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Well-ordered monoids - two numerical functions -

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1 Introduction

Compatible well-orders on monoids are used in the rewriting theory in algebra. They do not only guarantee the termination of reduction processes in a system [3], [4] but also are a base of the completion procedure [8]. In the Gröbner base theory of polynomial rings, such orders on free commutative monoids, and in the rewriting theory of groups and monoids, such orders on free monoids play a crucial role (see [1], [11]). Compatible orders on a free commutative monoid are characterized by weights [15], but in contrast, compatible orders on a free monoid are very diverse and complicated ([2], [9], [10]).

In this paper we study compatible well-orders on monoids. A well-ordered monoid is a monoid with a well-order that is strictly compatible with the operation of the underlying monoid. We introduce two numerical parameters associated with the order. The first one, which is discussed in Section 3, is related to weight functions on a monoid. The second, which is discussed in Section 4, comes from some effect of commutation of two elements in a monoid. With these parameters we study the ordered structure of well-ordered monoids, especially in the case of two-generator monoids in the last section.

2 Preliminary

A *quasi-order* $\succeq$ on a set X is a reflexive transitive relation on X such that $x \succeq y$ or $y \succeq x$ holds for any $x, y \in X$. For a quasi-order $\succeq$, define relations $\sim$ and $\succ$ as follows. For $x, y \in X$, $x \sim y$ if and only if $x \succeq y$ and $y \succeq x$, and $x \succ y$ if $y \succeq x$ does not hold, equivalently, $x \succeq y$ but $x \not\sim y$. We call $\succ$ the *strict part* of $\succeq$. It is easy to see that $\sim$ is an equivalence relation. A quasi-order $\succeq$ is an *order*, if $\sim$ is the equality relation, that is, $x \sim y$ if and only if $x = y$. A quasi-order $\succeq$ is *well-founded*, if there is no infinite decreasing sequence $x_1 \succ x_2 \succ \cdots$. A well-founded order is called a *well-order.*

Let $\succeq_1$ and $\succeq_2$ be two quasi-orders on X. The *lexicographic composition* $\succeq_1 \circ \succeq_2$ of $\succeq_1$ and $\succeq_2$ is defined by

$$x\,(\succeq_1 \circ \succeq_2)\,y \quad \Leftrightarrow \quad x \succ_1 y, \text{ or } x \sim_1 y \text{ and } x \succeq_2 y,$$

where $\sim_1$ is the equivalence relation induced by $\succeq_1$. Similarly, for n quasi-orders $\succeq_1, \ldots, \succeq_n$, the lexicographic composition $\succeq_1 \circ \cdots \circ \succeq_n$ is defined as follows. Let $x, y \in X$ and $\sim_i$ be the equivalence relation induced by $\succeq_i$ and $\succ_i$ be the strict part of $\succeq_i$. Then, $x\,(\succeq_1 \circ \cdots \circ \succeq_n)\,y$, if $x \sim_i y$ for $i = 1, \ldots, k-1$ and $x \succ_k y$ for some $1 \le k \le n$, or $x \sim_i y$ for all $i = 1, \ldots, n$.

Lemma 2.1. *If $\succeq_1, \ldots, \succeq_n$ are (well-founded) quasi-orders on X, then $\succeq_1 \circ \cdots \circ \succeq_n$ is (well-founded) quasi-orders on X.*

Let $\succeq_i$ be a quasi-order on a set X_i $(i = 1, 2)$. A mapping $f : X_1 \to X_2$ is *order-preserving,* if $x \succeq_1 y$ implies $f(x) \succeq_2 f(y)$ for all $x, y \in X_1$, equivalently, $f(x) \succ_2 f(y)$ implies $x \succ_1 y$. It is *strictly order-preserving* if $x \succeq_1 y \Leftrightarrow f(x) \succeq_2 f(y)$ for all $x, y \in X_1$.

A (*quasi-* (resp. *well-*))*ordered* set is a pair $(X, \succeq)$ of a set X and a (quasi- (resp. well-))order $\succeq$ on X. Quasi-ordered sets $(X_1, \succeq_1)$ and $(X_2, \succeq_2)$ are *isomorphic* if there is a bijection $f : X_1 \to X_2$ which is strictly order-preserving. A class of isomorphic well-ordered sets is an *order type.* If the class of $(X, \succeq)$ is α, we say $(X, \succeq)$ has order type α.

Let $\ge$ be the ordinary order of the set $\mathbb{N}$ of natural numbers. The order type of $(\mathbb{N}, \ge)$ is denoted by ω. For $n \ge 2$, let $\ge_{\text{lex}}$ be the lexicographic order on $\mathbb{N}^n$, that is, $(x_1, \ldots, x_n) >_{\text{lex}} (y_1, \ldots, y_n)$ if there is k such that $1 \le k \le n$ and $x_1 = y_1, \ldots, x_{k-1} = y_{k-1}$ and $x_k > y_k$. The order type of $(\mathbb{N}^n, \ge_{\text{lex}})$ is ω^n. Similarly we can consider the length-lexicographic order $\ge_{\text{llex}}$ on the set $\mathbb{N}^* = \bigcup_{k=1}^{\infty} \mathbb{N}^k$ of all finite sequences of natural numbers;

$$(x_1, \ldots, x_m) \ge_{\text{llex}} (y_1, \ldots, y_n) \;\Leftrightarrow\; m > n, \text{ or}$$

$$m = n \text{ and } (x_1, \ldots, x_m) \ge_{\text{lex}} (y_1, \ldots, y_m).$$

The ordered set $(\mathbb{N}^*, \ge_{\text{llex}})$ has order type ω^ω.

Let M be a monoid, a semigroup with identity element 1. A quasi-order $\succeq$ on M is *compatible* if $x \succeq y$ implies $zxw \succeq zyw$ for any $x, y, z, w \in M$, or equivalently, $zxw \succ zyw$ implies $x \succ y$. It is *strictly compatible* if $x \succ y \Leftrightarrow zxw \succ zyw$ for any $x, y, z, w \in M$, or equivalently $x \succeq y \Leftrightarrow zxw \succeq zyw$ for any $x, y, z, w \in M$. A pair $(M, \succeq)$ of a monoid and a compatible (quasi-)order $\succeq$ on M is a (*quasi-*)*ordered* monoid.

Lemma 2.2. *Let $\succeq$ be a compatible quasi-order on a monoid M. Then, the equivalence relation $\sim$ induced by $\succeq$ is a congruence and the quotient monoid $M/{\sim}$ is an ordered monoid with the order induced by $\succeq$. If $\succeq$ is strictly compatible (resp. well-founded) on M, then so is the induced order $\succeq$ on $M/{\sim}$.*

Note that the induced order $\succeq$ on the quotient $M/\sim$ in the above lemma is defined as $[x] \succeq [y] \Leftrightarrow x \succeq y$ for $x, y \in M$, where $[x]$ denotes the congruence class of x.

In this paper, by a *well-ordered monoid* we mean an ordered monoid $(M, \succeq)$ with strictly compatible well-order $\succeq$. As easily seen, a monoid $(M, \succeq)$ with compatible well-order $\succeq$ is a well-ordered monoid if it is cancellative.

Lemma 2.3. *If $\succeq_1, \ldots, \succeq_n$ are (strictly) compatible quasi-orders on M, then the lexicographic compositions $\succeq_1 \circ \cdots \circ \succeq_n$ is also a (strictly) compatible quasi-order on M.*

A *weight function* of M is a morphism of M to the additive group $\mathbb{R}$ of real numbers; $f(xy) = f(x) + f(y)$ for $x, y \in M$. Of course, the zero function 0 is a weight function. A weight function f is *non-negative* (resp. *positive*) if $f(x) \geq 0$ (resp. $f(x) > 0$) for all $x \in M \setminus \{1\}$. If M is generated by a subset Σ, a weight function f is determined by the values $f(a)$ for generators $a \in \Sigma$. In fact, for any $x = a_1 \cdots a_n$ with $a_i \in \Sigma$, we have $f(x) = \sum_i f(a_i)$.

We define a relation $\succeq_f$ associated with a weight function f on M by

$$x \succeq_f y \quad \Leftrightarrow \quad f(x) \geq f(y).$$

Lemma 2.4. *For a weight function f of a monoid M, $\succeq_f$ is a strictly compatible quasi-order on M. The congruence $\sim_f$ induced by $\succeq_f$ is given by $x \sim_f y \Leftrightarrow f(x) = f(y)$, and the quotient $M/\sim_f$ is a well-ordered monoid.*

A quasi-order $\succeq$ on a monoid M is *weight-sensitive*, if there is a nonzero order-preserving weight function f on M, that is, $x \succeq y$ implies $f(x) \geq f(y)$, or equivalently, $f(x) > f(y)$ implies $x \succ y$, for $x, y \in M$. In this case we say $\succeq$ is *f-sensitive* specifying f. If this f is nonzero non-negative (resp. positive), $\succeq$ is *non-negatively* (resp. *positively*) *weight-sensitive*.

Lemma 2.5. *Let M be a finitely generated monoid. If f is a non-negative weight function of M, the quasi-order $\succeq_f$ is well-founded. If f is positive, any f-sensitive quasi-order is well-founded.*

3 Weight sensitivity

In this section M is always a well-ordered monoid.

Lemma 3.1. *M is torsion-free, cancellative, and for any $x \in M \setminus \{1\}$, we have an infinite increasing sequence*

$$1 \prec x \prec x^2 \prec \cdots \prec x^n \prec \cdots.$$

Corollary 3.2. *Let $x, y \in M$ and $m, n \in \mathbb{N}$. Then,*

(1) $x^m \succ x^n \Leftrightarrow m > n$.

(2) $x^m \succ y^m \Leftrightarrow x \succ y$.

As we see in Lemma 3.1, $x \succeq 1$ for any $x \in M$. This is actually a sufficient condition for a compatible quasi-order $\succeq$ on M to be well-founded by Higman's well-known theorem [6].

An element a of M is a *pivot*, if for any $x \in M$ there is $n \in \mathbb{N}$ such that $x \prec a^n$. A monoid may not have a pivot, but a finitely generated monoid, which we are interested in, has one.

Lemma 3.3. *A nontrivial finitely generated monoid M has a pivot.*

We fix a pivot $a \in M$, and based on this element we define a function $\phi_a : M \to \mathbb{R}$ as follows:

$$\phi_a(x) = \inf \{ n/m \mid x^m \preceq a^n, \ m, n \in \mathbb{N}, m > 0 \}$$

for $x \in M$.

Now, we have the main results in this section.

Theorem 3.4. *ϕ_a is a nonzero non-negative order-preserving weight function on M.*

Corollary 3.5. *If a well-ordered monoid $(M, \succeq)$ has a pivot a, $\succeq$ is non-negatively weight-sensitive, specifically, ϕ_a-sensitive.*

Corollary 3.6. *A well-founded strictly compatible nontrivial quasi-order $\succeq$ on a finitely generated monoid M is non-negatively weight-sensitive.*

The above results were proved for free monoids in [13] (see [14] for a different proof), and are a variant of the classical embedding theorems of ordered semigroups into the nonnegative reals (see [5], [7]).

Lemma 3.7. *$\phi_a(a^n) = n$ for all $n \in \mathbb{N}$.*

Lemma 3.8 (approximation lemma). *For any $m > 0$ there is a positive integer n such that $x^m \preceq a^n$ and*

$$\frac{n-1}{m} \leq \phi_a(x) \leq \frac{n}{m}.$$

Lemma 3.9. *An order-preserving nonzero non-negative weight function on M (if exists) is unique up to constant factor.*

Theorem 3.10 (chain rule). *Let a and b be pivots of M. For any $x \in M$ we have*

$$\phi_a(x) = \phi_a(b) \cdot \phi_b(x).$$

Define a relation $\equiv$ on M as follows: For $x, y \in M$, $x \equiv y$ if and only if $y \preceq x^m$ and $x \preceq y^n$ for some positive integers m, n.

Lemma 3.11. *$\equiv$ is an equivalence relation on M.*

The equivalence class $A(x)$ of $x \in M$ is the *archimedean component* of x in M. As easily seen, if $x \succ y$ and $A(x) \neq A(y)$, then $x' \succ y'$ for all $x' \in A(x)$ and $y' \in A(y)$. Thus, we can define a relation $\succeq$ on the set $\mathcal{A}$ of all archimedean components by $A(x) \succeq A(y) \iff x \succeq y$. Actually $\succeq$ is a well-order of $\mathcal{A}$, and the set P of all pivots of M, if it is not empty, is the maximal archimedean component of M.

If P and $\{1\}$ are the only archimedean components of M, M is called *archimedean.* Accordingly, M is archimedean if and only if for any $x, y \in M \setminus \{1\}$ there are $m, n > 0$ such that $y \preceq x^m$ and $x \preceq y^n$.

Theorem 3.12. *For a nontrivial well-ordered monoid M, the following statements are equivalent.*

(1) M is archimedean.

(2) ϕ_a is positive for any (some) $a\,(\neq 1) \in M$.

(3) M has an order-preserving positive weight function.

If M is finitely generated, these are still equivalent to

(4) M has order type ω.

4 Position sensitive functions

The value of a weight function for an element is only determined by the weights of generators which appear in that element, but does not depend on the positions where the generators appear. In this section we introduce a function which depends not only on how many times a generator appears in the element but also on the places where it appears.

Let M_1 be a submonoid of a well-ordered monoid $(M, \succeq)$. M_1 is also a well-ordered monoid with the order $\succeq$ restricted to M_1. Let a be a pivot of M_1 and set $\phi = \phi_a$ be the weight function of M_1 based on a. For $x \in M$ define

$$\mu_\ell(x) = \inf\left\{\phi(u)/\phi(v) \;\middle|\; xv \preceq ux,\ u, v \in M_1,\ \phi(v) > 0\right\}.$$

Here, if there are no elements $u, v \in M$ such that $\phi(v) > 0$ and $xv \preceq ux$, $\mu_\ell(x)$ is defined to be ∞. These functions are considered to be a generalization of the functions introduced by Martin and Scott on the free monoid generated by two elements (see [12]).

For $r, s \in [0, \infty] = \mathbb{R}_+ \cup \{0, \infty\}$, the product $r \cdot s$ is defined in a conventional way, but $0 \cdot \infty$ and $\infty \cdot 0$ are not defined.

Theorem 4.1. *For all $x \in M$, $\mu_\ell(x) \in [0, \infty]$, and*

$$\mu_\ell(xy) = \mu_\ell(x) \cdot \mu_\ell(y)$$

holds for any $x, y \in M$ as far as the righthand side is defined.

The value $\mu_\ell(x)$ is, so to speak, the rate of change of the weight when an element of M_1 is transformed from right to left of x. We can consider also the rate from left to right as follows. For $x \in M$ define

$$\mu_r(x) = \inf\left\{\phi(v)/\phi(u) \mid ux \preceq xv,\ u, v \in M_1,\ \phi(u) > 0\right\}$$

here $\mu_r(x) = \infty$ if there are no $u, v \in M_1$ such that $ux \preceq xv$ and $\phi(u) > 0$.

Theorem 4.2. *For any $x \in M$ we have*

$$\mu_r(x) = 1/\mu_\ell(x).$$

Lemma 4.3. $\mu_\ell(x) = \mu_r(x) = 1$ *for any $x \in M_1$.*

For an element $x \in M$ expressed as

$$x = u_0 b_1 u_1 \cdots b_k u_k \tag{4.1}$$

with $k \geq 0$, $u_i \in M_1$ for $i = 0, 1, \ldots, k$ and $b_i \in M$ such that $0 \leq \mu_\ell(b_i) < \infty$, define

$$\psi(x) = \phi(u_0) + \phi(u_1)\mu_1 + \cdots + \phi(u_k)\mu_1 \cdots \mu_k,$$

where $\mu_i = \mu_\ell(b_i)$ and ϕ is the weight function on M_1 based on a pivot a of M_1.

For the main theorem of this section we need the following assumption:

(*) There is an element $b \in M$ such that $0 < \mu = \mu_\ell(b) < \infty$ and $\mu \neq 1$.

Theorem 4.4. *We assume the condition (*). Let x, y be two elements of M expressed as* (4.1);

$$x = u_0 b_1 u_1 \cdots b_k u_k \quad (u_i \in M_1,\ b_i \in M),$$
$$y = u'_0 b'_1 u'_1 \cdots b'_\ell u'_\ell \quad (u'_i \in M_1,\ b'_i \in M).$$

Then, $b_1 \cdots b_k = b'_1 \cdots b'_\ell$ and $\psi(x) > \psi(y)$ imply $x \succ y$.

We do not know if Theorem 4.4 remains true in general without assumption (*). But in some special situations we consider next, we can prove the assertion without (*).

First we consider the case where each b_i is in the cyclic submonoid b^* generated by a fixed element b in M. Consider an element x of M written as (4.1), where $b_i \in b^*$ for $i = 1, \ldots, k$. If $\mu_\ell(b) = 1$, the value of our weight sensitive function ψ is given by

$$\psi(x) = \phi(u_0) + \phi(u_1) + \cdots + \phi(u_k).$$

Let y be another element of M written as

$$y = u'_0 b'_1 u'_1 \cdots b'_\ell u'_\ell \quad (u'_i \in M_1,\ b'_i \in b^*).$$

Lemma 4.5. *Assume $\mu(b) = 1$. For elements x and y given above, if $b_1 \cdots b_k = b'_1 \cdots b'_\ell$ and*

$$\phi(u_0) + \phi(u_1) + \cdots + \phi(u_k) > \phi(u'_0) + \phi(u'_1) + \cdots + \phi(u'_k),$$

then $x \succ y$.

If M is generated by $M_1 \cup \{b\}$, any element $x \in M$ is written as (4.1) with $b_i \in b^*$. Now, using the function ψ we define a quasi-order $\succeq_\psi$ on M by

$$x \succeq_\psi y \ \Leftrightarrow\ \psi(x) \geq \psi(y).$$

Corollary 4.6. *If M is generated by $M_1 \cup \{b\}$ for some $b \in M$ such that $0 < \mu_\ell < \infty$, then*

$$\succeq = \succeq_{|\cdot|_b} \circ \succeq_\psi \circ \succeq'$$

for some quasi-order $\succeq'$.

If $\mu_\ell(b_i) = 0$ for some i in (4.1), letting $\bar{k}$ be the smallest such i, we have $\psi(x) = \psi(\bar{x})$, where $\bar{x} = u_0 b_1 u_1 \cdots b_{\bar{k}-1} u_{\bar{k}-1}$. So under the condition $(*)$, Theorem 4.4 asserts that $b_1 \cdots b_k = b'_1 \cdots b'_\ell$ and $\psi(\bar{x}) > \psi(\bar{y})$ imply $x \succ y$ for x, y given in the theorem. Here we consider without condition $(*)$ the case where $\mu_\ell(b_i) = 0$ for all i in (4.1) and M_1 is cyclic.

Lemma 4.7. *Let $a \in M_1$ and $b_1, \ldots, b_k \in M$ and suppose $\mu_\ell(b_1) = \cdots = \mu_\ell(b_k) = 0$. Let x and y be elements of M written as*

$$x = a^{m_0} b_1 a^{m_1} \cdots b_k a^{m_k} \tag{4.2}$$

and

$$y = a^{n_0} b_1 a^{n_1} \cdots b_k a^{n_k}. \tag{4.3}$$

Then, $(m_0, m_1, \ldots, m_k) \geq_{\text{lex}} (n_0, n_1, \ldots, n_k)$ implies $x \succ y$.

When $\mu_\ell(b_i) = \infty$ for all i in (4.1), then $\mu_r(b_i) = 0$. Thus, in the dual way we have

Corollary 4.8. *Let $a \in M_1$ and $b_1, \ldots, b_k \in M$ and suppose $\mu_\ell(b_1) = \cdots = \mu_\ell(b_k) = \infty$. Let x and y be elements of M written as (4.2) and (4.3) respectively. Then, $(m_k, m_{k-1}, \ldots, m_0) \geq_{\text{lex}} (n_k, n_{k-1}, \ldots, n_0)$ implies $x \succ y$.*

5 Monoids generated by two elements

In this section we apply the results obtained in Sections 3 and 4 to monoids generated by two elements. Let $(M, \succeq)$ be a well-ordered monoid generated by a and b. Suppose that b is a pivot and consider the weight function $\phi = \phi_b$ based on b. By Theorem 3.4

$$\succeq \, = \, \succeq_\phi \circ \succeq' \tag{5.1}$$

for some compatible quasi-order $\succeq'$, where $\succeq_\phi$ is the quasi-order associated with ϕ.

If $r = \phi(a) > 0$, a is also a pivot. By Theorem 3.12, M is archimedean and has order type ω. The quasi-order $\succeq_\phi$ is an order if and only if the congruence $\sim_\phi$ induced by $\succeq_\phi$ is the equality relation, that is, $x = y$ in M if and only if $\phi(x) = \phi(y)$. In this case, M is a commutative monoid isomorphic to $\{a,b\}^*/\sim$, where for $x, y \in \Sigma^*$, $x \sim y$ if and only if

$$\phi(x) = |x|_b + r \cdot |x|_a = \phi(y) = |y|_b + r \cdot |y|_a.$$

In particular, if r is irrational, the equality $\phi(x) = \phi(y)$ holds if and only if $x = y$ as abelian words over $\{a,b\}$. Hence, M is the free commutative monoid generated by a and b. If $r = n \in \mathbb{N}$, then $a \sim_\phi b^n$ and M is the infinite cyclic monoid generated by b. If $r = 1/n$ ($n \in \mathbb{N}$), then M is the infinite cyclic monoid generated by a. If $r = p/q$ with $p, q > 1$ and $(p,q) = 1$, then M is the commutative monoid generated by a, b subject to the relation $a^q = b^p$. But, if $\succeq_\phi$ is not an order, we need a quasi-order $\succeq'$ which is nontrivial on $\sim_\phi$ in (5.1), that is, there are two elements $x, y \in M$ such that $x \sim_\phi y$ and $x \succ' y$.

Suppose that $r = 0$, then M is not archimedean. Based on the weight function $|\cdot|_a$ on the submonoid $a^* = \{x \in M \mid \phi(a) = 0\}$ of M generated by a, we have the weight sensitive function μ_ℓ on M. Let $\mu = \mu_\ell(b)$.

First, suppose $0 < \mu < \infty$, then for an element

$$x = a^{m_0} b a^{m_1} \cdots b a^{m_k}$$

of M we have

$$\psi(x) = m_0 + m_1 \mu + \cdots m_k \mu^k.$$

By Corollary 4.6 we see

$$\succeq \; = \; \succeq_{|\cdot|_b} \circ \succeq_\psi \circ \succeq' \tag{5.2}$$

for some quasi-order $\succeq'$. For any $k \geq 0$, the subset $M_k = \{x \in M \mid |x|_b = k\}$ of M has order type ω, and hence M has order type ω^2.

If μ is transcendental, then $\psi(x) = \psi(y)$ implies $x = y$ as words over $\{a,b\}$. It follows that M is the free monoid generated by a and b, $\succeq_\psi$ is an order, and

$$\succeq \; = \; \succeq_{|\cdot|_b} \circ \succeq_\psi .$$

Next suppose that μ is algebraic. Let

$$P_\mu(X) = k_0 + k_1 X + \cdots + k_n X^n \quad (k_i \in \mathbb{Z})$$

be the minimal primitive polynomial of μ over $\mathbb{Z}$. Set

$$\ell_i = \begin{cases} k_i & \text{if } \; k_i > 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\ell_i' = \begin{cases} -k_i & \text{if } \; k_i < 0 \\ 0 & \text{otherwise,} \end{cases}$$

and define

$$x = a^{\ell_0} b a^{\ell_1} \cdots b a^{\ell_n},$$
$$x' = a^{\ell'_0} b a^{\ell'_1} \cdots b a^{\ell'_n}.$$

Then $x \neq x'$ as words but $\psi(x) = \psi(x')$. Let $\sim$ be the congruence induced by the quasi-order $\succeq_{|\cdot|_b} \circ \succeq_\psi$. Then, $\succeq_{|\cdot|_b} \circ \succeq_\psi$ is an order, if and only if $\sim$ is the equality relation, and in this case, M is isomorphic to $\{a,b\}^*/\sim_P$, where $\sim_P$ is the congruence on the free monoid $\{a,b\}$ defined as follows: for elements $x = a^{m_0} b a^{m_1} \cdots b a^{m_k}$ and $y = a^{n_0} b a^{n_1} \cdots b a^{n_\ell}$, $x \sim_P y$ if and only if $k = \ell$ and

$$(m_0 - n_0) + (m_1 - n_1)X + \cdots + (m_k - n_k)X^k \equiv 0 \pmod{P_\mu(X)}.$$

If $\sim$ is not the equality, $\succeq_{|\cdot|_b} \circ \succeq_\psi$ is not an order, and a quasi-order $\succeq'$ which is nontrivial on $\sim$ is necessary in (5.2).

Next suppose $\mu = 0$. Then, by Lemma 4.7, for two elements

$$x = a^{m_0} b a^{m_1} \cdots b a^{m_k} \tag{5.3}$$

and

$$y = a^{n_0} b a^{n_1} \cdots b a^{n_\ell} \tag{5.4}$$

in M, $x \succ y$ if and only if either $k > \ell$, or $k = \ell$ and $(m_0, m_1, \ldots, m_k) \succ_{\text{lex}} (n_0, n_1, \ldots, n_k)$. Hence, $x \succeq y$ and $y \succeq x$ if and only if $x = y$ as words. It follows that M is the free monoid generated by a and b, and

$$\succeq = \succeq_{|\cdot|_b} \circ \succeq_0,$$

where $\succeq_0$ is the quasi-order defined through the lexicographic order $\geq_{\text{lex}}$ on $\mathbb{N}^* = \bigcup_{k \geq 1} \mathbb{N}^k$. In other words, the ordered set M is isomorphic to $(\mathbb{N}^*, \geq_{\text{llex}})$ by the mapping which sends an element $x \in M$ written as (5.3) to the vector $(m_0, m_1, \ldots, m_k) \in \mathbb{N}^{k+1}$. The order type of M is ω^ω.

Finally, suppose $\mu = \infty$. Similarly to the case $\mu = 0$, M is free again, and by Corollary 4.8 we have

$$\succeq = \succeq_{|\cdot|_b} \circ \succeq_\infty,$$

where $\succeq_\infty$ is the quasi-order defined through the reverse-lexicographic order on $\mathbb{N}^*$. Again, M has order type ω^ω.

Summarizing, when M is non-archimedean and μ is transcendental or $\mu = 0$ or $\mu = \infty$, the structure of M is unique, that is, M is free, and the order on M is completely determined by r and μ. But in other cases, the two parameters r and μ are not sufficient to determine the structure of M. Actually uncountably many different structures for M are possible for each r and nonzero algebraic μ. Moreover, even if the algebraic structure of the underlying monoid M is fixed, uncountably many different well-orders on M are possible. We omit the details here.

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