The finite basis problem
for monoids of triangular boolean matrices

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Abstract

We show that the monoid of all $n \times n$ upper triangular boolean matrices has no finite identity basis whenever $n > 3$. The identities of its submonoid consisting of matrices in which all diagonal entries are 1 possess a finite basis if and only if $n \le 4$.

1   The finite basis problem for matrix monoids

An algebra A is said to be *finitely based* if all identities holding in A follow from a finite set of such identities (an *identity basis* of A); otherwise A is called *nonfinitely based*. While every finite group is finitely based (Oates and Powell [5]), among finite monoids there exist nonfinitely based ones. The first example of a nonfinitely based finite monoid (due to Perkins [6]) was the Brandt monoid B_2^1 formed by the six matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1}$$

under the usual multiplication of $n \times n$-matrices:

$$\left(\alpha_{ij} \right) \cdot \left(\beta_{ij} \right) = \left(\sum_{k=1}^{n} \alpha_{ik} \beta_{kj} \right). \tag{2}$$

We have not yet specified where 0 and 1, the entries of the matrices in (1), come from and how they add and multiply. In fact, there is a certain degree of freedom here: since the above matrices have at most one non-zero entry in each row, the multiplication (2) works just the same provided that the symbols 0 and 1 obey the following rules:

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0 + 0 = 0, \quad 1 \cdot 1 = 1 + 0 = 0 + 1 = 1, \qquad (3)$$

whereas the value of the sum $1 + 1$ does not matter at all. Therefore, one may conveniently think of B_2^1 as of a submonoid of the monoid $\mathcal{M}_2(F)$ of all 2×2-matrices over an arbitrary field F and, in the same time, as of a submonoid of the monoid $\mathcal{B}_2$ of all binary relations over the 2-element set because the latter monoid can be identified with the monoid of all 2×2-matrices over the boolean semiring $\mathcal{B} = \langle \{0,1\}; +, \cdot \rangle$ in which the operations satisfy the rules (3) and in addition $1 + 1 = 1$.

The twofold nature of B_2^1 is important in view of Sapir's powerful result [8] revealing that this monoid is inherently nonfinitely based. We recall that a finite monoid is said to be *inherently nonfinitely based* if it is contained in no locally finite finitely based variety. Since the variety generated by a finite monoid is locally finite, every inherently nonfinitely based finite monoid automatically is nonfinitely based; moreover, if the variety var M generated by a finite monoid M contains an inherently nonfinitely based monoid, then M itself is inherently nonfinitely based and so M is nonfinitely based. Therefore the fact that B_2^1 is inherently nonfinitely based immediately solves the finite basis problem for both the monoid $\mathcal{M}_n(F)$ of all $n \times n$-matrices over a finite field F and the monoid $\mathcal{B}_n$ of all binary relations over the n-element set: for $n \geq 2$ these monoids are nonfinitely based. (Indeed, one only has to take into account that if $n \geq 2$ then $\mathcal{M}_n(F)$ and $\mathcal{B}_n$ contain submonoids isomorphic to $\mathcal{M}_2(F)$ and respectively $\mathcal{B}_2$, whence they both possess submonoids isomorphic to B_2^1—then Sapir's theorem applies.) An alternative proof of this result which does not depend on the notion of an inherently nonfinitely based monoid has been published in [12]; again, this proof is a byproduct of a more general approach.

Since a complete solution of the finite basis problem for full matrix monoids has been found, the next logical step in the general programme of classifying finite monoids with respect to the finite basis property may consist in investigating certain distinguished submonoids in $\mathcal{M}_n(F)$ and $\mathcal{B}_n$. Here the monoids of triangular matrices naturally come into the play. Let $\mathcal{T}_n(F)$ denote the submonoid of all upper triangular matrices in $\mathcal{M}_n(F)$ and let $\mathcal{TB}_n$ be the monoid of all upper triangular boolean $n \times n$-matrices.

In [14] we have studied and partially solved the finite basis problem for the monoids $\mathscr{T}_n(F)$ where F is a finite field. We note that these monoids generate varieties which do not contain the monoid B_2^1 so Sapir's result does not apply here; neither does the approach from [12]. Nevertheless, we have proved that $\mathscr{T}_n(F)$ is nonfinitely based (and even inherently nonfinitely based) provided that $|F| > 2$ and $n \geq 4$. The cases when the field F has precisely 2 elements or when $|F| > 2$ but $n = 2, 3$ still remain open; we can only prove that under these conditions the monoid $\mathscr{T}_n(F)$ is not inherently nonfinitely based but we do not know whether or not it is finitely based.

The first main result of the present note is to some extent similar to the cited result of [14]: we show that the monoid $\mathscr{TB}_n$ is nonfinitely based (and even inherently nonfinitely based) provided that $n \geq 4$. The reason behind this fact is that in the boolean situation the monoid B_2^1 does belong to the varieties generated by the monoids of triangular matrices of sufficiently big size. Again, in the cases when $n = 2, 3$ we only know that $\mathscr{TB}_n$ is not inherently nonfinitely based but we do not know whether it is finitely based.

Our second main result solves the finite basis problem for a closely related family of submonoids in $\mathscr{B}_n$, namely, for the monoids $\mathscr{UTB}_n$ of all upper *unitriangular* boolean $n \times n$-matrices, that is, upper triangular boolean matrices in which all diagonal entries are 1. We show that the monoids $\mathscr{UTB}_n$ are nonfinitely based whenever $n \geq 5$ and provide an explicit finite identity basis for each monoid $\mathscr{UTB}_n$ with $n = 2, 3, 4$.

For the sake of completeness we mention that the monoids of all upper unitriangular $n \times n$-matrices over a finite field F are finitely based. Indeed, these monoids are groups, and Oates–Powell's theorem [5] applies.

We summarize this mini-survey of the current state of art in studying the finite basis problem for finite monoids of matrices in the following table.

Table 1: The finite basis property of some finite monoids of $n \times n$-matrices

The underlying structure:	Type of matrices:		
	arbitrary	triangular	unitriangular
a finite field F	nonfinitely based for all $n \geq 2$	nonfinitely based if $|F| > 2$ and $n \geq 4$; unknown if $|F| = 2$ or $n = 2, 3$	finitely based for all $n \geq 2$
the boolean semiring $\mathscr{B}$	nonfinitely based for all $n \geq 2$	nonfinitely based if $n \geq 4$; unknown if $n = 2, 3$	nonfinitely based if and only if $n \geq 5$

The reader who wants to place the results listed in Table 1 into a broader context can find a comprehensive report on the finite basis problem for finite monoids in [13].

2 The identities of the monoid $\mathscr{TB}_n$

Theorem 2.1. *The monoid $\mathscr{TB}_n$ is inherently nonfinitely based if and only if $n \geq 4$.*

Proof. First observe that for each $n \geq 4$ the monoid $\mathscr{TB}_n$ contains a submonoid isomorphic to $\mathscr{TB}_4$. Therefore, as explained in Section 1, it suffices to verify that the Brandt monoid B_2^1 belongs to the variety var $\mathscr{TB}_4$. Consider the submonoid B in $\mathscr{TB}_4$ generated by the following two matrices:

$$a = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since for each matrix $\left(\gamma_{ij}\right) \in B$ one has $\gamma_{11} = \gamma_{44} = 1$, the set I of all matrices $\left(\delta_{ij}\right) \in B$ such that $\delta_{14} = 1$ forms an ideal in B. A straightforward calculation shows that, besides a, b, and the identity matrix e, only the two matrices

$$ab = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad ba = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

belong to $B \setminus I$. This allows one to organize the following bijection between $B \setminus I$ and the set of non-zero matrices in B_2^1:

$$e \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad ab \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad a \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad ba \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and one easily checks that this bijection extends to an isomorphism between the Rees quotient B/I and the monoid B_2^1. We see that B_2^1 turns out to be a morphic image of a submonoid in $\mathscr{TB}_4$ whence B_2^1 belongs to the variety generated by $\mathscr{TB}_4$.

It remains to verify that the monoids $\mathscr{TB}_2$ and $\mathscr{TB}_3$ are not inherently nonfinitely based. Since $\mathscr{TB}_2$ embeds into $\mathscr{TB}_3$, it suffices to consider the

latter monoid. With a little help of a computer programme, we have checked that $\mathscr{TB}_3$ satisfies the identities

$$x^5 = x^6 \quad \text{and} \quad (xy)^5(yx)^5(xy)^5 = (xy)^5.$$

Clearly, the first of these identities fails in any non-trivial group, and the second one can be shown to fail in the monoid B_2^1: just substitute $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for x and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ for y. Hence the variety var $\mathscr{TB}_3$ contains neither B_2^1 nor non-trivial groups. From [9, Theorem 2] we conclude that the monoid $\mathscr{TB}_3$ is not inherently nonfinitely based. ∎

Remark 2.1. The monoids $\mathscr{TB}_n$ have been extensively studied by Pin and Straubing [7]. Using the techniques of formal language theory, they have proved that the pseudovariety generated by these monoids coincides with the well known pseudovariety **PJ** generated by power monoids of $\mathscr{J}$-trivial finite monoids. Since the Brandt monoid B_2^1 is known to belong to **PJ** (see, for instance, [1, Section 11.6]), this result implies that B_2^1 belongs to the variety var $\mathscr{TB}_m$ *for some* m, and thus, the monoids $\mathscr{TB}_n$ with $n \geq m$ are inherently nonfinitely based. Our direct and rather simple approach gives the exact value of this m, namely, $m = 4$.

3 The identities of the monoid $\mathscr{UTB}_n$

We obtain the main result of this section by identifying the collection of all identities of the monoid $\mathscr{UTB}_n$ with a set of identities previously studied by Blanchet-Sadri [2–4]. Let us first define the latter set. We fix a countably infinite alphabet $\Sigma = \{x, y, z, t, x_1, x_2, \ldots, y_1, y_2, \ldots\}$; as usual, Σ^* stands for the free monoid over Σ. If $u, v \in \Sigma^*$ are two words, we say that u is a *scattered subword* of v whenever there exist words $u_1, \ldots, u_n, v_0, v_1, \ldots, v_{n-1}, v_n \in \Sigma^*$ such that

$$u = u_1 \cdots u_n \quad \text{and} \quad v = v_0 u_1 v_1 \cdots v_{n-1} u_n v_n;$$

in other terms, this means that one can extract u treated as a sequence of letters from the sequence v. We denote by J_k the set of all identities $u = v$ such that the words u and v have the same set of scattered subwords of length $\leq k$. Clearly, we have

$$J_1 \supseteq J_2 \supseteq \cdots \supseteq J_k \supseteq \cdots.$$

Blanchet-Sadri [2–4] has established the following facts about the identity sets J_k:

Proposition 3.1. a) *The identities*

$$x^2 = x, \ xy = yx \tag{4}$$

form a basis of the set J_1.
  b) *The identities*

$$xyxzx = xyzx, \ (xy)^2 = (yx)^2 \tag{5}$$

form a basis of the set J_2.
  c) *The identities*

$$xyx^2zx = xyxzx, \ xyzx^2tz = xyxzx^2tz, \ zyx^2ztx = zyx^2zxtx,$$
$$(xy)^3 = (yx)^3 \tag{6}$$

form a basis of the set J_3.
  d) *The sets J_k with $k \geq 4$ are not finitely based.*

It should be stressed that speaking about identity bases we always mean bases in the sense of monoid identities. For instance, we have not included in the above base for J_2 the identity $x^3 = x^2$ (which certainly belongs to J_2) because in the monoid setting it can be deduced from the identity $xyxzx = xyzx$ by substituting 1 for y and z. However, as observed in [13, p.173], the property of a monoid to be finitely based or nonfinitely based does not depend on using the semigroup or the monoid deduction rules.

The following proposition relates the identity sets J_k and the monoid $\mathcal{UTB}_n$:

Proposition 3.2. *For each positive integer k, the set J_k coincides with the set I_k of all identities holding in the monoid $\mathcal{UTB}_{k+1}$.*

Proof. In order to prove the inclusion $I_k \subseteq J_k$ we use an argument due to Straubing [10]. Given a word $w = x_1 \cdots x_m$ with $x_1, \ldots, x_m \in \Sigma$ and $m \leq k$, we define the morphism $\varphi_w : \Sigma^* \to \mathcal{UTB}_{k+1}$ by the rule

$$(x\varphi_w)_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 1 & \text{if } x = x_i \text{ and } j = i+1, \\ 0 & \text{in the other cases.} \end{cases} \tag{7}$$

For instance, if $w = xyx$ and $k = 3$, then

$$x\varphi_w = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad y\varphi_w = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and } z\varphi_w = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for all letters $z \neq x, y$.

Lemma 3.3. *For each word $v \in \Sigma^*$, the word w is a scattered subword of v if and only if $(v\varphi_w)_{1\,m+1} = 1$.*

Proof. Let $v = y_1 \cdots y_n$ for some $y_1, \ldots, y_n \in \Sigma$, and let $y_s\varphi_w = \left(\alpha_{ij}^{(s)}\right)$. Then, using the multiplication rule (2) $n-1$ times, we arrive at the following expansion of the element $(v\varphi_w)_{1\,m+1}$:

$$(v\varphi_w)_{1\,m+1} = \sum_{1 \leq j_1 \leq \cdots \leq j_{n-1} \leq m+1} \alpha_{1j_1}^{(1)} \alpha_{j_1 j_2}^{(2)} \cdots \alpha_{j_{n-1}\,m+1}^{(n)}. \tag{8}$$

If $w = x_1 \cdots x_m$ is a scattered subword of v, then there exists a sequence $1 \leq s_1 < \cdots < s_m \leq n$ such that $x_i = y_{s_i}$ for every $i = 1, \ldots, m$. Then all factors in the product

$$\alpha_{11}^{(1)} \cdots \alpha_{11}^{(s_1-1)} \alpha_{12}^{(s_1)} \alpha_{22}^{(s_1+1)} \cdots \alpha_{22}^{(s_2-1)} \alpha_{23}^{(s_2)} \cdots$$
$$\cdots \alpha_{m\,m}^{(s_{m-1}+1)} \cdots \alpha_{m\,m}^{(s_m-1)} \alpha_{m\,m+1}^{(s_m)} \alpha_{m+1\,m+1}^{(s_m+1)} \cdots \alpha_{m+1\,m+1}^{(n)} \tag{9}$$

are equal to 1, whence the product itself equals 1 and so does the sum in the right hand side of (8) of which this product is one of the summands. Thus, $(v\varphi_w)_{1\,m+1} = 1$.

Conversely, if $(v\varphi_w)_{1\,m+1} = 1$, then at least one of the summands in the right hand side of (8) is equal to 1. Let

$$\alpha_{1j_1}^{(1)} \alpha_{j_1 j_2}^{(2)} \cdots \alpha_{j_{n-1}\,m+1}^{(n)} = 1.$$

By the definition of the mapping φ_w, we have $\alpha_{ij}^{(s)} = 0$ whenever $i \neq j, j+1$. Therefore the increasing sequence $j_0 = 1 \leq j_1 \leq \cdots \leq j_{n-1} \leq m + 1 = j_n$ contains exactly m "jumps" of the form $j_r < j_{r+1} = j_r + 1$. If we denote by s_i, $i = 1, \ldots, m$, the position of the ith such jump, then the product $\alpha_{1j_1}^{(1)} \alpha_{j_1 j_2}^{(2)} \cdots \alpha_{j_{n-1}\,m+1}^{(n)}$ can be written in the form (9), and since all the factors $\alpha_{i\,i+1}^{(s_i)}$ are equal to 1, we conclude from (7) that $y_{s_i} = x_i$ for every i. Thus, $w = x_1 \cdots x_m$ is indeed a scattered subword of $v = y_1 \cdots y_n$. ∎

Now in order to prove that $I_k \subseteq J_k$, take any identity $u = v$ holding in the monoid $\mathscr{U}\mathscr{T}\mathscr{B}_{k+1}$. Then $u\varphi = v\varphi$ under an arbitrary morphism $\varphi : \Sigma^* \to \mathscr{U}\mathscr{T}\mathscr{B}_{k+1}$. In particular, $u\varphi_w = v\varphi_w$ for every word $w \in \Sigma^*$ of length $m \leq k$, whence $(u\varphi_w)_{1\,m+1} = (v\varphi_w)_{1\,m+1}$. By Lemma 3.3, the latter equality means that w is a scattered subword of u if and only if it is a scattered subword of v. Therefore the identity $u = v$ belongs to set J_k.

The converse inclusion $J_k \subseteq I_k$ amounts to saying that for every identity $u = v$ in J_k and for every morphism $\varphi : \Sigma^* \to \mathscr{U}\mathscr{T}\mathscr{B}_{k+1}$ one has

$$(u\varphi)_{\ell m} = (v\varphi)_{\ell m}$$

for each $\ell \leq m$. By symmetry, it suffices to verify that $(u\varphi)_{\ell m} = 1$ whenever $(v\varphi)_{\ell m} = 1$. Thus, consider $v = y_1 \cdots y_n$ for some $y_1, \ldots, y_n \in \Sigma$, and let $y_s\varphi = (\alpha_{ij}^{(s)})$. Then

$$(v\varphi)_{\ell m} = \sum_{\ell \leq j_1 \leq \cdots \leq j_{n-1} \leq m} \alpha_{\ell j_1}^{(1)} \alpha_{j_1 j_2}^{(2)} \cdots \alpha_{j_{n-1} m}^{(n)},$$

and since the sum on the right hand side is equal to 1, one of its summands is equal to 1. Let the product $\alpha_{\ell j_1}^{(1)} \alpha_{j_1 j_2}^{(2)} \cdots \alpha_{j_{n-1} m}^{(n)}$ be such a summand. The number p of "jumps" $j_r < j_{r+1}$ in the increasing sequence $j_0 = \ell \leq j_1 \leq \cdots \leq j_{n-1} \leq m = j_n$ does not exceed $m - \ell \leq (k+1) - 1 = k$. If we denote the position of the ith such jump by s_i and its increment $j_{r+1} - j_r$ by d_i, $i = 1, \ldots, p$, then the chosen product can be written as

$$\alpha_{\ell \ell}^{(1)} \cdots \alpha_{\ell \ell}^{(s_1-1)} \alpha_{\ell \, \ell+d_1}^{(s_1)} \alpha_{\ell+d_1 \, \ell+d_1}^{(s_1+1)} \cdots \alpha_{\ell+d_1 \, \ell+d_1}^{(s_2-1)} \alpha_{\ell+d_1 \, \ell+d_1+d_2}^{(s_2)} \cdots$$
$$\cdots \alpha_{m-d_p \, m-d_p}^{(s_{p-1}+1)} \cdots \alpha_{m-d_p \, m-d_p}^{(s_{p-1}-1)} \alpha_{m-d_p \, m}^{(s_p)} \alpha_{m \, m}^{(s_p+1)} \cdots \alpha_{m \, m}^{(n)},$$

whence we conclude that

$$\alpha_{\ell \, \ell+d_1}^{(s_1)} = \alpha_{\ell+d_1 \, \ell+d_1+d_2}^{(s_2)} = \cdots = \alpha_{m-d_p \, m}^{(s_p)} = 1. \tag{10}$$

Consider the scattered subword $y_{s_1} y_{s_2} \cdots y_{s_p}$ of the word v. Since the length of this subword is $\leq k$ and since the identity $u = v$ belongs to J_k, the word u also has $y_{s_1} y_{s_2} \cdots y_{s_p}$ as a scattered subword. Thus, we can write u as

$$u = u_0 y_{s_1} u_1 y_{s_2} u_2 \cdots u_{p-1} y_{s_p} u_p$$

for some suitable words $u_0, u_1, \ldots, u_p \in \Sigma^*$. Therefore the entry $(u\varphi)_{\ell m}$ of the matrix $u\varphi$ has in its expansion the summand

$$(u_0)_{\ell \ell} \alpha_{\ell \, \ell+d_1}^{(s_1)} (u_1)_{\ell+d_1 \, \ell+d_1} \alpha_{\ell+d_1 \, \ell+d_1+d_2}^{(s_2)} \cdots (u_{p-1})_{m-d_p \, m-d_p} \alpha_{m-d_p \, m}^{(s_p)} (u_p)_{m m}$$

which is equal to 1 in view of (10) and of the fact that all diagonal entries of matrices in $\mathscr{UITB}_{k+1}$ are 1. Hence $\left(u\varphi\right)_{\ell m} = 1$ as required. ∎

From Propositions 3.1 and 3.2, we immediately obtain

Theorem 3.4. *The monoid $\mathscr{UITB}_n$ is nonfinitely based for all $n \geq 5$. The monoids $\mathscr{UITB}_2$, $\mathscr{UITB}_3$, and $\mathscr{UITB}_4$ are finitely based, and the identity systems (4), (5), and respectively (6) may serve as their identity bases.*

Remark 3.1. It is fairly easy to see that for each n the variety var $\mathscr{UITB}_n$ contains neither B_2^1 nor non-trivial groups whence none of the monoids $\mathscr{UITB}_n$ are inherently nonfinitely based by [9, Theorem 2].

Remark 3.2. The approach used in our proof of Theorem 3.4 can be also applied in order to provide a partial answer to Problem 6.3 in [13], namely, to solve the finite basis problem for the monoids $\mathscr{C}_n$ of order preserving and decreasing transformations of a chain with n elements. (Recall that a transformation α of the chain $\langle X, \leq \rangle$ is called *order preserving* if $i \leq j$ implies $i.\alpha \leq j.\alpha$ for all $i, j \in X$ and *decreasing* if $i.\alpha \leq i$ for every $i \in X$.) It can be shown that the monoid $\mathscr{C}_n$ satisfies precisely the same identities as the monoid $\mathscr{UITB}_n$, and therefore, $\mathscr{C}_n$ is finitely based if and only if $n \leq 4$. The proof of this result will be published elsewhere.

Remark 3.3. Using Simon's celebrated characterization of piecewise testable languages, Straubing [10] has shown that the monoids $\mathscr{UITB}_n$ generate the pseudovariety **J** of all $\mathscr{J}$-trivial finite monoids. This nice result, however, does not seem to shed any extra light on the finite basis problem for the monoids $\mathscr{UITB}_n$ (compare with Remark 2.1).

Remark 3.4. Straubing and Thérien [11] have observed that the monoid $\mathscr{UITB}_n$ admits the following compatible partial order: $\left(\alpha_{ij}\right) \leq \left(\beta_{ij}\right)$ if for i, j either $\alpha_{ij} = \beta_{ij}$ or $\alpha_{ij} = 1$ while $\beta_{ij} = 0$. On the other hand, from their elegant proof of Simon's theorem [11, p.397] one can extract the following

Proposition 3.5. *Let M be a monoid with a compatible partial order $\leq$ under which 1 is the greatest element. If $k + 1$ is the length of the longest chain in $\langle M, \leq \rangle$, then M satisfies every identity in the set J_k.*

It is easy to calculate that the length of the longest chain in the monoid $\mathscr{UITB}_n$ with the above ordering is $\frac{n(n-1)}{2} + 1$ so the application of Proposition 3.5 yields that $\mathscr{UITB}_n$ satisfies all identities in $J_{\frac{n(n-1)}{2}}$. Our Proposition 3.2 shows that in fact $\mathscr{UITB}_n$ satisfies all identities in J_{n-1} and this is the best possible result.

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