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Some polynomials on trice

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A polynomial function is expressed by the polynomials. Infinite polynomials can be made combining variables on one algebra. On any lattice, the polynomial functions of two variables are only four. However, polynomial functions have complex structure on trice. We tried one of the methods that create an example of trice easily. But, actual composition was complex more than the expectation and difficult. In this note we present some results on polynomial functions of the typical trice, which composed by 3 points.

1 Introduction

Definition 1 Let T be a set. And let $*_1$, $*_2$ and $*_3$ be three binary operations on T. If $(T, *_1)$, $(T, *_2)$ and $(T, *_3)$ are semilattices respectively, the quartet $(T, *_1, *_2, *_3)$ is called a **triple-semilattice**. We denote each order on T by

$$a \leq_i b \iff a *_i b = b, \tag{1}$$

respectively. Let S_3 be the symmetric group on $\{1, 2, 3\}$. The $(T, *_1, *_2, *_3)$ has the **roundabout-absorption law** if it satisfies the following 6 identities:

$$(((a *_{\sigma(1)} b) *_{\sigma(2)} b) *_{\sigma(3)} b) = b \tag{2}$$

for all $a, b \in T$ and for all $\sigma \in S_3$. The $(T, *_1, *_2, *_3)$ which satisfies the roundabout-absorption law is said to be a **trice**.

Definition 2 The set $P^{(2)}$ of **binary trice polynomials** is the smallest set satisfying (i) and (ii):

(i) variables $x, y \in P^{(2)}$.
(ii) If $p, q \in P^{(2)}$, then $p *_1 q$, $p *_2 q$, $p *_3 q \in P^{(2)}$.

Definition 3 Let T be a trice. A binary trice polynomial p defines **a binary polynomial function** on a trice T by following rules $(a, b, c, d \in T)$:

(i) If $p = x$, then $p(a, b) = a$ and if $p = y$, then $p(a, b) = b$.

(ii) If $p(a, b) = c$, $q(a, b) = d$, and $p *_1 q = r$, $p *_2 q = s$, $p *_3 q = t$, then $r(a, b) = c *_1 d$, $s(a, b) = c *_2 d$ and $t(a, b) = c *_3 d$.

The family of all binary trice polynomial functions on T is a trice. We denote it by $P^{(2)}(T)$.

2 Polynomial functions of two points trice

There are two triple-semilattices composed by two points. See Fig. 1. Suppose that arrowhead is larger than the other end. The right one (2) is the only trice. Two operations on this trice are the same. One operation is redundant. We can regard the trice as a lattice essentially. If T is this trice (2), then $P^{(2)}(T)$ has four elements.

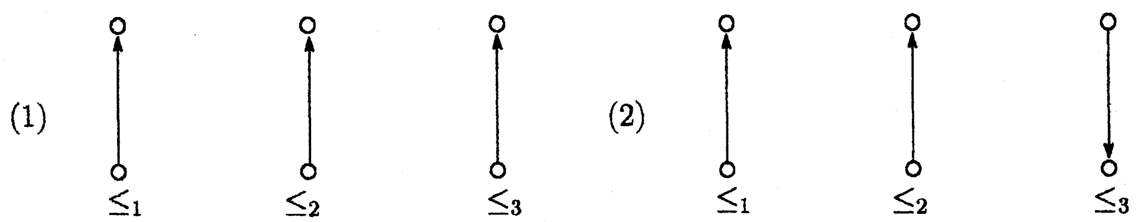

Figure 1: two points triple-semilattices

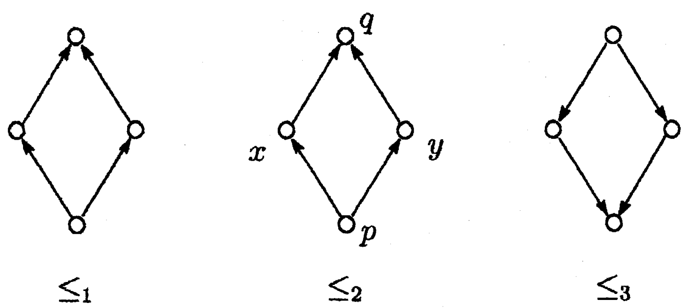

Figure 2: Redundant case

(See Fig. 2. Then, x and y are generators. That is $x(a, b) = a$, $y(a, b) = b$. And $p(x, y) = x *_3 y$, $q(x, y) = x *_1 y = x *_2 y$).

Three points triple-semilattice of III type

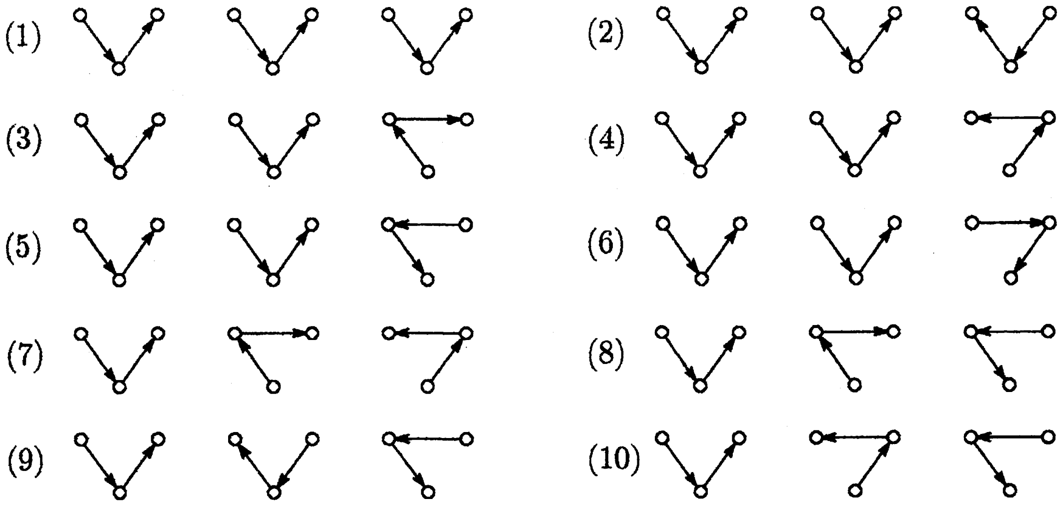

Figure 3: III type

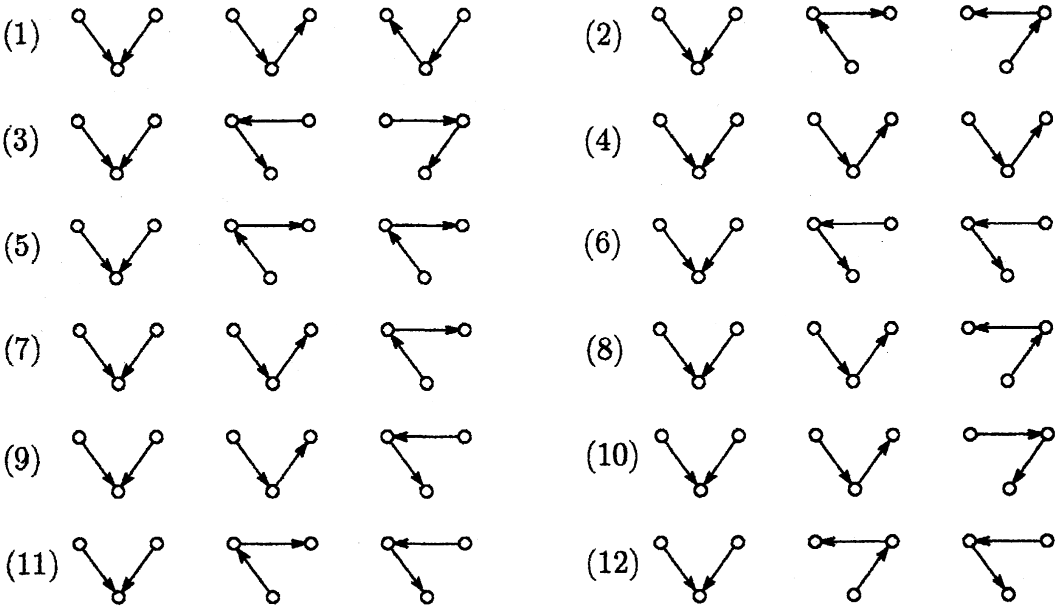

Figure 4: VII type

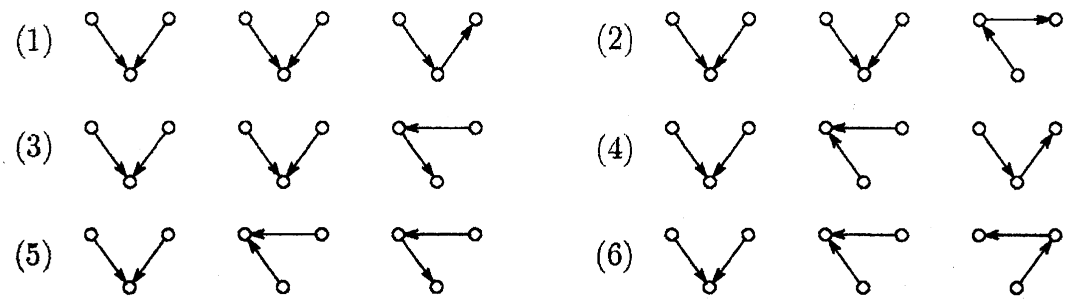

Figure 5: VVI type

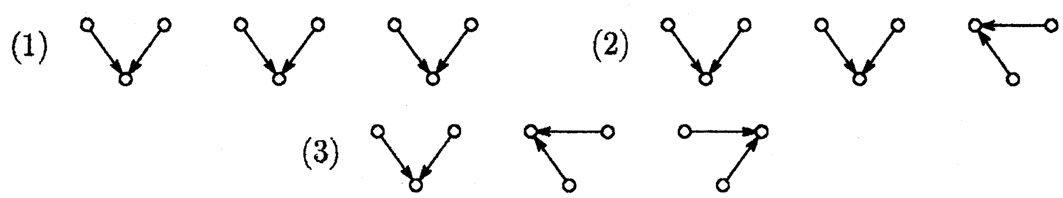

Figure 6: VVV type

3 polynomial functions of three points trice

The number of triple-semilattices, which composed by 3 points, is 31. The lists of them are Fig. 3 - Fig. 6. The symbols "I" and "V" are description of semilattice. Then, (2), (8), (9), (10) in III, (1), (2), (8) in VII, (4) in VVI and (3) in VVV are trices.

3.1 Redundant case

When two operations on a trice is the same, the trice becomes a lattice essentially. On any lattice, the polynomial functions of two variables are only four. See [1] or [2]. Hence, if T is (2) in III, then $P^{(2)}(T)$ is the trice of Fig. 2.

3.2 Case of linearly ordered set and another operation

Let L be a linearly ordered set, then $(L, \vee, \wedge)$ is a lattice. Denote $\vee$ by $*_1$ and $\wedge$ by $*_2$. If $*_3$ is a semilattice and averaging (i.e., $a \wedge b \leq a *_3 b \leq a \vee b$), then the $(L, *_1, *_2, *_3)$ is a trice. We say $*_3$ is a "mode-type operator" in [3].
Conversely, let $(L, *_1, *_2, *_3)$ be a trice and $(L, *_1, *_2)$ be a linearly ordered set. Then $*_3$ is a averaging for $*_1$ and $*_2$.

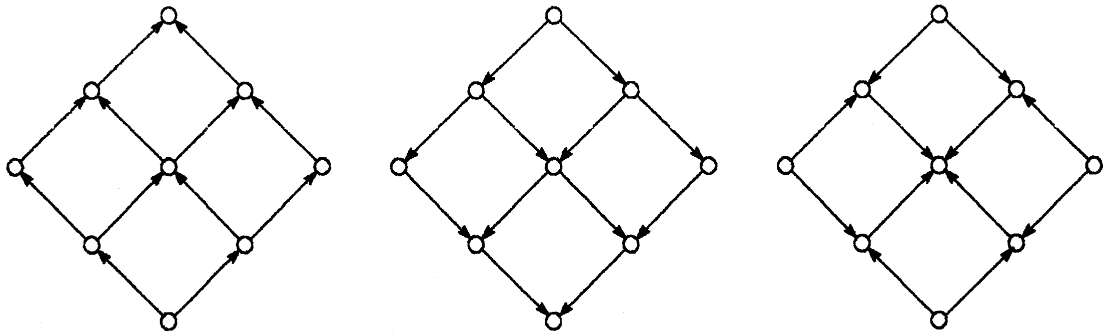

Figure 7: Linear + one case

Proposition 1 If $(L, *_1, *_2, *_3)$ be a trice and $(L, *_1, *_2)$ be a linearly ordered set, the $P^{(2)}(L)$ is the trice of Fig. 7.

The (8) in III, (9) in III and (1) in VII are this case, therefore $P^{(2)}(T)$ is the trice of Fig. 7.

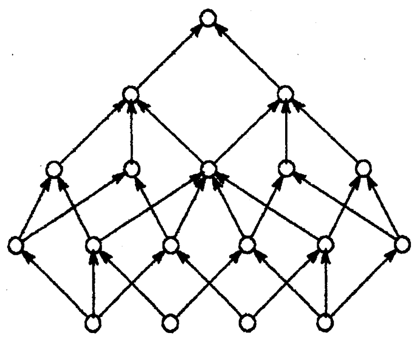

Figure 8: semilattice 12564

3.3 Case of 18 elements polynomial functions

When T is (10) in III, the trice $P^{(2)}(T)$ has 18 elements. This $P^{(2)}(T)$ is free distributive trice with two generators in [4]. The semilattice of $*_1$, $*_2$ and $*_3$ are the same type. (See Fig. 8).
When T is (2) in VII, the trice $P^{(2)}(T)$ has 18 elements, too. But it is different from free distributive trice with two generators. Only one of its semilattice is the same as

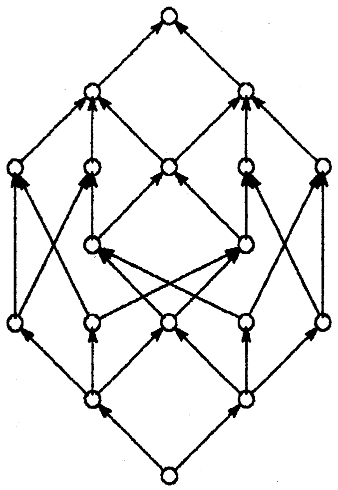

Figure 9: semilattce 1252521

Fig. 8. The other two are Fig. 9.
When T is (8) in VII, the trice $P^{(2)}(T)$ has 18 elements, too. But it is different from free distributive trice with two generators, neither. Two of its semilattice is the same as Fig. 8. The other is Fig. 9.

3.4 Case of 36 elements polynomial functions

When T is (4) in VVI, the trice $P^{(2)}(T)$ has 36 elements. After it was proved that 36 elements were necessary, we ascertained that it was actually made and it was sufficient. It is omitted writing it here because it is complicated and it isn't so impressive.

3.5 Polynomial functions of triangular situation

Let $(T, *_1, *_2, *_3)$ be a triple-semilattice and $a, b, c \in T(a \neq b \neq c \neq a)$. We say that an ordered triplex (a, b, c) is a **triangular situation** if (a, b, c) have the following properties:

$$a *_3 b = c \text{ and } a *_2 c = b \text{ and } b *_1 c = a. \tag{3}$$

To simplify explanation, we allow saying $\{a, b, c\}$ is triangular situation.
The (3) in VVV is the simplest trice, which has triangular situation. Therefore, it is very important and ineresting.

Proposition 2 If T is the trice (3) in VVV, then the $P^{(2)}(L)$ has 729 $(= 3^6)$ elements.

References

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