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Vanishing Theorems in Hyperasymptotic Analysis and Applications to Inhomogeneous Linear Differential Equations

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1 Introduction

The vanishing theorem of non-commutative case in asymptotic analysis is established by Sibuya([12], [13]) in 1970's to solve the so-called R-H-B problem. That was stated in terms of vector bundles, of which the origin is in a work on matricial functions of Birkhoff. Malgrange [6] translated Sibuya's theorem in terms of sheaves of germs of functions asymptotically developable on the S^1, the set of directions to a point in **C**. Malgrange proved also the vanishing theorem of commutative case in asymptotic analysis and, Malgrange and Deligne showed that it was usefull to study the structure of formal solutions to inhomogenous linear differential equations by using solutions asymptotic to the series 0 of the associated homogeneous linear differential equations. These are successively extended to the Gevrey asymptotic case in one variable(Ramis [11],..., [7], ...), to the general case of asymptotics in several variables (Majima [1], [2], [3]), the Gevrey case in several variables(Haraoka [?]), and some generalizations for these results (Mozo [8]).

These are also extended to the case of hyperasymptotics. The first attempt was done in [5](see also [4]).

In this paper, we give vanishing theorems in hyperasymptotic analysis of level 1 and 2.

2 Vanishing Theorems in Hyperasymptotic Analysis in the Commutative Case

In the following, we work at the infinity and for a real positive number R, real numbers a and b, we denote by $\mathcal{S}(R,\, a,\, b)$ the open sector at the infinity

$$\mathcal{S}(R,\, a,\, b) = \{\, z \,:\, |z| > R,\, a < \arg z < b\,\}. \tag{1}$$

Let $\{\mathcal{S}(R, a_\ell, b_\ell) \,|\, \ell = 1, \cdots, L\}$ be an open sectorial covering of the annulus

$$\mathcal{D}(R, \infty) = \{\, z \,|\, +\infty > |z| > R\,\}. \tag{2}$$

We say that $\{\mathcal{S}(R, a_\ell, b_\ell) \,|\, \ell = 1, \cdots, L\}$ is a good covering when the following condition is satisfied:

$$a_{L+1} = a_1,\ a_\ell < b_{\ell-1} < a_{\ell+1} < b_\ell, \qquad b_\ell - a_\ell < \pi, \qquad \ell = 1, \cdots, L. \tag{3}$$

We set, for fixed a_ℓ, b_ℓ, $\ell = 1, \cdots, L$,

$$\mathcal{S}_{\ell-1,\ell}(R) = \mathcal{S}(R, a_{\ell-1}, b_{\ell-1}) \cap \mathcal{S}(R, a_\ell, b_\ell) = \mathcal{S}(R, a_\ell, b_{\ell-1}), \tag{4}$$

and take

$$\tau_\ell = \frac{a_\ell + b_{\ell-1}}{2}. \tag{5}$$

These will be the directions of the Stokes lines in the next theorem, and these Stokes lines will be denoted by

$$\gamma_\ell = \left\{ te^{i\tau_\ell} \,\middle|\, t \in [0, \infty) \right\}, \qquad \gamma'_\ell = \left\{ tR'e^{i\tau_\ell} \,\middle|\, t \in [1, \infty) \right\}. \tag{6}$$

We will call $\{\lambda_k \,|\, k = 1, \cdots, K\}$ an acceptable set of exponentials for our covering when for each $1 \le k \le K$ there exists an ℓ such that $\arg(-\lambda_k) = -\tau_\ell$, that is, $\lambda_k z < 0$ when $\arg z = \tau_\ell$. For each ℓ we define

$$\mathcal{K}_\ell = \left\{ k \in \{1, \cdots, K\} \,\middle|\, \arg(-\lambda_k) = -\tau_\ell \right\}. \tag{7}$$

We will use the notation

$$\lambda_{jk} = \lambda_j - \lambda_k, \qquad \mu_{jk} = \mu_j - \mu_k. \tag{8}$$

Theorem 1 *Let* $\left\{\mathcal{S}(R, a_\ell, b_\ell) \,\middle|\, \ell = 1, \cdots, L\right\}$ *be a good open sectorial covering of* $\mathcal{D}(R, \infty)$ *and let* $\{\lambda_k \,|\, k = 1, \cdots, K\}$ *be an acceptable set of exponentials for this covering. For* $\ell = 1, \cdots, L$, *let*

$$U_{\ell-1,\ell}(z) = \sum_{k \in \mathcal{K}_\ell} \delta_k U^{(k)}_{\ell-1,\ell}(z) \tag{9}$$

be a finite sum of functions defined in $S_{\ell-1,\ell}(R)$ that are in that sector asymptotically developable to the formal power-series

$$U^{(k)}_{\ell-1,\ell}(z) \sim e^{\lambda_k z} \sum_{s=0}^{\infty} u_{sk} z^{\mu_k - s}, \tag{10}$$

where μ_k are complex constants. In (9) δ_k are constants that are either 1 or 0.

Then, there exist a positive number $R''(\geq R)$, a formal power-series $\widehat{V}(z) = \sum_{r=0}^{\infty} T_r z^{-r}$ and functions V_ℓ defined in $S_\ell(R'')$, $\ell = 1, \cdots, L$, such that

(i) the relation

$$U_{\ell-1,\ell}(z) = V_\ell(z) - V_{\ell-1}(z) \tag{11}$$

holds for $z \in S_{\ell-1,\ell}(R'')$.

(ii) V_ℓ is aymptotically developable to the formal power-series $\widehat{V}(z)$ in $S_\ell(R'')$, and if we write

$$V_\ell(z) = \sum_{r=1}^{M-1} T_r z^{-r} + \tilde{R}^{(0)}_\ell(z, M), \tag{12}$$

then

$$\tilde{R}^{(0)}_\ell(z, M) = e^{-\alpha_0 |z|} \mathcal{O}\left(|z|^{\tilde{\mu}_0 + 1/2}\right), \tag{13}$$

as $|z| \to \infty$ in the sector $\tau_\ell \leq \arg z \leq \tau_{\ell+1}$, where we have taken the optimum number of terms

$$M = \alpha_0 |z| + \mathcal{O}(1), \tag{14}$$

where

$$\alpha_0 = \min\left\{ |\lambda_k| \,\middle|\, k = 1 \cdots K, \delta_k \neq 0 \right\}, \tag{15}$$

$$\tilde{\mu}_0 = \max\left\{ \Re\mu_k \,\middle|\, k = 1 \cdots K \right\}. \tag{16}$$

(iii) As $r \to \infty$

$$T_r \sim \frac{-1}{2\pi i} \sum_{k=1}^{K} \sum_{s=0}^{\infty} \delta_k u_{sk} \Gamma(r + \mu_k - s) \left(-\lambda_k\right)^{s - \mu_k - r}, \tag{17}$$

Remark 1: The lines $\arg z = \tau_\ell, \tau_{\ell+1}$ are Stokes lines for the function $V_\ell(z)$.
Remark 2: The constant α_0 defined in (15) is the distance from the origin to the nearest active λ_k in the complex plane. By changing the values of δ_k the value of α_0 might change.

The next theorem is the hyperasymptotic level 1 version. For this theorem we need some extra information in the asymptotic expansions of the functions $U^{(k)}_{\ell-1,\ell}(z)$.

Theorem 2 *In addition to the assumption of Theorem 1, we moreover assume that there exist constants $\tilde{\alpha}_k$ and ν_k, $k = 1, \cdots, K$, such that*

$$U^{(k)}_{\ell-1,\ell}(z) = e^{\lambda_k z} \sum_{s=0}^{N_k-1} u_{sk} z^{\mu_k - s} \quad + \quad R^{(0)}_k(z, N_k), \tag{18}$$

where for all z 'near' γ'_ℓ and large N_k we have

$$R^{(0)}_k(z, N_k) = e^{\lambda_k z} z^{\mu_k - N_k + 1} \frac{\Gamma(N_k + \nu_k)}{(\tilde{\alpha}_k)^{N_k}} \mathcal{O}(1). \tag{19}$$

Define

$$\alpha_1 = \min\left\{\tilde{\alpha}_k + |\lambda_k| \,\middle|\, k = 1 \cdots K, \delta_k \neq 0\right\}, \tag{20}$$

$$\tilde{\mu}_1 = \max\left\{\nu_k + \Re\mu_k \,\middle|\, k = 1 \cdots K\right\}. \tag{21}$$

Then

$$T_r \sim \frac{-1}{2\pi i} \sum_{k=1}^{K} \sum_{s=0}^{N_k-1} \delta_k u_{sk} \Gamma(r + \mu_k - s) \left(-\lambda_k\right)^{s - \mu_k - r} \quad + \hat{R}^{(0)}(r, N_p), \tag{22}$$

where, when we take the optimal choice

$$N_k = \frac{\max(\alpha_1 - |\lambda_k|, 0)}{\alpha_1} r + \mathcal{O}(1), \tag{23}$$

we have

$$\hat{R}^{(0)}(r, N_p) = \frac{\Gamma(r)}{(\alpha_1)^r} \mathcal{O}\left(r^{\tilde{\mu}_1 + 1/2}\right), \tag{24}$$

as $r \to \infty$. For the remainder in (12) we have

$$\tilde{R}^{(0)}_\ell(z, M) = -\frac{z^{1-M}}{2\pi i} \sum_{k=1}^{K} \sum_{s=0}^{N_k-1} \delta_k u_{sk} F^{(1)}\left(z; \frac{M + \mu_k - s}{\lambda_k}\right) \quad + \quad \tilde{R}^{(1)}_\ell(z, M, N_p), \tag{25}$$

where, when we take the optimal choice

$$M = \alpha_1 |z| + \mathcal{O}(1), \qquad N_k = \max(\alpha_1 - |\lambda_k|, 0)|z| + \mathcal{O}(1), \tag{26}$$

we have

$$\tilde{R}^{(1)}_\ell(z, M, N_p) = e^{-\alpha_1 |z|} \mathcal{O}\left(|z|^{\tilde{\mu}_1 + 1}\right), \tag{27}$$

as $|z| \to \infty$ in the sector $\tau_\ell \leq \mathrm{ph} z \leq \tau_{\ell+1}$.

In the definition we shall use the notation

$$\int_{\lambda}^{[\eta]} = \int_{\lambda}^{\infty e^{i\eta}}, \qquad \eta \in \mathbf{R}.$$

Let l be a nonnegative integer, $\Re M_j > 1$, $\sigma_j \in \mathbf{C}$, $\sigma_j \neq 0$, $j = 0, \cdots, l$. Then

$$\begin{aligned} F^{(0)}(z) &= 1, \\ F^{(1)}\left(z; \begin{matrix} M_0 \\ \sigma_0 \end{matrix}\right) &= \int_0^{[\pi-\theta_0]} \frac{e^{\sigma_0 t_0} t_0^{M_0-1}}{z - t_0}\, dt_0, \\ F^{(l+1)}\left(z; \begin{matrix} M_0, \cdots, M_l \\ \sigma_0, \cdots, \sigma_l \end{matrix}\right) &= \int_0^{[\pi-\theta_0]} \cdots \int_0^{[\pi-\theta_l]} \frac{e^{\sigma_0 t_0 + \cdots + \sigma_l t_l} t_0^{M_0-1} \cdots t_l^{M_l-1}}{(z-t_0)(t_0-t_1)\cdots(t_{l-1}-t_l)}\, dt_l \cdots dt_0, \end{aligned}$$

where $\theta_j = \arg\sigma_j$, $j = 0, 1, \cdots, l$. In the case $\arg\sigma_j = \arg\sigma_{j+1} \pmod{2\pi}$ we have to make the choice between the t_j-contour being on the 'left' or 'right' of the t_{j+1}-contour. We make the choice via the definition

$$F^{(l+1)}\left(z; \begin{matrix} M_0, \cdots, M_l \\ \sigma_0, \cdots, \sigma_l \end{matrix}\right) = \lim_{\varepsilon \downarrow 0} F^{(l+1)}\left(z; \begin{matrix} M_0, & M_1, & \cdots, & M_{l-1}, & M_l \\ \sigma_0 e^{-l\varepsilon i}, & \sigma_1 e^{-(l-1)\varepsilon i}, & \cdots, & \sigma_{l-1} e^{-\varepsilon i}, & \sigma_l \end{matrix}\right),$$

which means that once again we prefer 'right' over 'left'.

The multiple integrals converge when $-\pi - \theta_0 < \arg z < \pi - \theta_0$.

The next theorem is the hyperasymptotic level 2 version. For this theorem we need some extra information on the re-expansions of the functions $U_{\ell-1,\ell}^{(k)}(z)$.

Theorem 3 *In addition to the assumption of Theorem 1, we moreover assume that there exist constants $\tilde{\alpha}_{kj}$ and ν_{kj}, $k, j = 1, \cdots, K$, $j \neq k$ such that*

$$R_k^{(0)}(z, N_k) = \sum_{j \neq k} e^{\lambda_k z} z^{1-N_k+\mu_k} \frac{K_{jk}}{2\pi i} \sum_{s=0}^{\tilde{N}_{kj}-1} u_{skj} F^{(1)}\left(z; \begin{matrix} N_k + \mu_{jk} - s \\ \lambda_{jk} \end{matrix}\right) + R_{kj}^{(1)}(z, N_k, \tilde{N}_{kj}), \tag{28}$$

where for all z 'near' γ'_ℓ and large $N_k - \tilde{N}_{kj}$ and large $\tilde{N}_{kj}$ we have

$$R_{kj}^{(1)}(z, N_k, \tilde{N}_{kj}) = e^{\lambda_k z} z^{\mu_k - N_k + 2} \frac{\Gamma(N_k - \tilde{N}_{kj} + \mu_{jk})\Gamma(\tilde{N}_{kj} + \nu_{kj})}{|\lambda_{kj}|^{N_k - \tilde{N}_{kj}} (\tilde{\alpha}_{kj})^{\tilde{N}_{kj}}} \mathcal{O}(1). \tag{29}$$

Define

$$\begin{aligned} \alpha_2 &= \min\{\tilde{\alpha}_{kj} + |\lambda_{kj}| + |\lambda_k| \mid k, j = 1, \cdots, K,\ j \neq k,\ \delta_k \neq 0,\ K_{jk} \neq 0\}, & (30) \\ \tilde{\mu}_2 &= \max\{\nu_{kj} + \Re\mu_k \mid k, j = 1, \cdots, K,\ j \neq k\}. & (31) \end{aligned}$$

Then

$$T_r = \frac{-1}{2\pi i}\sum_{k=1}^{K}\left\{\sum_{s=0}^{N_k-1}\delta_k u_{sk}\Gamma(r+\mu_k-s)\,(-\lambda_k)^{s-\mu_k-r}\right. \tag{32}$$

$$\left.+\sum_{j\neq k}\frac{K_{jk}}{2\pi i}\sum_{s=0}^{\tilde{N}_{kj}-1}u_{skj}F^{(2)}\left(0;\begin{matrix}r+\mu_k-N_k+2, & N_k+\mu_{jk}-s\\ \lambda_k, & \lambda_{jk}\end{matrix}\right)\right\} \tag{33}$$

$$+\hat{R}^{(1)}(r,N_p,\tilde{N}_{pq}), \tag{34}$$

where, when we take the optimal choice

$$N_k = \frac{\max(\alpha_2-|\lambda_k|,0)}{\alpha_2}r+\mathcal{O}(1), \qquad \tilde{N}_{kj} = \frac{\max(\alpha_2-|\lambda_k|-|\lambda_{kj}|,0)}{\alpha_2}r+\mathcal{O}(1), \tag{35}$$

we have

$$\hat{R}^{(1)}(r,N_p,\tilde{N}_{pq}) = \frac{\Gamma(r)}{(\alpha_2)^r}\mathcal{O}\left(r^{\tilde{\mu}_2+1}\right), \tag{36}$$

as $r\to\infty$*. For the remainder in (12) we have*

$$\tilde{R}_\ell^{(0)}(z,M) = -\frac{z^{1-M}}{2\pi i}\sum_{k=1}^{K}\left\{\sum_{s=0}^{N_k-1}\delta_k u_{sk}F^{(1)}\left(z;\begin{matrix}M+\mu_k-s\\ \lambda_k\end{matrix}\right)\right. \tag{37}$$

$$\left.+\sum_{j\neq k}\frac{K_{jk}}{2\pi i}\sum_{s=0}^{\tilde{N}_{kj}-1}u_{skj}F^{(2)}\left(z;\begin{matrix}M+\mu_k-N_k+1, & N_k+\mu_{jk}-s\\ \lambda_k, & \lambda_{jk}\end{matrix}\right)\right\} \tag{38}$$

$$+\tilde{R}_\ell^{(2)}(z,M,N_p,\tilde{N}_{pq}), \tag{39}$$

where, when we take the optimal choice

$$M=\alpha_2|z|+\mathcal{O}(1),\quad N_k=\max(\alpha_2-|\lambda_k|,0)|z|+\mathcal{O}(1),\quad \tilde{N}_{kj}=\max(\alpha_2-|\lambda_k|-|\lambda_{kj}|,0)|z|+\mathcal{O}(1), \tag{40}$$

we have

$$\tilde{R}_\ell^{(2)}(z,M,N_p,\tilde{N}_{pq}) = e^{-\alpha_2|z|}\mathcal{O}\left(z^{\tilde{\mu}_2+3/2}\right), \tag{41}$$

as $|z|\to\infty$ *in the sector that is bounded (from the right) by the Stokes line* $\arg z=\tau_\ell$ *and (on the left) by the Stokes line* $\arg z=\tau_{\ell+1}$ *or one of the other Stokes lines* $\arg(\lambda_{kj}z)=0$*, such that this sector doesn't contain any of these Stokes lines.*

The proof of Theorem 1 is given in [MHO]([5]) except for estimate (13). We can prove these theorems by using the integral representation:

$$V_\ell(z) = \sum_{j=1}^{L}\frac{1}{2\pi i}\int_{\gamma_j'}\frac{U_{j-1,j}(\zeta)}{\zeta-z}\,d\zeta = \sum_{j=1}^{L}\sum_{k\in K_j}\frac{\delta_k}{2\pi i}\int_{\gamma_j'}\frac{U_{j-1,j}^{(k)}(\zeta)}{\zeta-z}\,d\zeta, \tag{42}$$

where $z \in \mathcal{S}(R'', \tau_\ell, \tau_{\ell+1})$. Hence,

$$T_r = \sum_{j=1}^{L} \sum_{k \in \mathcal{K}_j} \frac{-\delta_k}{2\pi i} \int_{\gamma'_j} U^{(k)}_{j-1,j}(\zeta)\zeta^{r-1}\, d\zeta \tag{43}$$

and

$$\tilde{R}^{(0)}_\ell(z, M) = \sum_{j=1}^{L} \sum_{k \in \mathcal{K}_j} \delta_k \frac{z^{1-M}}{2\pi i} \int_{\gamma'_j} \frac{U^{(k)}_{j-1,j}(\zeta)\zeta^{M-1}}{\zeta - z}\, d\zeta, \tag{44}$$

where, again, $z \in \mathcal{S}(R'', \tau_\ell, \tau_{\ell+1})$.

3 Application: inhomogeneous linear ordinary differential equations

Let

$$Pw := \frac{d^K w}{dz^K} + f_{K-1}(z)\frac{d^{K-1} w}{dz^{K-1}} + \cdots + f_0(z)w = 0, \tag{45}$$

be a linear differential equation with a singularity of rank one at infinity, and let

$$\hat{u}_k(z) = e^{\lambda_k z} z^{\mu_k} \sum_{s=0}^{\infty} u_{sk} z^{-s}, \qquad k = 1, \cdots, K, \tag{46}$$

be all the formal solutions. We assume that all λ_k are nonzero and $\lambda_j \neq \lambda_k$, if $j \neq k$. With these exponentials we can construct our covering.

The complete hyperasymptotic expansions of solutions of (45) are given in [9], and with the theory and proofs in that paper it can be checked that all assumptions of Theorems 2 and 3 are satisfied when we take $U^{(k)}_{\ell-1,\ell}(z)$ as follows.

For the moment we fix $k \in \{1, \cdots, K\}$ take ℓ such that $k \in \mathcal{K}_\ell$ and let $U^{(k)}_{\ell-1,\ell}(z)$ be the solution of (45) with asymptotic behaviour $\hat{u}_k(z)$ as its complete asymptotic expansion in a sector that either contains $\arg z = \tau_\ell$, or has this line as its boundary on the 'right-hand side'. In other words, $U^{(k)}_{\ell-1,\ell}(z)$ is supposed to be the Borel-Laplace transform of $\hat{u}_k(z)$.

Define

$$W^{(+)}_k(z) = V_\ell(z) = \frac{1}{2\pi i} \int_{\gamma'_\ell} \frac{U^{(k)}_{\ell-1,\ell}(\zeta)}{\zeta - z}\, d\zeta, \qquad z \in \mathcal{S}(R'' a_\ell, b_{\ell-1} + 2\pi), \tag{47}$$

$$W^{(-)}_k(z) = V_{\ell-1}(z) = \frac{1}{2\pi i} \int_{\gamma'_\ell} \frac{U^{(k)}_{\ell-1,\ell}(\zeta)}{\zeta - z}\, d\zeta, \qquad z \in \mathcal{S}(R'' a_\ell - 2\pi, b_{\ell-1}). \tag{48}$$

Compare (42). Thus we have taken all δ_k zero except one. In (47) we integrate to the 'right' of z and in (48) we integrate to the 'left' of z. Note that we have the relations

$$W_k^{(+)}(z) = W_k^{(-)}(ze^{-2\pi i}) \quad \text{and} \quad W_k^{(+)}(z) - W_k^{(-)}(z) = U_{\ell-1,\ell}^{(k)}(z). \tag{49}$$

Compare (11). Hence,

$$PW_k^{(+)}(z) = PW_k^{(+)}(ze^{2\pi i}). \tag{50}$$

Thus

$$p_k(z) = PW_k^{(+)}(z) \tag{51}$$

is analytic at infinity.

Let $\hat{W}_k^{(+)}(z)$ be the asymptotic expansion of function $W_k^{(+)}(z)$ for $k = 1, \ldots, K$. Then, $\hat{W}_k^{(+)}(z)$ is a formal solution of the inhomogeneous equations $P\hat{W} = p_k$ for $k = 1, \ldots, K$. If we consider P as an operator on $\hat{\mathcal{O}}/\mathcal{O}$, we see that $< \hat{W}_1^{(+)}(z), \cdots, \hat{W}_K^{(+)}(z) >$ mod.$\mathcal{O}$ form a basis of $\mathrm{Ker}(P; \hat{\mathcal{O}}/\mathcal{O}) \simeq H^1(S^1, \mathcal{K}er(P : \mathcal{A}_0))$(see, for example [4]).

Namely, for any analytic function $p(z)$ at infinity and a formal solution of $Pw = p$, there exist constants C_k and an analytic function $h(z)$ at infinity, such that

$$\hat{W}(z) = C_1 W_1^{(+)}(z) + \cdots + C_K W_K^{(+)}(z) + h(z). \tag{52}$$

Put

$$\hat{W}(z) = \sum_{r=0}^{\infty} t_r z^{-r}. \tag{53}$$

According to Theorem 1 we have

$$t_r \sim \frac{-1}{2\pi i} \sum_{k=1}^{K} C_k \sum_{s=0}^{\infty} u_{sk} \Gamma(r + \mu_k - s) \left(-\lambda_k\right)^{s-\mu_k-r}, \tag{54}$$

as $r \to \infty$, with re-expansions in Theorems 2 and 3. The constants C_k can be computed via this relation, or the higher level versions of this relation. For more details on the computation of the connection coefficients C_k see [10].

At the moment that we have these connection coefficients, for an actual solution $w(z)$ of $Pw = p$, we can use them in the approximation

$$w(z) \sim \sum_{r=0}^{M-1} t_r z^{-r} - \frac{z^{1-M}}{2\pi i} \sum_{k=1}^{K} C_k \sum_{s=0}^{N_k-1} u_{sk} F^{(1)} \left(z; \begin{matrix} M + \mu_k - s \\ \lambda_k \end{matrix}\right), \tag{55}$$

or higher order level versions of this approximation.

NOTE: that in (55) only the Poincaré part depends on $p(z)$. The re-expansions are the same for any $p(z)$, and once we have computed these re-expansions for one function $p(z)$, we can use them for any other function!

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