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On Commutative Semigroup Rings

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I am now making a book on commutative semigroup rings. It will appear before long. This is an introduction to the book.

Thus let G be an abelian additive group which is torsion-free. A subsemigroup S of G which contains 0 is called a grading monoid (or a g-monoid). Let R be a commutative ring, and let $R[X;S] = \{\sum_{\text{finite}} a_iX^{s_i} \mid a_i \in R, s_i \in S\}$ be the semigroup ring of S over R. Let Π be a ring-theoretical property. We will determine conditions for $R[X;S]$ to have property Π. For the present, within my knowledge and within my interest, there are 71 Theorems and 38 Propositions on $R[X;S]$ by a number of authors. We confer a number of references. The following is a part of them:

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Now we will note some theorems on commutative semigroup rings. Let G be a non-zero torsion-free abelian additive group, S be a non-zero grading monoid, R be a commutative ring, and D be an integral domain.

Let $\mathrm{q}(S) = \{a - b \mid a, b \in S\}$. Then it is called the quotient group of

S.

Let $\alpha \in \mathrm{q}(S)$. If $n\alpha \in S$ for some positive integer n, then α is called integral over S. If each integral element of $\mathrm{q}(S)$ belongs to S, then S is called integrally closed.

Theorem 1 The followings are equivalent.

(1) $D[X;S]$ is integrally closed.

(2) D is integrally closed, and S is integrally closed.

Let $\alpha \in \mathrm{q}(S)$. Then α is called almost integral over S, if there exists $s \in S$ such that $s+n\alpha \in S$ for each positive integer n. If each almost integral element belongs to S, then S is called completely integrally closed.

Theorem 2 The followings are equivalent.

(1) $D[X;S]$ is completely integrally closed.

(2) D is completely integrally closed, and S is completely integrally closed.

A non-zero divisor of R is also called a regular element. An ideal of R which contains regular elements is called a regular ideal.

The total quotient ring of R is denoted by $\mathrm{q}(R)$.

If each finitely generated regular ideal of R is invertible, then R is called a Prüfer ring.

If each finitely generated ideal of R is principal, then R is called a Bezout ring.

If, for each $a \in R$, there exists $b \in R$ such that $a = a^2b$, then R is called a von Neumann regular ring.

Theorem 3 Let $\mathbf{Q}_0$ be the non-negative rational numbers. The followings are equivalent.

(1) $R[X;S]$ is a Prüfer ring.

(2) R is a von Neumann regular ring, and S is isomorphic onto either a subgroup of $\mathbf{Q}$ or a subsemigroup S' of $\mathbf{Q}_0$ such that $\mathrm{q}(S') \cap \mathbf{Q}_0 = S'$.

(3) $R[X;S]$ is a Bezout ring.

If G satisfies ascending chain condition on cyclic subgroups, then G is said to satisfy ACCC.

Theorem 4 Let $G = \mathrm{q}(S)$. The followings are equivalent.

(1) $D[X;S]$ is a unique factorization ring.

(2) D is a unique factorization ring, S is a unique factorization semigroup, and G satisfies ACCC.

If R satisfies ascending chain condition on regular ideals, then R is called an r-Noetherian ring.

Theorem 5 The followings are equivalent.

(1) $R[X;S]$ is a Noetherian ring.

(2) $R[X;S]$ is an r-Noetherian ring.

(3) R is a Noetherian ring, and S is a finitely generated g-monoid.

Let I be a non-empty subset of $\mathrm{q}(R)$. We set $I^{-1} = \{x \in \mathrm{q}(R) \mid xI \subset R\}$. We set $I^v = (I^{-1})^{-1}$.

Let I be a fractional ideal of R. If $I^v = I$, then I is called divisorial.

If each divisorial ideal of D is principal, then D is called a pseudo-principal ring.

If each divisorial ideal of S is principal, then S is called a pseudo-principal semigroup.

Theorem 6 Let $G = \mathrm{q}(S)$. The followings are equivalent.

(1) $D[X;S]$ is a pseudo-principal ring.

(2) D is a pseudo-principal ring, S is a pseudo-principal semigroup, and G satisfies ACCC.

Let I be an ideal of R such that $I^{k+1} = 0$ for some positive integer k. We set $\mathrm{d}(I^i/I^{i+1}) = \min \{| X || X$ is a set of generators of the R-module $I^i/I^{i+1}\}$ for each i (d(0)= 0). We set $\nu(I) = \mathrm{d}(I/I^2) + \cdots + \mathrm{d}(I^{k-1}/I^k) + \mathrm{d}(I^k)$.

If each finitely generated ideal of R is generated by n-elements, then R is said to have n-generator property.

Let S be a finitely generated subsemigroup of $\mathbf{Q}_0$, and let $\mathrm{q}(S) = \mathbf{Z}r$ ($r \in \mathrm{Q}_0$). Then min $\{(1/r)S - \{0\}\}$ is called the order of S, and is denoted by $\mathrm{o}(S)$.

Theorem 7 Let N be the nil radical of R. The followings are equivalent.

(1) $R[X;S]$ has the n-generator property.

(2) One of the followings holds.

(i) S is isomorphic onto a subgroup of $\mathbf{Q}$, and dim $(R) = 0$. If I is a finitely generated ideal contained in N, there exists a decomposition $R = Re_1 \oplus \cdots \oplus Re_h$ such that $\nu(Ie_j) < n$ for each j.

(ii) S is isomorphic onto a subsemigroup of $\mathbf{Q}_0$, $\mathrm{o}(S) < \infty$, and dim $(R) = 0$. If I is a finitely generated ideal contained in N, there exists a decomposition $R = Re_1 \oplus \cdots \oplus Re_h$ such that $(\nu(Ie_j) + 1)\mathrm{o}(S) \leq n$ for each j.

If each finitely generated regular ideal is generated by n-elements, then R is said to have r-n-generator property.

If, for each regular non-unit a of R, $R/(a)$ has n-generator property, then R is said to have r-$n(1/2)$-generator property.

If, for each non-zero and non-unit a of R, $R/(a)$ has n-generator property, then R is said to have $n(1/2)$-generator property.

Theorem 8 The followings are equivalent.

(1) $R[X;S]$ has $n(1/2)$-generator property.

(2) $R[X;S]$ has n-generator property.

(3) $R[X;S]$ has r-$n(1/2)$-generator property.

(4) $R[X;S]$ has r-n-generator property.

If each ideal of R is generated by n-elements, then R is said to have rank n.

Theorem 9 Let $\mathbf{Z}_0$ be the non-negative integers. The followings are equivalent.

(1) $R[X;S]$ has rank n.

(2) One of the followings holds.

(i) S is isomorphic onto $\mathbf{Z}$, and there exists a decomposition $R = R_1 \oplus \cdots \oplus R_h$ which satisfieds the following condition: If N_i is the nil radical of R_i, then $\nu(N_i) < n$, and R_i is a Noetherian local ring with maximal ideal N_i for each i.

(ii) S is isomorphic onto a subsemigroup of $\mathbf{Z}_0$, and there exists a decomposition $R = R_1 \oplus \cdots \oplus R_h$ which satisfies the following condition: If N_i is the nil radical of R_i, then $(\nu(N_i) + 1)\mathrm{o}(S) \leq n$, and R_i is a Noetherian local ring with maximal ideal N_i for each i.

Let K be a commutative ring with $K = \mathrm{q}(K)$, and let Γ be a totally ordered abelian additive group. A mapping v of K onto $\Gamma \cup \{\infty\}$ is called a valuation on K if $v(x+y) \geq \inf\,(v(x), v(y))$, and $v(xy) = v(x) + v(y)$ for all $x, y \in K$. The subring $V = \{x \in K \mid v(x) \geq 0\}$ of K is called a valuation ring on K. t.f.r. (Γ) is called the rank of v (or of V), where t.f.r. $(\Gamma) = \max\,\{|\,X\,| \mid X$ is a subset of Γ which is linearly independent over $\mathbf{Z}\}$.

If there exists a family $\{V_\lambda \mid \Lambda\}$ of valuation rings on $\mathrm{q}(R)$ which satisfies the following conditions, then R is called a Krull ring: $R = \cap_\lambda V_\lambda$, each V_λ is rank 1 and discrete, and each regular element of R is a unit of V_λ for almost all λ.

Let Γ be a totally ordered abelian additive group. A mapping v of G onto Γ is called a valuation on G, if $v(x+y) = v(x) + v(y)$ for all $x, y \in G$. The subsemigroup $V = \{x \in G \mid v(x) \geq 0\}$ of G is called a valuation semigroup on G. t.f.r. (Γ) is called the rank of v (or of V).

Theorem 10 Let $G = \mathrm{q}(S)$. The followings are equivalent.

(1) $D[X;S]$ is a Krull ring.

(2) D is a Krull ring, S is a Krull semigroup, and G satisfies ACCC.

Let L be an abelian additive group, and let p be a prime number. The subgroup $\{x \in L \mid p^n x = 0$ for some positive integer $n\}$ is called the p-primary component of L.

If R_M is a Noetherian ring for each maximal ideal M of R, then R is called a locally Noetherian ring.

Theorem 11 Let H be the unit group of S, and let F be a free subgroup of H such that H/F is torsion. Let Ω be the set of prime numbers p such that $p1_R$ is a non-unit of R. The followings are equivalent.

(1) $R[X;S]$ is a locally Noetherian ring.

(2) t.f.r. $(H) < \infty$, R is locally Noetherian, S is of the form $H+\mathbf{Z}_0 s_1 + \cdots + \mathbf{Z}_0 s_n$, and the p-primary component of H/F is finite for each $p \in \Omega$.

Theorem 12 Assume that $D[X;S]$ is a Krull ring. Then

$C(D[X;S]) \cong C(D) \oplus C(S)$,

where $C(\ )$ denotes the divisor class group.

R is called a v-ring, if it satisfies the following condition: If I, J_1, J_2 are finitely generated ideals of R with I regular, and $(IJ_1)^v \subset (IJ_2)^v$, then $J_1^v \subset J_2^v$.

We may naturally define v-semigroup.

Theorem 13 The followings are equivalent.

(1) $D[X;S]$ is a v-ring.

(2) D is a v-ring, and S is a v-semigroup.

Theorem 14 Assume that D is integrally closed, and S is integrally closed. The followings are equivalent.

(1) For each finite number of finitely generated non-zero ideals $I_1, \cdots, I_n$ of $D[X;S]$, we have $(I_1 \cap \cdots \cap I_n)^v = I_1^v \cap \cdots \cap I_n^v$.

(2) For each finite number of finitely generated non-zero ideals $I_1, \cdots, I_n$ of D, we have $(I_1 \cap \cdots \cap I_n)^v = I_1^v \cap \cdots \cap I_n^v$, and for each finite number of finitely generated ideals $I_1, \cdots, I_m$ of S, we have $(I_1 \cap \cdots \cap I_m)^v = I_1^v \cap \cdots \cap I_m^v$.

(3) $D[X;S]$ is a v-ring.

If, for each finitely generated regular ideal I of R, there exists a finitely generated regular fractional ideal J such that $(IJ)^v = R$, then R is called a Prüfer v-multiplication ring.

Theorem 15 The followings are equivalent.

(1) $D[X;S]$ is a Prüfer v-multiplication ring.

(2) D is a Prüfer v-multiplication ring, and S is a Prüfer v-multiplication semigroup.

Let I be a non-zero fractional ideal of R. We set $I^t = \cup\{J^v \mid J$ is a finitely generated fractional ideal contained in $I\}$.

Theorem 16 Assume that D is integrally closed, and S is integrally closed. The followings are equivalent.

(1) For each finite number of non-zero ideals $I_1, \cdots, I_n$ of D[X;S], we have $(I_1 \cap \cdots \cap I_n)^t = I_1^t \cap \cdots \cap I_n^t$.

(2) For each finite number of non-zero ideals $I_1, \cdots, I_n$ of D, we have $(I_1 \cap \cdots \cap I_n)^t = I_1^t \cap \cdots \cap I_n^t$, and for each finite number of ideals $I_1, \cdots, I_m$ of S, we have $(I_1 \cap \cdots \cap I_m)^t = I_1^t \cap \cdots \cap I_m^t$.

(3) $D[X;S]$ is a Prüfer v-multiplication ring.

If R satisfies the following condition, then R is called a root closed ring: If $x \in \mathrm{q}(R)$ and $x^n \in R$ for some positive integer n, then $x \in R$.

Theorem 17 The followings are equivalent.

(1) $D[X;S]$ is a root closed ring.

(2) D is a root closed ring, and S is an integrally closed semigroup.

If R satisfies the following condition, then R is called a seminormal ring: If $x \in \mathrm{q}(R)$ and $x^2, x^3 \in R$, then $x \in R$.

If S satisfies the following condition, then S is calld a seminormal semigroup: If $x \in \mathrm{q}(S)$ and $2x, 3x \in S$, then $s \in S$.

Theorem 18 The followings are equivalent.
(1) $D[X;S]$ is seminormal.
(2) D is seminormal, and S is seminormal.

R is called a u-closed ring, if it satisfies the following condition: If $x \in q(R)$, and $x^2 - x \in R$, $x^3 - x^2 \in R$, then $x \in R$.

Theorem 19 If D is u-closed, then $D[X;S]$ is u-closed.

An ideal of R (resp. S) is also called an integral ideal.
If D satisfies the ascending chain condition on divisorial integral ideals of D, then D is called a Mori-ring.
If D is a Mori-ring, and if, for all $a, b \in D - \{0\}$, the ideal (a, b) is divisorial, then D is called an M-ring.
We may naturally define Mori-semigroup and M-semigroup.

Theorem 20 The followings are eqiovalent.
(1) $D[X;S]$ is an M-ring.
(2) D is a field, and S is isomorphic onto an M-subsemigroup of $\mathbf{Z}$.

Let $F(R)$ be the set of non-zero fractinal ideals of R. A mapping $*$ of $F(R)$ to $F(R)$ is called a star operation on R, if, for regular $a \in q(R)$ and $I, J \in F(R)$,
$(a)^* = (a)$.
$(aI)^* = aI^*$.
$I \subset I^*$.
If $I \subset J$, then $I^* \subset J^*$.
$(I^*)^* = I^*$.
The mapping $I \longmapsto I^v = (I^{-1})^{-1}$ is a star operation called v-operation.
Assume that R is integrally closed, and let $\{V_\lambda \mid \Lambda\}$ be the set of valuation overrings of R. The mapping $I \longmapsto I^b = \cap_\lambda I V_\lambda$ is a star operation called b-operation.
A star operation $*$ is called an e.a.b., if it satisfies the following condition: If I, J_1, J_2 are finitely generated non-zero ideals of R with I regular,

and $(IJ_1)^* \subset (IJ_2)^*$, then $J_1^* \subset J_2^*$.

Let $F'(R)$ be the set of non-zero R-submodules of $q(R)$. A mapping $*$ of $F'(R)$ to $F'(R)$ is called a semistar operation on R, if it satisfies the following condition: For regular $a \in q(R)$ and $I, J \in F'(R)$,

$(aI)^* = aI^*$.

$I \subset I^*$.

If $I \subset J$, then $I^* \subset J^*$.

$(I^*)^* = I^*$.

A semistar operation $*$ of R is called e.a.b., if it satisfies the following condition: If I, J_1, J_2 are non-zero finitely generated ideals with I regular, and $(IJ_1)^* \subset (IJ_2)^*$, then $J_1^* \subset J_2^*$.

The mapping $I \longmapsto I^{v}$ of $F'(R)$ is a semistar operation called v'-operation.

Let $\{V_\lambda \mid \Lambda\}$ be the set of valuation overrings of R. The mapping $I \longmapsto I^{b'} = \cap_\lambda IV_\lambda$ of $F'(R)$ is a semistar operation called b'-operation.

If each finitely generated regular ideal is principal, then R is called an r-Bezout ring.

Let $f = \sum a_i X^{s_i}$, where each $a_i \neq 0$, and $s_i \neq s_j$ for $i \neq j$. We set $\sum Ra_i = c(f)$.

If each regular ideal of R is generated by regular elements, then R is called a Marot ring. If R satisfies the following condition, then R is said to have Property (A): If f is a regular element of $R[X]$, then $c(f)$ is a regular ideal of R.

A denotes a Marot ring with Property (A).

Theorem 21 Let $*$ be an e.a.b. star operation on A.

Set $A_* = \{f/g \in q(A[X;S]) \mid f, g \in A[X;S] - \{0\}$, g is regular, and $c(f)^* \subset c(g)^*\} \cup \{0\}$. Then,

(1) A_* is an overring of $A[X;S]$, and $A_* \bigcap K = A$, where $K = q(A)$.

(2) A_* is an r-Bezout ring.

(3) If I is a finitely generated regular ideal of A, then $IA_* \bigcap K = I^*$ and $IA_* = I^*A_*$.

A multiplicative subset T of R is called a regular multiplicative subset, if each element of T is regular.

Theorem 22 Assume that A is integrrly closed. Let $T = \{f \in A[X;S] \mid c(f) = A\}$. The followings are equivalent.

(1) A is a Prüfer ring.

(2) $A[X;S]_T = A_b$.

(3) $A[X;S]_T$ is a Prüfer ring.

(4) A_b is a quotient ring of $A[X;S]$ with respect to a regular multiplicative subset.

(5) Each prime ideal of $A[X;S]_T$ is the contraction of a prime ideal of A_b.

(5)′ Each regular prime ideal of $A[X;S]_T$ is the contraction of a prime ideal of A_b.

(6) Each regular prime ideal of $A[X;S]_T$ is the extension of a prime ideal of A.

If each regular ideal is the product of prime ideals, then R is called a Dedekind ring.

If each regular ideal of R is principal, then R is called an r-principal ideal ring.

Theorem 23 Assume that A is integrally closed. Let $T = \{f \in A[X;S] \mid c(f) = A\}$. The followings are equivalent.

(1) A is a Dedekind ring.

(2) $A[X;S]_T$ is a Dedekind ring.

(3) A_b is a Dedekind ring.

(4) A_b is an r-Noetherian ring.

(5) A_b is a Krull ring.

(6) A_b is an r-principal ideal ring.

Let $*$ be a star operation on R. If, for each finitely generated regular ideal I of R, there exixts a finitely generated regular fractional ideal J such that $(IJ)^* = R$, then R is called a Prüfer $*$-mltiplication ring.

Let P be a prime ideal of R. Then we set $R_{[P]} = \{x \in q(R) \mid sx \in R$

for some $s \in R - P\}$.

Theorem 24 Let $*$ be an e.a.b. star operation on A. Let $N = \{g \in A[X;S] \mid g$ is regular, and $c(g)^* = A\}$. The followings are equivalent.

(1) A is a Prüfer $*$-multiplication ring.

(2) A_* is a quotient ring of $A[X;S]$ with respect to a regular multiplicative subset.

(3) If V is a valuation overring of A_*, there exists a prime ideal P of A which satisfies the following condition: $A_{[P]}$ is a valuation overring of A, and $V = A[X;S]_{[PA[X;S]]}$.

(4) A_* is a flat $A[X;S]$-module.

(5) $A[X;S]_N$ is a Prüfer ring.

Let $f = \sum a_i X^{s_i}$, where each $a_i \neq 0$ and $s_i \neq s_j$ for $i \neq j$. We set $e(f) = \cup(S + s_i)$.

Theorem 25 Let $*$ be an e.a.b. star operation on S, $G = q(S)$, and let K be a field. We set $S_* = \{f/g \mid f, g \in K[X;S] - \{0\},\ e(f)^* \subset e(g)^*\} \cup \{0\}$.

(1) S_* is an overring of $K[X;S]$, and $S_* \bigcap G = S$.

(2) S_* is a Bezout ring.

(3) If I is a finitely generated ideal of S, then $(IS_*) \bigcap G = I^*$, and $IS_* = I^* S_*$.

For an e.a.b. semiatar operation $*$ on A (or on S), we may naturally define Kronecker function ring A_* (or S_*). Moreover, we may show the similar results to those for star operations.