[S(A;v,i)]_{min}$) of A by

$$[S(A;v,i)]_{max} = S(A;v,i) \quad , \quad (\; [S(A;v,i)]_{min} = S(A;v,i) \;)$$

if $A(A;v,i) = \emptyset$ or $S(A;v,i)$ has a maximal (minimal) element, and by

$$[S(A,v,i)]_{max} = S(A;v,i) \cup \{(...v_{i-1}, \infty, v_{i+1}, ...)\},$$

$$(\; [S(A;v,i)]_{min} = S(A;v;) \cup \{(...v_{i-1}, -\infty, v_{i+1}, ...)\} \;),$$

if $A(A;v,i) = \emptyset$ and $S(A;v,i)$ doesn't have any maximal (minimal) element. Let

$$[A]_{max} = \bigcup_{i=1}^{m} \bigcup_{v \in A} [S(A;v,i)]_{max} \; , \tag{6}$$

$$(\; [A]_{min} = \bigcup_{i=1}^{m} \bigcup_{v \in A} [S(A;v,i)]_{min} \;). \tag{7}$$

□

Obviously, $[A]_{max}$ ($[A]_{min}$) is closed under maximality (minimality).

Definition 10. Let u and v be two elements of the unit grid U. The operations $u \sqcap v$ and $u \sqcup v$ are defined by

$$u \sqcap v = \max\{w \mid (w \leq u) \wedge (w \leq v)\} \; , \tag{8}$$

$$u \sqcup v = \min\{w \mid (w \geq u) \wedge (w \geq v)\} \; . \tag{9}$$

Lemma 11. *Let $u = (u_1, ..., u_m)$ and $v = (v_1, ..., v_m)$ two vectors in the unit grid U. Then*

$$u \sqcap v \;\; = \;\; (\min(u_1, v_1), ..., \min(u_m, v_m)) \; , \tag{10}$$

$$u \sqcup v \;\; = \;\; (\max(u_1, v_1), ..., \max(u_m, v_m)) \; . \tag{11}$$

□

Definition 12. Let A be a nonempty subset of the unit grid U. A is said to be *closed* under the operation $\sqcap$ (under $\sqcup$) or shortly $\sqcap$ *-closed* ($\sqcup$ *-closed*) if $u, v \in A$ implies $u \sqcap v \in A$ $(u \sqcup v \in A)$. □

Definition 13. Let A be a nonempty subset of the unit grid U. The $\sqcap$*-closure* ($\sqcup$*-closure*) $A_\sqcap$ ($A_\sqcup$) of A is the smallest set closed under $\sqcap$ ($\sqcup$) containing A. □

Definition 14. Let A be a nonempty $\sqcap$-closed ($\sqcup$-closed) subset of the unit grid U. A subset $B \subseteq A$ is a generator set for A if $A = B_\sqcap$ ($A = B_\sqcup$). □

Lemma 15. *Let A be a nonempty subset of the unit grid U. Suppose that A is closed under section maximality (minimality) and under $\sqcap$ ($\sqcup$). Denote by $M(A)$ ($m(A)$) the set of section maximal (section minimal) elements of A, i.e., let*

$$M(A) = \{w \mid w = \max S(A;v,i), v \in A, i = 1, ..., m\}, \tag{12}$$

$$(\; m(A) = \{w \mid w = \min S(A;v,i), v \in A, i = 1, ..., m\} \;). \tag{13}$$

Then $M(A)$ ($m(A)$) is a generator system for A.

Proof. For every $v \in A$

$$v = \sqcap_{i=1}^{m} \max S(A; v, i), \tag{14}$$

$$(\; v = \sqcup_{i=1}^{m} \min S(A; v, i) \;). \tag{15}$$

□

The vectors $v_1, ..., v_r \in \mathbb{N}^m$ are said to be *linearly independent* if

$$\lambda_1 v_1 + \cdots + \lambda_r v_r = 0 \tag{16}$$

implies $\lambda_1, ..., \lambda_r = 0$. (Here $\lambda_1, ..., \lambda_r$ are integers and the operations in (16) are meant to be operations over vectors with integer components.)

Lemma 16. *Let $F \subset \mathbb{N}^m$ be a finite vector set and v any vector in $\mathbb{N}^m$. Then,*

$$\min(F + v) = \min(F) + v, \tag{17}$$

and for any $\lambda \in \mathbb{N}$ holds

$$\min(\lambda F) = \lambda \min(F). \tag{18}$$

□

Lemma 17. *Let $E \subset \mathbb{N}^m$ and $F \subset \mathbb{N}^m$ be finite vector sets. Then*

$$\min(E \cup F) = \min(\min(E), \min(F)). \tag{19}$$

□

3 Closure properties

Theorem 18. *Let*

$$F = \{v_0 + k_1 v_1 + ... + k_r v_r \mid k_i \in \mathbb{N}, i = 1, \cdots, r\} \tag{20}$$

where v_0 is the preperiod, and $\{v_1, ..., v_r\}$ the set of periods associated to F. Assume that $v_0, v_1, ..., v_r \in \mathbb{N}^2$. Then $F_\sqcap$ is semilinear.

Proof. Without loss of generality we may assume that $v_0 = 0$, and for $i \in [r]$ holds

$$v_{i,2}/v_{i,1} \geq v_{i+1,2}/v_{i+1,1} \; . \tag{21}$$

(If $v_{i,1} = 0$, then we write $v_{i,2}/v_{i,1} = \infty$).

For $u \in F$ and $i \neq j$ we define $T(u; v_i, v_j)$ and $Q(u; v_i, v_j)$ by

$$T(u; v_i, v_j) = \{u + t_i v_i + t_j v_j \mid 0 \leq t_i \leq 1, \; 0 \leq t_j \leq 1\}, \tag{22}$$

and

$$Q(u; v_i, v_j) = F_\sqcap \cap T(u; v_i, v_j) \; . \tag{23}$$

If we write simply $Q(u)$, then we mean $Q(u) = Q(u; v_1, v_r)$. Obviously,

$$F_{\sqcap} = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} Q(iv_1 + jv_r) \ . \tag{24}$$

Consider the function $\mu(u)$ defined on the elements of F as follows:

$$\mu(u) \doteq |Q(u)| \tag{25}$$

The function $\mu(u)$ is monotone in some sense, i.e. : for $j = 1, r$ holds that if $v \in Q(u)$ then $v + v_j \in Q(u + v_j)$, therefore $\mu(u) \leq \mu(u + v_j)$. Let u' be a vector from F such that

$$\mu(u') = \max\{\mu(u) \mid u \in F\} \tag{26}$$

where $u' = k'_0 v_1 + l'_0 v_r$, and $k'_0 + l'_0$ is minimal ($k'_0, l'_0 \in \mathbb{N}$).

For any fixed $l \in \{0, ..., l'_0\}$ let

$$k_0(l) = \min\{k \mid \mu(kv_1 + lv_r) \ maximal\}, \tag{27}$$

and

$$k_0 = \max\{k_0(l) \mid l = 0, ..., l'_0\}. \tag{28}$$

Similarly, for any fixed $k \in \{0, ..., k'_0\}$, let

$$l_0(k) = \min\{l \mid \mu(kv_1 + lv_r) \ maximal\}, \tag{29}$$

and

$$l_0 = \max\{l_0(k) \mid k = 0, ..., k'_0\}. \tag{30}$$

Note that by definition, $k_0 \geq k'_0$ and $l_0 \geq l'_0$.
We shall show that

$$F_{\sqcap} \ = \ Q_0 \cup Q_1 \cup Q_{1,2} \cup Q_2 \tag{31}$$

where

$$Q_0 \ = \ \bigcup_{k=0}^{k_0-1} \bigcup_{l=0}^{l_0-1} Q(kv_1 + lv_r) \ , \tag{32}$$

$$Q_1 \ = \ \bigcup_{l=0}^{l_0-1} \ \bigcup_{u \in Q(k_0 v_1 + lv_r)} \{u + kv_1 \mid k \geq 0\} \ , \tag{33}$$

$$Q_{1,2} = \bigcup_{u \in Q(k_0 v_1 + l_0 v_r)} \{u + kv_1 + lv_r \mid k, l \geq 0\} \ , \tag{34}$$

$$Q_2 = \bigcup_{k=0}^{k_0-1} \ \bigcup_{u \in Q(kv_1 + l_0 v_r)} \{u + lv_r \mid l \geq 0\} \ . \tag{35}$$

By (24), for any $u \in F_{\sqcap}$ there are integers k and l such that $u \in Q(kv_1 + lv_r)$ holds. If $k < k_0$ and $l < l_0$ then $u \in Q_0$. If $l < l_0$, but $k \geq k_0$, then there is a $k' \geq 0$ such that $k = k_0 + k'$. Because of the maximality of $\mu(k_0 v_1 + lv_r)$ and the monotony of $\mu(v)$ "in direction v_1", there is a v' in $Q(k_0 v_1 + lv_r)$ such that $u = v' + k'v_1$ holds, hence $u \in Q_1$. (Recall that $0 \leq l < l_0$). Similarly, if $k \geq k_0$ and $l \geq l_0$, then $u \in Q_{1,2}$, and if $k < k_0$ and $l \geq l_0$, then $u \in Q_2$.

□

Lemma 19. *Let $m_1, ..., m_r$ be integers with the property* $\gcd(m_1, ..., m_r) = 1$. *(As usually,* $\gcd(v_{1,i}, ..., v_{r,i})$ *denotes the greatest common divisor of the numbers in question). Then for any fixed $i \in \{1, ..., r\}$ there exist integers K and L such that any number k with $k \geq K$ may be written in the form*

$$k = l_1 m_1 + ... + l_r m_r \tag{36}$$

such that for any $j \neq i$ holds that

$$0 \leq l_j \leq L. \tag{37}$$

Proof. Using the algorithm of Euclid, it is easy to prove that there are numbers $\alpha_1, ..., \alpha_r$ such that

$$1 = \alpha_1 m_1 + ... + \alpha_r m_r \tag{38}$$

holds. Let $M = lcm(m_1, ..., m_r)$, the least common multiple of numbers $m_1, ..., m_r$, and let α be the least integer such that $\alpha M + \alpha_\nu m_\nu \geq 0$ holds for $\nu = 1, ..., r$. Then

$$\begin{aligned} r\alpha M + 1 &= \sum_{\nu=1}^{r} (\alpha M + \alpha_\nu m_\nu) \\ &= \sum_{\nu=1}^{r} (\alpha M / m_\nu + \alpha_\nu) m_\nu && (39) \\ &= \sum_{\nu=1}^{r} \gamma_\nu m_\nu \,. && (40) \end{aligned}$$

Let $N = r\alpha M$. Then

$$\begin{aligned} N + 1 &= \sum_{\nu=1}^{r} \gamma_\nu m_\nu \\ 2N + 2 &= \sum_{\nu=1}^{r} 2\gamma_\nu m_\nu \\ &\vdots \\ \mu N + \mu &= \sum_{\nu=1}^{r} \mu\gamma_\nu m_\nu \\ &\vdots \\ (N-1)N + N - 1 &= \sum_{\nu=1}^{r} (N-1)\gamma_\nu m_\nu \,. \end{aligned}$$

It is easy to check that for $\mu = 0, 1, ..., N-1$ holds that

$$\begin{aligned} (N-1)N + \mu &= (N-1)N - \mu N + \mu N + \mu && (41) \\ &= (N - (1+\mu))N + \sum_{\nu=1}^{r} \mu\gamma_\nu m_\nu \,. && (42) \end{aligned}$$

Let $K = (N-1)N$, $L = \max\{(N-1)\gamma_\nu \mid \nu = 1, ..., r\}$, and assume that $k \geq K$. There are integers I and J such that $k = K + IN + J$ holds with $0 \leq J \leq N-1$. Then, by (42), it follows that

$$\begin{aligned} k &= IN + (N-1)N + J = IN + (N-(1+J))N + \sum_{i=\nu}^{r} J\gamma_\nu m_\nu \\ &= (I + (N-(1+J))N/m_i)m_i + \sum_{\nu=1}^{r} J\gamma_\nu m_\nu \ . \end{aligned} \tag{43}$$

Thus (36) and (37) hold with numbers $l_i = (I + (N-(1+J))N/m_i + J\gamma_i)$ and $l_j = J\gamma_j \ (j \neq i)$.
□

In the sequel we will prove that $Q(k_0 v_1 + l_0 v_r)$ is of a special structure.

Theorem 20. *Let*

$$F = \{v_0 + k_1 v_1 + ... + k_r v_r \mid k_1, ..., k_r \geq 0\} \tag{44}$$

where v_0 is the preperiod, and $\{v_1, ..., v_r\}$ the set of periods associated to F. Assume that $v_0, v_1, ..., v_r \in \mathbb{N}^2$. Let $b_1 = \Delta_1 e_1$ and $b_2 = \Delta_2 e_2$ with unit vectors $e_1 = (1,0)$, $e_2 = (0,1)$, and $\Delta_i = \gcd(v_{1,i}, ..., v_{r,i})$ for $i = 1, 2$. (By definition $\gcd(0, ..., 0) = 1$). Let us define the linear vector set G by

$$G = G(v_0; b_1, b_2) = \{v_0 + k_1 b_1 + k_2 b_2 \mid k_1, k_2 \geq 0\} \ . \tag{45}$$

Then the set $Q_{1,2}$ in the decomposition of $F_\sqcap$ given in (31) may be chosen as

$$Q_{1,2} = T_{1,2}(V_0; v_1, v_r) \cap G, \tag{46}$$

where $V_0 \in F$ is a suitable vector and

$$T_{1,2}(V_0; v_1, v_r) = \{V_0 + t_1 v_1 + t_r v_r \mid t_1, t_2 \in \mathbf{R}, 0 \leq t_1 \leq 1, \ 0 \leq t_2 \leq 1\}. \tag{47}$$

Proof. For $t = 1, 2$ let $\Delta_t = \gcd(v_{1,t}, v_{2,t}, ..., v_{r,t})$. As in the proof of (18) we assume that for $i \in \underline{r-1}$

$$v_{i,2}/v_{i,1} \geq v_{i+1,2}/v_{i+1,1} \ , \tag{48}$$

and $v_0 = 0$.
($v_{i,2}/v_{i,1} = \infty$, whenever $v_{i,1} = 0$.) By Lemma 19, for $t = 1, 2$ there exist integers K_t and L_t such that any number k_t with $k_t \geq K_t$ may be written in the form

$$k_t = (l_{1,t}/\Delta_t)v_{1,t} + ... + (l_{r,t}/\Delta_t)v_{r,t} \ . \tag{49}$$

Thus

$$\Delta_t k_t = l_{1,t} v_{1,t} + ... + l_{r,t} v_{r,t} \tag{50}$$

such that for any $j \neq i$ holds that

$$0 \leq l_{j,t} \leq L_t. \tag{51}$$

Let us choose the vector V_0 in (46) as

$$V_0 = (V_{0,1}, V_{0,2}) = (K_1 + (r-1)L_1)v_r + (K_2 + (r-1)L_2)v_1 \ . \tag{52}$$

It is easy to prove that

$$F_{\sqcap} \subseteq G. \tag{53}$$

We shall show that

$$T_{1,2}(V_0; v_1, v_r) \cap G \subseteq F_{\sqcap} \tag{54}$$

holds as well. Let

$$z \in T_{1,2}(V_0; v_1, v_r) \cap G, \quad z = (z_1, z_2). \tag{55}$$

Since $z \in G$, we have $z = k_1\Delta_1 e_1 + k_2\Delta_2 e_2$ for some $k_1, k_2 \in \mathbb{N}$. By the choice of V_0, $k_t \geq K_t$ for $t = 1, 2$. Thus k_t may be written in the form (50) such that (51) holds. Consider the vectors

$$u_t = (u_{t,1}, u_{t,2}) = l_{1,t}v_1 + \ldots + l_{r,t}v_r, \quad (t = 1, 2) \tag{56}$$

By definition, $u_{1,1} = k_1\Delta_1$, and $u_{2,2} = k_2\Delta_2$. Using (48),(50), and (51) it is easy to see that

$$u_{1,2} \geq z_2. \tag{57}$$

Similarly,

$$u_{2,1} \geq z_1. \tag{58}$$

Thus $u_1 \sqcap u_2 = (u_{1,1}, u_{2,2}) = z$. □

Theorem 21. *Let A and B be "one dimensional" linear sets, i.e., let*

$$\begin{aligned} A &= \{u_0 + ku_1 \mid k \geq 0\}, \\ B &= \{v_0 + lv_1 \mid l \geq 0\}. \end{aligned}$$

Then $A \sqcap B$ is a semilinear set.

Proof. For $i = 0, 1$ let $u_i = (u_{i,1}, u_{i,2})$ and $v_i = (v_{i,1}, v_{i,2})$. If $u_1 = (0, u_{1,2})$ and $v_1 = (v_{1,1}, 0)$, then

$$A \sqcap B = u_0 \sqcap v_0.$$

Assume that $u_1 = (0, u_{1,2})$ and $v_1 = (v_{1,1}, v_{1,2})$, where $v_{1,2} > 0$. Then

$$A \sqcap B = A \cup \{u_{0,1}e_1 + (v_{0,2} + kv_{1,2})e_2 \mid k \geq 0\},$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. In the sequel we will assume that for $i = 1, 2$ $\ u_{1,i} > 0$ and $v_{1,i} > 0$. We distinguish two further cases.

Case 1. $u_{1,2}/u_{1,1} = v_{1,2}/v_{1,1}$. Consider the "lines" f and g defined by the equations

$$f(t) = u_0 + u_1 t \quad t \in \mathbf{R}\ , \tag{59}$$

$$g(t) = v_0 + v_1 t \quad t \in \mathbf{R}\ , \tag{60}$$

respectively. We may assume that line f lies "above" line g, i.e., if $(x,y) \in f$ and $(x,z) \in g$, then $y \geq z$. Let k_0 be the least integer such that

$$u_{0,1} + k_0 u_{1,1} \geq \max(u_{0,1}, v_{0,1}) \tag{61}$$

holds, and

$$c = lcm(u_{1,1}, v_{1,1}) \ . \tag{62}$$

Consider the sets

$$Q_0 = \{(x,y) \in A \sqcap B \mid x \leq k_0 u_{1,1}\} \ , \tag{63}$$

$$Q_1 = \{(x,y) \in A \sqcap B \mid k_0 u_{1,1} < x \leq k_0 u_{1,1} + c\} \ . \tag{64}$$

We shall show that

$$A \sqcap B = Q_0 + \bigcup_{z \in Q_1} \{z + kw \mid k \geq 0\} \tag{65}$$

where

$$w = (c/u_{1,1})u = (c/v_{1,1})v. \tag{66}$$

Let

$$(p,q) \in A \sqcap B \ . \tag{67}$$

If $p \leq k_0 u_{1,1}$, then $(p,q) \in Q_0$. (See : (63)). If $p > k_0 u_{1,1}$, it is easy to see that there is a $z = (z_1, z_2)$, and an integer k such that $k_0 u_{1,1} < z_1 \leq k_0 u_{1,1} + c$ and $(p,q) = z + kw$. By (67), there are vectors $a \in A$ and $b \in B$ such that

$$(p,q) = a \sqcap b \ . \tag{68}$$

By the choice of w, $a - kw \in A$ and $b - kw \in B$, thus $z = (a - kw) \sqcap (b - kw)$ and $z \in Q_1$. Assume now that

$$(p,q) \in Q_0 + \bigcup_{z \in Q_1} \{z + kw \mid k \geq 0\} \ . \tag{69}$$

If $(p,q) \in Q_0$ then $(p,q) \in A \sqcap B$. (See : (63)). Let $(p,q) \in Q_1$. There are vectors $a' \in A$ and $b' \in B$ with $(p,q) = a' \sqcap b'$. Then for any integer k for the vector $(p,q) + kw$ holds that

$$(p,q) + kw = (a' + kw) \sqcap (b' + kw), \tag{70}$$

where $(a' + kw) \in A$ and $(b' + kw) \in B$.

Case 2. $u_{1,2}/u_{1,1} > v_{1,2}/v_{1,1}$.

Claim 1. *Let* $c = lcm(u_{1,1}, v_{1,1})$, *and define the two vectors* $u_1' = (u_{1,1}', u_{1,2}') = (c/u_{1,1})u_1$, $v_1' = (v_{1,1}', v_{1,2}') = (c/v_{1,1})v_1$. *Consider the halflines* f *and* g *defined by the equations*

$$f(t) = u_0 + u_1 t \quad t \in \mathbf{R}, t > 0, \tag{71}$$

$$g(t) = v_0 + v_1 t \quad t \in \mathbf{R}, t > 0, \tag{72}$$

respectively. Assume that f *lies above* g.

1. *Then* $z \in A \sqcap B$ *implies* $z + v_1' \in A \sqcap B$.
2. *If* $z = a \sqcap b$ *with* $a, a - u_1' \in A$ *and* $b, b - v_1' \in B$, *then* $z - v_1' \in A \sqcap B$.

Let $z = a \sqcap b$ where $a \in A$ and $b \in B$. For $u'_{1,1}$ and $v'_{1,1}$ we have

$$u'_{1,1} = (c/u_{1,1})u_{1,1} = c = (c/v_{1,1})v_{1,1} = v'_{1,1}, \tag{73}$$

thus

$$z + v_1' = (a + u_1') \sqcap (b + v_1'). \tag{74}$$

Here $a + u_1' \in A$ and $b + v_1' \in B$. Therefore $z + v_1' \in A \sqcap B$. If additionally $a - u_1' \in A$ and $b - v_1' \in B$, then

$$z - v_1' = (a - u_1') \sqcap (b - v_1') \tag{75}$$

holds as well.

Claim 2. *Let* $d = lcm(u_{1,2}, v_{1,2})$, *and define the two vectors* $u_1'' = (u''_{1,1}, u''_{1,2}) = (d/u_{1,2})u_1$, $v_1'' = (v''_{1,1}, v''_{1,2}) = (d/v_{1,2})v_1$. *Consider the halflines* f *and* g *defined by the equations*

$$f(t) = u_0 + u_1 t \quad t \in \mathbf{R}, t > 0, \tag{76}$$

$$g(t) = v_0 + v_1 t \quad t \in \mathbf{R}, t > 0, \tag{77}$$

respectively. Assume that f *lies over* g.

1. *Then* $z \in A \sqcap B$ *implies* $z + u_1'' \in A \sqcap B$.
2. *If* $z = a \sqcap b$ *with* $a, a - u_1'' \in A$ *and* $b, b - v_1'' \in B$, *then* $z - v_1'' \in A \sqcap B$.

Claim 2 may be proved similarly to *Claim* 1.

In the sequel we shall show that $A \sqcap B$ may be decomposed as follows :

$$A \sqcap B = Q_0 \cup \bigcup_{z \in Q_{1,2}} \{z + kv_1' + lu_1'' \mid k, l \geq 0\} . \tag{78}$$

Here $v_1' = (v'_{1,1}, v'_{1,2})$ and $u_1'' = (u''_{1,1}, u''_{1,2})$ are defined in *Claim* 1 and *Claim* 2. In order to define Q_0 and $Q_{1,2}$, consider the halflines f and g given by equations (76) and (77) respectively. Let $u_0^* \in A$ and $v_0^* \in B$ be vectors such that the halfline f^* lies over the halfline g^*, where f^* and g^* are defined by the equations

$$f^*(t) = u_0^* + u_1 t \quad t \in \mathbf{R}, t > 0, \tag{79}$$

$$g^*(t) = v_0^* + v_1 t \quad t \in \mathbf{R}, t > 0, \tag{80}$$

respectively. Note that such u_0^* and v_0^* always exist by $u_{1,2}/u_{1,1} > v_{1,2}/v_{1,1}$. Let $u_0^* = (u^*_{0,1}, u^*_{0,2})$, $v_0^* = (v^*_{0,1}, v^*_{0,2})$, $m = \max(u^*_{0,1}, v^*_{0,1})$, $n = u''_{1,1} + v'_{1,1}$, and

$$Q_0 = \{(x, y) \in A \sqcap B \mid x < m\} , \tag{81}$$

$$Q_{1,2} = \{(x,y) \in A \sqcap B \mid m \leq x \leq m+n\} \ . \tag{82}$$

Let $(p,q) = a \sqcap b$ where $a \in A$ and $b \in B$. We have to prove

$$(p,q) \in Q_0 \cup \bigcup_{z \in Q_{1,2}} \{z + kv_1' + lu_1'' \mid k,l \geq 0\}. \tag{83}$$

If $p < m$, then $(p,q) \in Q_0$. If $p \geq m$, then there is a vector $z = (z_1, z_2)$, and there are integers $k,\ l$ such that $(p,q) = z + kv_1' + lu_1''$ holds with $m \leq z_1 \leq m+n$. But then $z \in Q_{1,2}$ by *Claim* 1 and *Claim* 2, and by the choice of $Q_{1,2}$.

Similarly, if $(p,q) \in Q_0$, then by definition, $(p,q) \in A \sqcap B$, and if $(p,q) \in Q_{1,2}$, then $(p,q) + kv_1' + lu_1'' \in A \sqcap B$ by *Claim* 1 and *Claim* 2. □

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