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A Remark on k-summability of divergent solution of a non-Kowalevski type equation with Cauchy data of entire functions

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1 Introduction

Throughout this paper, we shall discuss on the k-summability of formal solutions of the Cauchy problem for a partial differential equation of non-Kowalevski type

$$(\mathbf{CP}) \qquad \begin{cases} \partial_t^p u(t,x) = \partial_x^q u(t,x), \\ u(0,x) = \varphi(x), \quad \partial_t^j u(0,x) = 0 \ (1 \le j \le p-1), \end{cases}$$

where $(t,x) \in \mathbb{C}^2$, p and q are natural numbers with $1 \le p < q$. We first assume that the Cauchy data $\varphi(x)$ is holomorphic in a neighbourhood of the origin. This Cauchy problem (**CP**) has a unique formal solution

$$(1.1) \qquad \hat{u}(t,x) = \sum_{n=0}^{\infty} \varphi^{(qn)}(x) \frac{t^{pn}}{(pn)!} \underset{\text{put}}{=} \sum_{n=0}^{\infty} u_n(x) t^n.$$

From the local analyticity of the Cauchy data, this series is formal power series of Gevrey order $(q-p)/p$ with respect to t variable, which means that the following Gevrey estimates for the coefficients hold for some positive constants r, C and K.

$$(1.2) \qquad \max_{|x| \le r} |u_n(x)| \le CK^n \Gamma\left(1 + (q-p)n/p\right), \quad n = 0, 1, 2, \ldots.$$

In this case, we write it by

$$(1.3) \qquad \hat{u}(t,x) \in \mathcal{O}_x[[t]]_{(q-p)/p}.$$

We put $\sigma(0) = (q-p)/p$ and $k(0) = 1/\sigma(0)$. The results of $k(0)$-summability and $k(0)$-sum are given by M. Miyake [Miy] and K. Ichinobe [Ich 1,2] (cf. [LMS] where they considered the case of $(p,q) = (1,2)$). The definitions of these terminologies will be given in the next section. In [Miy], Miyake gave the characterization of $k(0)$-summability of the formal solution (1.1) in terms of the property of analytic continuation and its exponential growth condition for the Cauchy data. Under these conditions for the Cauchy data, Ichinobe obtained an explicit integral representation of the $k(0)$-sum by using a kernel function (cf. [Ich 1,2]).

Therefore our interest in this paper is that we discuss the same problems when the Cauchy data $\varphi(x)$ is an entire function of finite exponential growth order. We shall show that under some conditions the formal solution (1.1) is k-summable in a direction and the integral representation of k-sum by using a kernel function is obtained.

2 Preparations

2.1 Definitions

In order to discuss the k-summability and k-sum, we give some definitions.

For $d \in \mathbb{R}$, $\beta > 0$ and $\rho\,(0 < \rho \leq \infty)$, we define a sector $S(d, \beta, \rho)$ by

$$S(d,\beta,\rho) := \left\{ t \in \mathbb{C}; |\arg t - d| < \frac{\beta}{2}, 0 < |t| < \rho \right\}, \tag{2.1}$$

and d, β and ρ are called the direction, the opening angle and the radius of this sector, respectively.

Let $\sigma > 0$ and $k = 1/\sigma$. Let $\hat{u}(t,x) \in \mathcal{O}_x[[t]]_\sigma$ and $u(t,x)$ be an analytic function on $S(d,\beta,\rho) \times B(r)$, where $B(r) := \{x \in \mathbb{C}; |x| \leq r\}$. Then we say that $u(t,x)$ has the Gevrey asymptotic expansion $\hat{u}(t,x)$ of order σ in $S(d,\beta,\rho)$, if for any relatively compact subsector S' of $S(d,\beta,\rho)$, there exists a positive small constant r_1 such that for any non negative integer N, we have

$$\max_{|x|\leq r_1} \left| u(t,x) - \sum_{n=0}^{N-1} u_n(x) t^n \right| \leq C K^N \Gamma(1+\sigma N) |t|^N, \quad t \in S', \tag{2.2}$$

by some positive constants C and K.

When the opening angle β is less than $\sigma\pi$, for any direction $d \in \mathbb{R}$ there exists an analytic functions $u(t,x)$ on $S(d,\beta,\rho) \times B(r)$ such that (2.2) holds, and actually there are infinitely many analytic functions on the region.

When $\beta > \sigma\pi$ for the opening angle β, there does not exist such an analytic function $u(t,x)$ on $S(d,\beta,\rho) \times B(r)$ such that (2.2) holds in general. But if there exit such functions $u(t,x)$, then it is unique. In this sense, such an function $u(t,x)$ is called the *k-sum of* $\hat{u}(t,x)$ in d direction ($k = 1/\sigma$). We write it by $u^d(t,x)$, and we say that $\hat{u}(t,x)$ is *k-summable* in d direction. (For the detail of the k-summability, see [Bal 1].)

We remark that if the formal solution $\hat{u}(t,x)$ of (**CP**) is $k(0)$-summable in d direction, then the $k(0)$-sum $u^d(t,x)$ is an actual analytic solution of the equation $\partial_t^p u(t,x) = \partial_x^q u(t,x)$ satisfying the Cauchy data in the asymptotic meaning as $t \to 0$ along the sector $S(d,\beta,\rho)$ with $\beta > \sigma(0)\pi$.

2.2 Known Results

We give the known results of $k(0)$-summability and $k(0)$-sum of the formal solution $\hat{u}(t,x)$. First we give the theorem for the $k(0)$-summability in Miyake's paper [Miy].

Theorem 2.1 (Miyake) *Let $k(0) = p/(q-p)$ and $\varphi(x)$ be holomorphic in a neighbourhood of the origin. Then the formal solution $\hat{u}(t,x)$ of* (**CP**) *is $k(0)$-summable in d direction if and only if there exists a positive constant ε such that*

(1) *the Cauchy data φ can be continued analytically in a domain*

$$\Omega_\varepsilon(d;p,q) := \bigcup_{m=0}^{q-1} S\left(\frac{dp+2\pi m}{q}, \varepsilon, \infty\right), \tag{2.3}$$

(2) *the Cauchy data φ has a growth condition of exponential order at most $q/(q-p)$ in $\Omega_\varepsilon(d;p,q)$, which means*

$$|\varphi(x)| \le C \exp\left(\gamma |x|^{q/q-p}\right), \quad x \in \Omega_\varepsilon(d;p,q), \tag{2.4}$$

for some positive constants C and γ.

We write the condition (2) by

$$\varphi(x) \in \mathrm{Exp}\left(\frac{q}{q-p}; \Omega_\varepsilon(d;p,q)\right). \tag{2.5}$$

Next, in order to give an explicit formula for the $k(0)$-sum, we need a preparation for Meijer's G-function.

Meijer's G-Function. (cf. [MS, p. 2]) For $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_p) \in \mathbb{C}^p$ and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_q) \in \mathbb{C}^q$ with $\alpha_\ell - \gamma_j \notin \mathbb{N}$ $(\ell = 1,2,\dots,n; j = 1,2,\dots,m)$ such that $0 \le n \le p$, $0 \le m \le q$, we define

$$G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} \boldsymbol{\alpha} \\ \boldsymbol{\gamma} \end{matrix}\right) = \frac{1}{2\pi i} \int_I \frac{\prod_{j=1}^m \Gamma(\gamma_j + \tau) \prod_{\ell=1}^n \Gamma(1 - \alpha_\ell - \tau)}{\prod_{j=m+1}^q \Gamma(1 - \gamma_j - \tau) \prod_{\ell=n+1}^p \Gamma(\alpha_\ell + \tau)} z^{-\tau} d\tau, \tag{2.6}$$

where the path of integration I runs from $\kappa - i\infty$ to $\kappa + i\infty$ for any fixed $\kappa \in \mathbb{R}$ in such a manner that, if $|\tau|$ is sufficiently large, then $\tau \in I$ lies on the line $\mathrm{Re}\,\tau = \kappa$, all poles of $\Gamma(\gamma_j+\tau)$, $\{-\gamma_j-k\,;\,k\geq 0, j=1,2,\ldots,m\}$, lie to the left of the path I and all poles of $\Gamma(1-\alpha_\ell-\tau)$, $\{1-\alpha_\ell+k\,;\,k\geq 0,\ell=1,2,\ldots,n\}$, lie to the right of the path I, which is enable us by the conditions that $\alpha_\ell-\gamma_j\notin\mathbb{N}$.

In the following, the integration $\int_0^{\infty(\theta)}$ denotes the integration from 0 to ∞ along the half line of argument θ, and we use the following notations.

$$\boldsymbol{p}=(1,2,\ldots,p)\in\mathbb{N}^p,\quad \boldsymbol{q}=(1,2,\ldots,q)\in\mathbb{N}^q$$

$$\frac{\boldsymbol{p}}{p}=\left(\frac{1}{p},\frac{2}{p},\ldots,\frac{p}{p}\right),\quad \boldsymbol{p}+c=(1+c,2+c,\ldots,p+c)\quad(c\in\mathbb{C}),$$

$$\mathbf{1}_\ell=(1,1,\ldots,1)\in\mathbb{N}^\ell,$$

$$\Gamma\left(\boldsymbol{p}/p\right)=\prod_{j=1}^{p}\Gamma(j/p),\quad \Gamma\left(\boldsymbol{p}/p+c\right)=\prod_{j=1}^{p}\Gamma\left(j/p+c\right).$$

We give the theorem for the integral representation of the $k(0)$-sum in Ichinobe's papers [Ich 1, 2].

Theorem 2.2 (Ichinobe) *Under the conditions* (1) *and* (2) *in Theorem* 2.1, *the* $k(0)$*-sum* $u^d(t,x)$ *is obtained by the following function*

$$u^d(t,x)=\int_0^{\infty(pd/q)}\Phi_q(x,\zeta)E_0(t,\zeta;p,q)d\zeta, \tag{2.7}$$

where $(t,x)\in S(d,\beta,\rho)\times B(r)$ *with* $\beta>(q-p)\pi/p$ *and sufficiently small* r,

$$\Phi_q(x,\zeta)=\sum_{m=0}^{q-1}\varphi(x+\zeta\omega_q^m),\quad \omega_q=\exp(2\pi i/q), \tag{2.8}$$

and the kernel function $E_0(t,\zeta;p,q)$ *is given by*

$$E_0(t,\zeta;p,q)=\frac{\Gamma(\boldsymbol{p}/p)}{\Gamma(\boldsymbol{q}/q)}\frac{1}{\zeta}G^{q,0}_{p,q}\left(\frac{p^p}{q^q}\frac{\zeta^q}{t^p}\,\middle|\,\begin{matrix}\boldsymbol{p}/p\\ \boldsymbol{q}/q\end{matrix}\right). \tag{2.9}$$

3 Result

When the Cauchy data $\varphi(x)$ is an entire function of finite exponential growth order, we consider the k-summability of the formal solution of (**CP**) which is given by

$$\hat{u}(t,x)=\sum_{n=0}^{\infty}\varphi^{(qn)}(x)\frac{t^{pn}}{(pn)!}. \tag{3.1}$$

We first remark that the formal solution $\hat{u}(t,x)$ is convergent in a neighbourhood of the origin in $\mathbb{C}^2$ if and only if

$$\varphi(x) \in \mathrm{Exp}\left(\frac{q}{q-p};\mathbb{C}\right), \tag{3.2}$$

which means that $\varphi(x)$ is an entire function of exponential order at most $q/(q-p)$. In fact, the condition (3.2) is equivalent to the condition

$$|\varphi^{(qn)}(0)| \le CK^n(pn)!, \quad n = 0,1,2,\ldots, \tag{3.3}$$

for some positive constants C and K.

Therefore instead of the assumption that $\varphi(x)$ is holomorphic in a neighbourhood of the origin, we assume in the following that

$$\varphi(x) \in \mathrm{Exp}\left(\frac{q}{\ell};\mathbb{C}\right), \tag{3.4}$$

where ℓ is a natural number with $1 \le \ell \le q-p-1$, which means

$$\left|\varphi^{(qn)}(0)\right| \le CK^n((q-\ell)n)!, n = 0,1,2,\ldots. \tag{3.5}$$

Then we have

$$\hat{u}(t,x) \in \mathcal{O}_x[[t]]_{(q-p-\ell)/p}. \tag{3.6}$$

We put $\sigma(\ell) = (q-p-\ell)/p$ and $k(\ell) = 1/\sigma(\ell) = p/(q-p-\ell)$.

Our results of the $k(\ell)$-summability of $\hat{u}(t,x)$ and its $k(\ell)$-sum are stated as follows.

Theorem 3.1 *Let*

$$\varphi(x) \in \mathrm{Exp}\left(\frac{q}{\ell};\mathbb{C}\right). \tag{3.7}$$

Assume that

$$\Phi_q(x,\zeta) = \sum_{m=0}^{q-1} \varphi(x+\zeta\omega_q^m) \in \mathrm{Exp}_\zeta\left(\frac{q}{q-p}; S\left(\frac{pd}{q},\varepsilon,\infty\right)\right), \tag{3.8}$$

uniformly for small $|x|$, which means that there exist positive constants r, C and γ such that

$$\max_{|x|\le r} |\Phi_q(x,\zeta)| \le C\exp\left(\gamma|\zeta|^{q/(q-p)}\right), \quad \zeta \in S\left(\frac{pd}{q},\varepsilon,\infty\right). \tag{3.9}$$

Then the formal solution $\hat{u}(t,x)$ is $k(\ell)$-summable in d direction, and the integral representation of $k(\ell)$-sum just coincides with the one of $k(0)$-sum in Theorem 2.2. Exactly speaking, the $k(\ell)$-sum $u^d(t,x)$ is given by

$$u^d(t,x) = \int_0^{\infty(dp/q)} \Phi_q(x,\zeta)E_0(t,\zeta;p,q)d\zeta, \tag{3.10}$$

where $(t,x) \in S(d,\beta,\rho) \times B(r_1)$ *with* $\beta > (q-p-\ell)\pi/p$ *and small* r_1, *and the kernel function* $E_0(t,\zeta;p,q)$ *is the same one as in Theorem* 2.2.

In order to prove Theorem 3.1, we use the following important lemma for the k-summability.

Lemma 3.2 *Let* $\sigma > 0$, $k = 1/\sigma$ *and* $d \in \mathbb{R}$. *Let* $\hat{u}(t,x) = \sum_{n=0}^{\infty} u_n(x)t^n \in \mathcal{O}_x[[t]]_\sigma$. *Then the following three statements are equivalent.*

(i) $\hat{u}(t,x)$ *is* k*-summable in* d *direction.*

(ii) *Let* $v_1(s,x)$ *be the formal* k*-Borel transformation of* $\hat{u}(t,x)$

$$v_1(s,x) = (\hat{\mathcal{B}}_k\hat{u})(s,x) := \sum_{n=0}^{\infty} u_n(x)\frac{s^n}{\Gamma(1+n/k)}, \tag{3.11}$$

which is holomorphic in a neighbourhood of the origin in $\mathbb{C}^2$. *Then* $v_1(s,x)$ *can be continued analytically in a sector* $S(d,\varepsilon,\infty)$ *in* s*-plane for some positive constant* ε *and satisfies*

$$v_1(s,x) \in \mathrm{Exp}_s\left(k; S(d,\varepsilon,\infty)\right), \tag{3.12}$$

uniformly for small $|x|$.

(iii) *Let* $j \geq 2$ *and* $k_1 > 0, \ldots, k_j > 0$ *satisfy* $1/k = 1/k_1 + \cdots + 1/k_j$. *Let* $v_2(s,x)$ *be the following iterated formal Borel transformations of* $\hat{u}(t,x)$

$$v_2(s,x) = (\hat{\mathcal{B}}_{k_j} \circ \cdots \circ \hat{\mathcal{B}}_{k_1}\hat{u})(s,x). \tag{3.13}$$

Then $v_2(s,x)$ *has the same properties as* $v_1(s,x)$ *above.*

Under these conditions, the k-sum $u^d(t,x)$ is given by the analytic continuation of the following k-Laplace integral of v_1.

$$u^d(t,x) = (\mathcal{L}_{k,d}v_1)(t,x) := \frac{1}{t^k}\int_0^{\infty(d)} \exp\left[-\left(\frac{s}{t}\right)^k\right] v_1(s,x)d(s^k), \tag{3.14}$$

where $(t,x) \in S(d,\beta,\rho) \times B(r)$ with $\beta < \sigma\pi$. Exactly speaking, the analytic continuation of $u^d(t,x)$ in t variable is done by rotating the argument d of the path of integration.

The k-sum $u^d(t,x)$ is also obtained as the following iterated Laplace integrals of v_2.

$$u^d(t,x) = (\mathcal{L}_{k_1,d} \circ \cdots \circ \mathcal{L}_{k_j,d}v_2)(t,x). \tag{3.15}$$

Proof of Theorem 3.1

Let $v(s,x)$ be the $(q-p-\ell)$ times iterated formal p-Borel transformations of $\hat{u}(t,x)$

$$v(s,x) = \left(\hat{\mathcal{B}}_p^{q-p-\ell}\hat{u}\right)(s,x) = \sum_{n=0}^{\infty} \varphi^{(qn)}(x)\frac{s^{pn}}{(pn)!n!^{q-p-\ell}}, \tag{3.16}$$

which is convergent in a neighbourhood of $(s,x)=(0,0)$. Then under the condition (3.8) for $\Phi_q(x,\zeta)$, it is enough to prove that for some positive constant ε

$$(3.17)\qquad v(s,x)\in \mathrm{Exp}_s\left(\frac{p}{q-p-\ell}; S(d,\varepsilon,\infty)\right),$$

uniformly for small $|x|$.

In doing so, we first consider the following function $w(\eta,x)$

$$(3.18)\quad w(\eta,x)=\left(\mathcal{B}_p^\ell v\right)(\eta,x)\left(=\left(\hat{\mathcal{B}}_p^{q-p}\hat{u}\right)(\eta,x)\right)=\sum_{n=0}^{\infty}\varphi^{(qn)}(x)\frac{\eta^{pn}}{(pn)!n!^{q-p}}.$$

By Cauchy's integral formula, for sufficiently small $|\eta|$ and $|x|$, we have

$$(3.19)\qquad w(\eta,x)=\frac{1}{2\pi i}\oint_{|\zeta|=r}\frac{\varphi(x+\zeta)}{\zeta}\sum_{n=0}^{\infty}\frac{(\boldsymbol{q}/q)_n}{(\boldsymbol{p}/p)_n(n!)^{q-p}}\left(\frac{q^q}{p^p}\frac{\eta^p}{\zeta^q}\right)^n d\zeta,$$

where $r>\sqrt[q]{(q^q/p^p)|\eta|^p}$. We put

$$(3.20)\qquad \begin{aligned} f(\eta,\zeta) &:= \sum_{n=0}^{\infty}\frac{(\boldsymbol{q}/q)_n}{(\boldsymbol{p}/p)_n(n!)^{q-p}}(X_\eta{}^{-1})^n \\ &= {}_qF_{q-1}\left(\begin{matrix}\boldsymbol{q}/q\\ \boldsymbol{p}/p,\mathbf{1}_{q-p-1}\end{matrix};X_\eta{}^{-1}\right),\quad X_\eta{}^{-1}=\frac{q^q}{p^p}\frac{1}{\zeta^q}\eta^p,\end{aligned}$$

where ${}_qF_{q-1}$ is called the generalized hypergeometric series.

We notice that $f(\eta,\zeta)(={}_qF_{q-1})$ has q singular points in ζ-plane at q roots of $\zeta^q=(q^q/p^p)\eta^p$ since ${}_qF_{q-1}(\cdots;z)$ is a holomorphic solution at the origin of a Fuchsian ordinary differential equation with three regular singular points at $z=0,1,\infty$. We put $\zeta(\eta)=(q^q/p^p)^{1/q}\eta^{p/q}$ (the root with argument $p\arg\eta/q$) for any fixed $\eta\neq 0$ and we denote by $[0,\zeta(\eta)]$ the segment joining the origin and $\zeta(\eta)$. Then $f(\eta,\zeta)$ is singled-valued and analytic in $\mathbb{C}_\zeta\setminus\cup_{j=0}^{q-1}[0,\zeta(\eta)\omega_q^j]$ (cf. [Ich 1,2]). Therefore by deforming the contour of integration (3.19) on q segments, we get the following expression

$$(3.21)\qquad w(\eta,x)=\frac{1}{2\pi i}\int_0^{\zeta(\eta)}\frac{\Phi_q(x,\zeta)}{\zeta}\left\{f(\eta,\zeta)-f(\eta,\zeta\omega_q^{-1})\right\}d\zeta.$$

We remark that th integrability at the end points of each segments is assured from the properties of ${}_qF_{q-1}$. From this expression and the assumption that $\varphi(x)\in\mathrm{Exp}(q/\ell;\mathbb{C})$, we obtain

$$(3.22)\qquad \max_{|x|\le r_1}|w(\eta,x)|\le C\exp\left(\delta|\eta|^{p/\ell}\right),\quad \eta\in\mathbb{C},$$

by some positive constants r_1, C and δ.

Therefore the function $v(s,x)$ is given by the following ℓ times iterated p-Laplace integrals of $w(\eta,x)$ for any direction θ over η-plane

$$(3.23)\qquad v(s,x)=\left(\mathcal{L}^{\ell}_{p,\theta}w\right)(s,x).$$

We fix $\arg\eta=\theta=d$ for $\eta\neq 0$. By exchanging the order of integrations, we have

$$(3.24)\quad v(s,x)=\frac{1}{2\pi i}\int_0^{\infty(pd/q)}\frac{\Phi_q(x,\zeta)}{\zeta}\left\{\left(\mathcal{L}^{\ell}_{p,d}\right)\left(f(\cdot,\zeta)-f(\cdot,\zeta\omega_q^{-1})\right)\right\}(s)d\zeta.$$

Now, we calculate the explicit integral representation of the kernel functions of w and v. For that purpose, we employ the Barnes type integral representation for the function ${}_qF_{q-1}$. Then we always assume $(p,q)\neq(1,2),(1,3)$ since we can remove such the restriction by employing the Euler type integral representation instead of Barnes type, but we shall not go into details about this here. (See [Ich 2, Appendix].)

We recall the following Barnes type integral representation of $f(\eta,\zeta)={}_qF_{q-1}$

$$(3.25)\quad f(\eta,\zeta)=\frac{C_{pq}}{2\pi i}\int_I\frac{\Gamma(\boldsymbol{q}/q+\tau)\Gamma(-\tau)}{\Gamma(\boldsymbol{p}/p+\tau)\Gamma(1+\tau)^{q-p-1}}(-X_\eta{}^{-1})^\tau d\tau,\quad X_\eta{}^{-1}=\frac{q^q}{p^p}\frac{\eta^p}{\zeta^q},$$

where $|\arg(-X_\eta{}^{-1})|\leq\pi$, the path of integration I runs from $\kappa-i\infty$ to $\kappa+i\infty$ with $-1/q<\kappa<0$ and $C_{pq}=\Gamma(\boldsymbol{p}/p)/\Gamma(\boldsymbol{q}/q)$. (cf. [IKSY])

Since the function $f(\eta,\zeta)-f(\eta,\zeta\omega_q^{-1})$, which is the kernel function of w, is well-defined on the line of $\arg\zeta=p\arg\eta/q$ for any fixed $\eta\neq 0$, we have after easy calculation

$$(3.26)\quad \begin{aligned} f(\eta,\zeta)-f(\eta,\zeta\omega_q^{-1}) &= C_{pq}\int_I\frac{\Gamma(\boldsymbol{q}/q+\tau)}{\Gamma(\boldsymbol{p}/p+\tau)\Gamma(1+\tau)^{q-p}}X_\eta^{-\tau}d\tau\\ &= 2\pi iC_{pq}G^{q,0}_{q,q}\left(X_\eta\,\middle|\,\begin{matrix}\boldsymbol{p}/p,\mathbf{1}_{q-p}\\ \boldsymbol{q}/q\end{matrix}\right)\\ &\underset{\text{put}}{=} 2\pi i\zeta E_{q-p}(\eta,\zeta;p,q),\end{aligned}$$

and for the kernel function of v, we get

$$(3.27)\quad \begin{aligned}&\left\{\left(\mathcal{L}^{\ell}_{p,d}\right)\left(f(\cdot,\zeta)-f(\cdot,\zeta\omega_q^{-1})\right)\right\}(s)\\ &= C_{pq}\int_I\frac{\Gamma(\boldsymbol{q}/q+\tau)}{\Gamma(\boldsymbol{p}/p+\tau)\Gamma(1+\tau)^{q-p-\ell}}X_s^{-\tau}d\tau\\ &= 2\pi iC_{pq}G^{q,0}_{q-\ell,q}\left(X_s\,\middle|\,\begin{matrix}\boldsymbol{p}/p,\mathbf{1}_{q-p-\ell}\\ \boldsymbol{q}/q\end{matrix}\right)=2\pi i\zeta E_{q-p-\ell}(s,\zeta;p,q).\end{aligned}$$

In order to prove (3.17) under the condition (3.8) for $\Phi_q(x,\zeta)$, we use the asymptotic expansion of the G-function (cf. [Luk, p. 179])

$$
\begin{aligned}
(3.28)\qquad & G^{q,0}_{q-\ell,q}\left(z\,\middle|\,\begin{matrix} \mathbf{p}/p, \mathbf{1}_{q-p-\ell} \\ \mathbf{q}/q \end{matrix}\right) \\
&= \frac{(2\pi)^{(\ell-1)/2}}{\ell^{1/2}} \exp\left(-\ell z^{1/\ell}\right) z^{-(q-p-\ell)/2\ell}\left[1+O\left(z^{-1/\ell}\right)\right], \\
&\qquad \text{as } |z|\to\infty, \quad |\arg z|\le \ell\pi.
\end{aligned}
$$

From this property of the G-function and expressions (3.24) and (3.27) for $v(s,x)$, we have for sufficiently small r

$$
\begin{aligned}
\max_{|x|\le r}|v(s,x)| &\le \int_0^\infty \max_{|x|\le r}|\Phi_q(x,\zeta)|\;|E_{q-p-\ell}(s,\zeta;p,q)|\;|d\zeta| \\
&\le C\int_0^\infty p(|\zeta|,|s|)\exp\left(\gamma|\zeta|^{q/(q-p)}\right)\exp\left(-c\frac{|\zeta|^{q/\ell}}{|s|^{p/\ell}}\right)|d\zeta|,
\end{aligned}
$$

where $p(a,b)$ denotes a power function of a and b, and C, γ and c are some positive constants.

There is no problem if $|\zeta|$ is sufficiently large, since $q/(q-p) < q/\ell$. Hence on $0\le|\zeta|\le M$ for any large fixed $M>0$, we calculate the maximum with respect to ζ of the function

$$
(3.29)\qquad F(|\zeta|;|s|) := \exp\left(\gamma|\zeta|^{q/(q-p)} - c|\zeta|^{q/\ell}|s|^{-p/\ell}\right).
$$

We see that the function $F(|\zeta|;|s|)$ takes the maximum at

$$
|\zeta| = c_1|s|^{\frac{p(q-p)}{q(q-p-\ell)}} =: |\zeta_0|
$$

and

$$
(3.30)\qquad F(|\zeta_0|;|s|) = \exp\left(c_2|s|^{p/(q-p-\ell)}\right),
$$

where c_1 and c_2 are some positive constants. From this we have the desired properties (3.17) of $v(s,x)$.

Finally, the $k(\ell)$-sum is given by the following iterated p-Laplace integrals of $v(s,x)$

$$
\begin{aligned}
(3.31)\qquad & u^d(t,x) = \left(\mathcal{L}^{q-p-\ell}_{p,d}v\right)(t,x) \\
&= C_{pq}\int_0^{\infty(pd/q)} \frac{\Phi_q(x,\zeta)}{\zeta}\left[(\mathcal{L}^{q-p-\ell}_p)G^{q,0}_{q-\ell,q}\left(X_s\,\middle|\,\begin{matrix} \mathbf{p}/p, \mathbf{1}_{q-p-\ell} \\ \mathbf{q}/q \end{matrix}\right)\right](t)d\zeta \\
&= C_{pq}\int_0^{\infty(pd/q)} \frac{\Phi_q(x,\zeta)}{\zeta}G^{q,0}_{p,q}\left(X_t\,\middle|\,\begin{matrix} \mathbf{p}/p \\ \mathbf{q}/q \end{matrix}\right)d\zeta.
\end{aligned}
$$

This completes the proof of Theorem 3.1.

Remark 3.3 Comparing the condition for $k(0)$-summability in Theorem 2.1

$$\varphi(x) \in \mathrm{Exp}\left(\frac{q}{q-p}; \Omega_\varepsilon(d;p,q)\right) \tag{3.32}$$

and the condition for $k(\ell)$-summability in Theorem 3.1

$$\Phi_q(x,\zeta) = \sum_{m=0}^{q-1} \varphi(x+\zeta\omega_q^m) \in \mathrm{Exp}_\zeta\left(\frac{q}{q-p}; S(pd/q,\varepsilon,\infty)\right), \tag{3.33}$$

uniformly for small $|x|$, we seem that they are different ones. But we see that these conditions are equivalent each other. In fact, it follows from the relation formula

$$\varphi^{(q-1)}(x+\zeta) = \frac{1}{q}\sum_{j=0}^{q-1} \partial_x^j \partial_\zeta^{q-1-j} \Phi_q(x,\zeta). \tag{3.34}$$

Hence Theorem 3.1 actually follows from the above fact and a property of the k-summability (cf. [Bal 1, p. 31, Exercises 2]). The claim of our theorem is the fact that the integral representations of $k(\ell)$-sum and $k(0)$-sum are the same ones!

Remark 3.4 In Theorem 3.1, the condition (3.33) is only a sufficient condition for $k(\ell)$-summability though it is necessary and sufficient condition for $k(0)$-summability from Theorem 2.1 and Remark 3.3. We do not know yet whether it is also a necessary condition.

In the paper [Bal 2], W. Balser gave the necessary and sufficient condition by using different conditions with ours in case of heat equation which is the case of $(p,q) = (1,2)$ in our notation. According to his argument, we can get the necessary and sufficient condition as follows:

Let $\sigma > 0$ and $k = 1/\sigma$, and let $\hat{u}(t,x) = \sum_{n=0}^{\infty} \varphi^{(qn)}(x)t^{pn}/(pn)! \in \mathcal{O}_x[[t]]_\sigma$ be a formal solution of (**CP**). *We put*

$$\hat{\psi}_j(x) := \partial_x^j \hat{u}(x,0) = \sum_{n=0}^{\infty} \varphi^{(qn+j)}(0)\frac{x^{pn}}{(pn)!} \quad (0 \le j \le q-1). \tag{3.35}$$

Then $\hat{u}(t,x)$ is k-summable in d direction if and only if $\hat{\psi}_j(x)$ $(0 \le j \le p-1)$ is k-summable in d direction.

In fact, the necessity is trivial. For the proof of the sufficiency, let $\psi_j(x)$ be the k-sum of $\hat{\psi}_j(x)$ in d direction. We put

$$u(t,x) = \sum_{j=0}^{q-1}\sum_{n=0}^{\infty} \partial_t^{pn}\psi_j(t)\frac{x^{qn+j}}{(qn+j)!}. \tag{3.36}$$

Then one can prove that $u(t,x)$ has the Gevrey asymptotic expansion $\hat{u}(t,x)$ of order σ in a sector whose direction is d and its opening angle is greater that $\sigma\pi$, which shows that $u(t,x)$ is the k-sum of $\hat{u}(t,x)$ in d direction.

4 Hierarchy of equations

Let $\hat{u}(t,x) = \sum_{n=0}^{\infty} \varphi^{(qn)}(x)t^{pn}/(pn)!$ be the formal solution of (**CP**). We put

$$\hat{v}(s,x) := \left(\hat{\mathcal{B}}_p^{\ell}\hat{u}\right)(s,x) = \sum_{n=0}^{\infty} \varphi^{(qn)}(x)\frac{s^{pn}}{(pn)!n!^{\ell}}, \tag{4.1}$$

where ℓ is a natural number with $1 \le \ell \le q-p-1$.

Then it is seen that $\hat{v}(s,x)$ satisfies the following Cauchy problem

$$(\mathbf{CP})_{\ell} \qquad \begin{cases} \partial_s^p\left(\dfrac{1}{p}s\partial_s\right)^{\ell} v(s,x) = \partial_x^q v(s,x), \\ v(0,x) = \varphi(x), \quad \partial_s^j v(0,x) = 0 \ (1 \le j \le p-1). \end{cases}$$

In fact, it follows from the following commutative diagram

$$\begin{array}{ccc} \dfrac{t^{pn}}{(pn)!} & \xrightarrow{(\hat{\mathcal{B}}_p)^{\ell}} & \dfrac{s^{pn}}{(pn)!(n!)^{\ell}} \\ \Big\downarrow \partial_t^p & & \Big\downarrow \partial_s^p\left(\dfrac{1}{p}s\partial_s\right)^{\ell} \\ \dfrac{t^{p(n-1)}}{\{p(n-1)\}!} & \xrightarrow{(\hat{\mathcal{B}}_p)^{\ell}} & \dfrac{s^{p(n-1)}}{\{p(n-1)\}!\{(n-1)!\}^{q-p}} \end{array}$$

Then we can prove the following proposition in a similar way to Theorems 2.1 and 2.2.

Proposition 4.1 *Let $\sigma(\ell) = (q-p-\ell)/p$ and $k(\ell) = 1/\sigma(\ell)$. Let $\hat{v}(s,x)$ be the formal solution of the Cauchy problem $(\mathbf{CP})_{\ell}$ where the Cauchy data $\varphi(x)$ is holomorphic in a neighbourhood of the origin. Then $\hat{v}(s,x) \in \mathcal{O}_x[[s]]_{\sigma(\ell)}$. Then the formal solution $\hat{v}(s,x)$ is $k(\ell)$-summable in d direction if and only if*

$$\Phi_q(x,\zeta) \in \mathrm{Exp}_{\zeta}\left(\frac{q}{q-p-\ell}; S\left(\frac{pd}{q},\varepsilon,\infty\right)\right), \tag{4.2}$$

uniformly for small $|x|$. Moreover, the $k(\ell)$-sum $v^d(s,x)$ is given by

$$v^d(s,x) = \int_0^{\infty(pd/q)} \Phi_q(x,\zeta)E_{\ell}(s,\zeta;p,q)d\zeta, \tag{4.3}$$

where the kernel function $E_\ell(s,\zeta;p,q)$ *is given by*

$$E_\ell(s,\zeta;p,q) = \frac{\Gamma(\mathbf{p}/p)}{\Gamma(\mathbf{q}/q)}\,\frac{1}{\zeta}\,G^{q,0}_{p+\ell,q}\left(\frac{p^p}{q^q}\frac{\zeta^q}{s^p}\,\middle|\,\begin{matrix}\mathbf{p}/p,\mathbf{1}_\ell\\ \mathbf{q}/q\end{matrix}\right). \tag{4.4}$$

We fix ℓ as above. We take a natural number m with $1 \le m \le q-p-\ell$. Then we can prove the following proposition in a similar way to Theorem 3.1.

Proposition 4.2 *Suppose* $\varphi(x) \in \mathrm{Exp}\,(q/m;\mathbb{C})$. *Then* $\hat{v}(s,x) \in \mathcal{O}_x[[s]]_{\sigma(m+\ell)}$. *Assume that*

$$\Phi_q(x,\zeta) \in \mathrm{Exp}_\zeta\left(\frac{q}{q-p-\ell};S\left(\frac{pd}{q},\varepsilon,\infty\right)\right), \tag{4.5}$$

uniformly for small $|x|$. *Then the formal solution* $\hat{v}(s,x)$ *of* $(\mathbf{CP})_\ell$ *is* $k(m+\ell)$*-summable in* d *direction, and the integral representation of* $k(m+\ell)$*-sum just coincides with the one of* $k(\ell)$*-sum in Proposition* 4.1.

5 Singular Perturbation

The idea developed in the previous sections can be extended directly to the study of k-summability for divergent solutions to singular perturbed ordinary differential equations. In this section, we shall present the idea.

Let p and q be two natural numbers, and consider the following equation

$$(I - t^p\partial_x^q)\,u(t,x) = \varphi(x), \tag{5.1}$$

where $\varphi(x)$ is assumed to be holomorphic in a neighbourhood of the origin and I denotes the identity operator. Then a unique formal solution is obtained by

$$\hat{u}(t,x) = \sum_{n=0}^{\infty}\varphi^{(qn)}(x)t^{pn} \in \mathcal{O}_x[[t]]_{q/p}. \tag{5.2}$$

From this expression of formal solution, we easily see that $\hat{u}(t,x)$ is convergent in a neighbourhood of the origin in $\mathbb{C}^2$ if and only if $\varphi(x) \in \mathrm{Exp}(1;\mathbb{C})$.

As a fundamental result on the p/q-summability and the integral expression of the p/q-sum, we can prove the following proposition which corresponds to Theorems 2.1 and 2.2.

Proposition 5.1 *Let* $\varphi(x)$ *be holomorphic in a neighbourhood of the origin. Then* $\hat{u}(t,x)$ *is* p/q*-summable in* d *direction if and only if*

$$\Phi_q(x,\zeta) = \sum_{m=0}^{q-1}\varphi(x+\zeta\omega_q^m) \in \mathrm{Exp}_\zeta\left(1;S\left(\frac{pd}{q},\varepsilon,\infty\right)\right), \tag{5.3}$$

uniformly for small $|x|$ for some constant $\varepsilon > 0$. Moreover, in this case, the p/q-sum $u^d(t,x)$ is given by the following integral formula

$$u(t,x) = \int_0^{\infty(pd/q)} \Phi_q(x,\zeta) F_0(t,\zeta)\, d\zeta, \tag{5.4}$$

where the kernel function $F_0(t,\zeta)$ is given by

$$F_0(t,\zeta) = \frac{1}{\Gamma(\mathbf{q}/q)} \frac{1}{\zeta} G^{q,0}_{0,q}\left(\frac{1}{q^q}\frac{\zeta^q}{t^p} \,\middle|\, \mathbf{q}/q \right). \tag{5.5}$$

The proof of this proposition is reduced to those of Theorems 2.1 and 2.2 for the Cauchy problem (**CP**). In fact, by taking formal 1-Borel transformation to the equation (5.1), we obtain an integro-differential equation

$$\left(I - \partial_s^{-p}\partial_x^q\right) v(s,x) = \varphi(x), \tag{5.6}$$

since the multiplier by t^p is send to an integral operator ∂_s^{-p}. Therefore by operating ∂_s^p to the obtained equation, we get the following Cauchy problem.

$$\begin{cases} \partial_s^p v(s,x) = \partial_x^q v(s,x), \\ v(0,x) = \varphi(x), \quad \partial_s^j v(0,x) = 0 \ (1 \le j \le p-1). \end{cases} \tag{5.7}$$

Thus, when $q > p$, the problem is reduced to the Cauchy problem (**CP**) and the previous results are applicable. On the other hand, when $p \ge q$, by putting $\tau = t^p$ we change the problem to the following one

$$(I - \tau\partial_x^q)\, w(\tau,x) = \varphi(x), \tag{5.8}$$

by which the problem is reduced to the above case.

Now, instead of the assumption that $\varphi(x)$ is holomorphic in a neighbourhood of the origin, we assume that

$$\varphi(x) \in \mathrm{Exp}\left(\frac{q}{q-\ell}; \mathbb{C}\right) \quad (1 \le \ell \le q-1). \tag{5.9}$$

Then we have

$$\hat{u}(t,x) \in \mathcal{O}_x[[t]]_{\ell/p}. \tag{5.10}$$

In this case, we can prove the following proposition for the p/ℓ-summability and the p/ℓ-sum of the formal solution $\hat{u}(t,x)$ of the singular perturbed equation (5.1), which corresponds to Theorem 3.1.

Proposition 5.2 *Let $\varphi(x) \in \mathrm{Exp}(q/\ell; \mathbb{C})$ and assume* (5.3) *for $\Phi_q(x,\zeta)$. Then the formal solution $\hat{u}(t,x)$ of* (5.1) *is p/ℓ-summable in d direction, and the integral representation of p/ℓ-sum just coincides with the one of p/q-sum in Proposition* 5.1.

We can present the corresponding results to Propositions 4.1 and 4.2, but we omit to write them down in explicit form, since they will be easily recognized.

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