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Borel Summability of Divergent Solutions for Singularly Perturbed First Order Linear Ordinary Differential Equations

名城大学理工学部数学科
(Department of Mathematics, Meijo University)
日比野 正樹 (Masaki HIBINO)

1 Introduction and Main Result.

In this paper we are concerned with the following first order linear ordinary differential equation with a parameter ε ($\in \mathbf{C}$):

$$a(x,\varepsilon)D_x u(x,\varepsilon)+b(x,\varepsilon)u(x,\varepsilon)=f(x,\varepsilon), \tag{1.1}$$

where $x \in \mathbf{C}$, $D_x = d/dx$. a, b and f are holomorphic at $(x,\varepsilon)=(0,0)\in \mathbf{C}^2$.

First of all we give two fundamental assumptions. The first one demands that ε is a perturbation parameter, that is, we assume the following:

$$a(x,0)\equiv 0. \tag{1.2}$$

The second one is

$$a_\varepsilon(0,0)\neq 0, \tag{1.3}$$

where $a_\varepsilon(x,\varepsilon)=(d/d\varepsilon)a(x,\varepsilon)$. These two assumptions imply that $a(0,\varepsilon)\neq 0$ for sufficiently small $\varepsilon\neq 0$, which means that the equation (1.1) has a regularity at $x=0$.

Throughout this paper we always assume (1.2) and (1.3).

It follows from (1.2) and (1.3) that solutions of (1.1) can be expressed by convergent power series around $x=0$. Here, however, let us consider solutions expressed by power series in the parameter ε. Then we shall see that under a suitable condition the equation

(1.1) has a unique formal power series solution $u(x,\varepsilon) = \sum_{n=0}^{\infty} u_n(x)\varepsilon^n$ ($u_n(x)$ are holomorphic in a common neighborhood of $x = 0$), which is divergent in general (cf. Definition 1.1, (3) and Theorem 1.1).

So in this paper we shall deal with the summability problem for such divergent solutions. Our main purpose is to obtain the conditions under which such formal solutions are Borel summable (cf. Definition 1.1, (5)). Those conditions will be given in Theorem 1.2.

1.1 Definition and Fundamental Result.

Firstly, in order to state our problem precisely, let us introduce some notations.

Definition 1.1 (1) For $R > 0$, $\mathcal{O}[R]$ denotes the ring of holomorphic functions on the closed ball $B(R) := \{x \in \mathbf{C};\ |x| \leq R\}$.

(2) The ring of formal power series in ε ($\in \mathbf{C}$) over the ring $\mathcal{O}[R]$ is denoted as $\mathcal{O}[R][[\varepsilon]]$: $\mathcal{O}[R][[\varepsilon]] = \{u(x,\varepsilon) = \sum_{n=0}^{\infty} u_n(x)\varepsilon^n;\ u_n(x) \in \mathcal{O}[R]\}$.

(3) We say that $u(x,\varepsilon) = \sum_{n=0}^{\infty} u_n(x)\varepsilon^n \in \mathcal{O}[R][[\varepsilon]]$ *belongs to* $\mathcal{O}[R][[\varepsilon]]_2$ if there exist some positive constants C and K such that $\max_{|x|\leq R} |u_n(x)| \leq CK^n n!$ for all $n \in \mathbf{N}$. Therefore an element of $\mathcal{O}[R][[\varepsilon]]_2$ diverges in general.

(4) For $\theta \in \mathbf{R}$ and $T > 0$, we define the region $O(\theta, T)$ by

$$O(\theta,T) = \{\varepsilon;\ |\varepsilon - Te^{i\theta}| < T\}. \tag{1.4}$$

(5) Let $u(x,\varepsilon) = \sum_{n=0}^{\infty} u_n(x)\varepsilon^n \in \mathcal{O}[R][[\varepsilon]]_2$. We say that $u(x,\varepsilon)$ *is Borel summable in* θ if there exists a holomorphic function $U(x,\varepsilon)$ on $B(r) \times O(\theta,T)$ for some $0 < r \leq R$ and $T > 0$ which satisfies the following asymptotic estimates: There exist some positive constants C and K such that

$$\max_{|x|\leq r} \left| U(x,\varepsilon) - \sum_{n=0}^{N-1} u_n(x)\varepsilon^n \right| \leq CK^N N! |\varepsilon|^N, \quad \varepsilon \in O(\theta,T), \quad N = 1, 2, \ldots. \tag{1.5}$$

In general a given power series $u(x,\varepsilon) \in \mathcal{O}[R][[\varepsilon]]_2$ is not necessarily Borel summable. However, if $u(x,\varepsilon)$ is Borel summable in θ, we see that the above holomorphic function $U(x,\varepsilon)$ is unique by a general theory of Gevrey asymptotic expansion (cf. Balser[1][2], Lutz-Miyake-Schäfke[5] and Malgrange[6]). So we call this $U(x,\varepsilon)$ the *Borel sum of* $u(x,\varepsilon)$ *in* θ.

The following theorem is fundamental in the argument below.

Theorem 1.1 (cf. Hibino[4]) *Let us assume $b(0,0) \neq 0$. Then the equation* (1.1) *has a unique formal power series solution $u(x,\varepsilon) = \sum_{n=0}^{\infty} u_n(x)\varepsilon^n \in \mathcal{O}[R][[\varepsilon]]_2$ for some* $R > 0$.

In the following we always assume $b(0,0) \neq 0$. On the basis of Theorem 1.1, let us study the Borel summability of the formal solution.

1.2 Main Result.

Before stating the main theorem in this paper, let us rewrite the equation (1.1).

By the condition $b(0,0) \neq 0$, we may assume that $b(x,\varepsilon) \neq 0$ in the neighborhood of $(x,\varepsilon) = (0,0)$. Theorefore by dividing $b(x,\varepsilon)$ into both sides of (1.1), we may assume that $b(x,\varepsilon) \equiv 1$. Then it follows from (1.2) and (1.3) that the equation (1.1) is rewritten in the following form:

$$\{\alpha(x) + \gamma(x,\varepsilon)\}\varepsilon D_x u(x,\varepsilon) + u(x,\varepsilon) = f(x,\varepsilon), \tag{1.6}$$

where $\alpha(x)$ and $\gamma(x,\varepsilon)$ are holomorphic at $x = 0$ and $(x,\varepsilon) = (0,0)$, respectively. Moreover they satisfy

$$\alpha(0) \neq 0, \tag{1.7}$$

$$\gamma(x,0) \equiv 0. \tag{1.8}$$

Furthermore in this paper we assume for simplicity that $\alpha(x)$ is the constant. That is, we consider the Borel summability of the formal solution for the following equation:

$$\{\alpha + \gamma(x,\varepsilon)\}\varepsilon D_x u(x,\varepsilon) + u(x,\varepsilon) = f(x,\varepsilon), \tag{1.9}$$

where α is the constant satisfying $\alpha \neq 0$. On the general case, see Hibino[3].

Now let us give the conditions under which the formal solution of (1.9) is Borel summable.

First we define the region $E_+(\theta,\kappa)$ $(\kappa > 0)$ by

$$E_+(\theta,\kappa) := \{\xi;\ \mathrm{dist}\,(\xi, \mathbf{R}_+ e^{i\theta}) \leq \kappa\}, \tag{1.10}$$

where $\mathbf{R}_+ = [0, +\infty)$. Then the first assumption is stated as follows:

(A1) $f(x,\varepsilon)$ can be continued analytically to $E_+(\theta + \pi + \arg(\alpha), \kappa) \times \{\varepsilon \in \mathbf{C};\ |\varepsilon| \leq c\}$ $(^{\exists}\kappa,\ ^{\exists}c > 0)$. Moreover $f(x,\varepsilon)$ has the following exponential growth estimate on

$E_+(\theta+\pi+\arg(\alpha),\kappa)\times\{\varepsilon\in\mathbf{C};\ |\varepsilon|\le c\}$: There exist some positive constants C and δ such that

$$\max_{|\varepsilon|\le c}|f(x,\varepsilon)|\le Ce^{\delta|x|},\quad x\in E_+(\theta+\pi+\arg(\alpha),\kappa). \tag{1.11}$$

Next we assume the following for $\gamma(x,\varepsilon)$:

(A2) $\gamma(x,\varepsilon)$ can be continued analytically to $E_+(\theta+\pi+\arg(\alpha),\kappa)\times\{\varepsilon\in\mathbf{C};\ |\varepsilon|\le c\}$. Moreover $\gamma(x,\varepsilon)$ is bounded on $E_+(\theta+\pi+\arg(\alpha),\kappa)\times\{\varepsilon\in\mathbf{C};\ |\varepsilon|\le c\}$:

$$M:=\sup_{E_+(\theta+\pi+\arg(\alpha),\kappa)\times\{\varepsilon\in\mathbf{C};\ |\varepsilon|\le c\}}|\gamma(x,\varepsilon)|<\infty. \tag{1.12}$$

Then we obtain the following main result in this paper.

Theorem 1.2 *Under the assumptions* (A1) *and* (A2) *the formal solution $u(x,\varepsilon)$ of the equation* (1.9) *is Borel summable in θ.*

Remark 1.1 When the formal solution $u(x,\varepsilon)$ of (1.9) is Borel summable, we see that its Borel sum is a holomorphic solution of (1.9). This is an immediate consequence of the uniqueness of the Borel sum.

We will prove Theorem 1.2 in §3. In the proof, we consider an differential convolution equation (the equation (2.5) in §2) which is obtained by applying the formal Borel transform (cf. Definition 2.1) to (1.9), and prove an analytic continuation property and an exponential growth estimate for solutions of (2.5) by using the iteration method. Lemma 3.1 in §3 will play the most important role in the proof.

2 Formal Borel Transform of Equations.

Before proving Theorem 1.2, we give some preliminaries.

Definition 2.1 For $u(x,\varepsilon)=\sum_{n=0}^{\infty}u_n(x)\varepsilon^n\in\mathcal{O}[R][[\varepsilon]]_2$, we define a convergent power series $\mathcal{B}(u)(x,\eta)$ in a neighborhood of $(x,\eta)=(0,0)$ by

$$\mathcal{B}(u)(x,\eta):=\sum_{n=0}^{\infty}u_n(x)\frac{\eta^n}{n!}. \tag{2.1}$$

We call $\mathcal{B}(u)(x,\eta)$ the *formal Borel transform of* $u(x,\varepsilon)$.

When we want to check the Borel summability of formal power series $u(x,\varepsilon) = \sum_{n=0}^{\infty} u_n(x)\varepsilon^n \in \mathcal{O}[R][[\varepsilon]]_2$, the following theorem plays a fundamental role in general.

Theorem 2.1 (Lutz-Miyake-Schäfke[5], Malgrange[6]) *The following two conditions* (i) *and* (ii) *are equivalent:*

(i) $u(x,\varepsilon) = \sum_{n=0}^{\infty} u_n(x)\varepsilon^n \in \mathcal{O}[R][[\varepsilon]]_2$ *is Borel summable in* θ.

(ii) $\mathcal{B}(u)(x,\eta)$ *can be continued analytically to* $B(r_0) \times E_+(\theta,\kappa_0)$ *for some* $r_0 > 0$ *and* $\kappa_0 > 0$, *and has the following exponential growth estimate for some positive constants* C *and* δ:

$$(2.2)\qquad \max_{|x|\le r_0} |\mathcal{B}(u)(x,\eta)| \le Ce^{\delta|\eta|}, \quad \eta \in E_+(\theta,\kappa_0).$$

When the condition (i) *or* (ii) *(therefore both) is satisfied, the Borel sum* $U(x,\varepsilon)$ *of* $u(x,\varepsilon)$ *in* θ *is given by*

$$(2.3)\qquad U(x,\varepsilon) = \frac{1}{\varepsilon}\int_{\mathbf{R}_+e^{i\theta}} e^{-\eta/\varepsilon}\mathcal{B}(u)(x,\eta)d\eta.$$

Therefore in order to prove Theorem 1.2, it is sufficient to prove that the formal Borel transform $\mathcal{B}(u)(x,\eta)$ of the formal solution $u(x,\varepsilon)$ satisfies the above condition (ii) under the conditions (A1) and (A2). In order to do that, firstly let us write down the equation which $\mathcal{B}(u)(x,\eta)$ should satisfy. By operating the formal Borel transform to (1.9), we see that $\mathcal{B}(u)(x,\eta)$ is a solution of the following equation:

(2.4)
$$\alpha D_\eta^{-1} D_x v(x,\eta) + \int_0^\eta \mathcal{B}(\gamma)(x,\eta-t)D_x v(x,t)dt + v(x,\eta) = \mathcal{B}(f)(x,\eta),$$

where $D_\eta^{-1} = \int_0^\eta$, and $\mathcal{B}(\gamma)(x,\eta)$ and $\mathcal{B}(f)(x,\eta)$ are the formal Borel transforms of $\gamma(x,\varepsilon) = \sum_{n=1}^{\infty}\gamma_n(x)\varepsilon^n$ and $f(x,\varepsilon) = \sum_{n=0}^{\infty} f_n(x)\varepsilon^n$, respectively, that is,

$$\mathcal{B}(\gamma)(x,\eta) = \sum_{n=1}^{\infty}\gamma_n(x)\frac{\eta^n}{n!} \quad\text{and}\quad \mathcal{B}(f)(x,\eta) = \sum_{n=0}^{\infty} f_n(x)\frac{\eta^n}{n!}.$$

Furthermore by operating D_η to the equation (2.4) from the left, we see that $\mathcal{B}(u)(x,\eta)$ is a solution of the following initial value problem:

$$(2.5)\qquad \begin{cases} \{D_\eta + \alpha D_x\}v(x,\eta) = -\int_0^\eta \mathcal{B}(\gamma)_\eta(x,\eta-t)v_x(x,t)dt + g(x,\eta), \\ v(x,0) = f(x,0), \end{cases}$$

where $g(x,\eta) = D_\eta \mathcal{B}(f)(x,\eta)$.

It is easy to prove that $\mathcal{B}(u)(x,\eta)$ is the unique locally holomorphic solution of (2.5). Hence Theorem 1.2 will be proved by showing that the solution $v(x,\eta)$ of the equation (2.5) satisfies the condition (ii) in Theorem 2.1.

3 Proof of Theorem 1.2.

Let us prove that the solution $v(x,\eta)$ of the equation (2.5) satisfies the condition (ii) in Theorem 2.1. Firstly we remark that in general the solution $V(x,\eta)$ of the initial value problem of the following first order linear partial differential equation

$$\left\{\begin{array}{l}\{D_\eta + \alpha D_x\}V(x,\eta) = k(x,\eta),\\ V(x,0) = l(x)\end{array}\right. \tag{3.1}$$

is given by

$$V(x,\eta) = \int_0^\eta k(x-\alpha(\eta-t),t)dt + l(x-\alpha\eta). \tag{3.2}$$

Proof of Theorem 1.2. First, let us transform the equation (2.5). It follows from (3.2) that the equation (2.5) is equivalent to the following equation:

$$\begin{aligned} v(x,\eta) &= f(x-\alpha\eta,0) + \int_0^\eta g(x-\alpha(\eta-t),t)dt \\ &\quad - \int_0^\eta\int_0^t \mathcal{B}(\gamma)_\eta(x-\alpha(\eta-t),t-s)v_x(x-\alpha(\eta-t),s)dsdt. \end{aligned}$$

Let us transform the third term of the right hand side. By using Fubini's Theorem, we write $\int_0^\eta\int_0^t \cdots dsdt = \int_0^\eta\int_s^\eta \cdots dtds$. Here we remark that

$$\begin{aligned} &\int_s^\eta \mathcal{B}(\gamma)_\eta(x-\alpha(\eta-t),t-s)v_x(x-\alpha(\eta-t),s)dt \\ &= \frac{1}{\alpha}\int_s^\eta \mathcal{B}(\gamma)_\eta(x-\alpha(\eta-t),t-s)\frac{d}{dt}v(x-\alpha(\eta-t),s)dt. \end{aligned}$$

Therefore by an integration by parts and Fubini's Theorem again we see that (2.5) is equivalent to the following equation:

$$v(x,\eta) = f(x-\alpha\eta,0) + \int_0^\eta g(x-\alpha(\eta-t),t)dt + \sum_{i=1}^4 J_i v(x,\eta), \tag{3.3}$$

where each operator J_i is given by

$$\begin{aligned}
J_1 v(x,\eta) &= -\frac{1}{\alpha}\int_0^\eta \mathcal{B}(\gamma)_\eta(x,\eta-t)v(x,t)dt,\\
J_2 v(x,\eta) &= \frac{1}{\alpha}\int_0^\eta \mathcal{B}(\gamma)_\eta(x-\alpha(\eta-t),0)v(x-\alpha(\eta-t),t)dt,\\
J_3 v(x,\eta) &= \frac{1}{\alpha}\int_0^\eta\int_0^t \mathcal{B}(\gamma)_{\eta\eta}(x-\alpha(\eta-t),t-s)v(x-\alpha(\eta-t),s)dsdt,\\
J_4 v(x,\eta) &= \int_0^\eta\int_0^t \mathcal{B}(\gamma)_{x\eta}(x-\alpha(\eta-t),t-s)v(x-\alpha(\eta-t),s)dsdt.
\end{aligned}$$

In order to prove that the solution $v(x,\eta)$ of (3.3) satisfies the condition (ii) in Theorem 2.1 we employ the iteration method. Let us define $\{v_n(x,\eta)\}_{n=0}^\infty$ as follows:

$$v_0(x,\eta) := f(x-\alpha\eta,0) + \int_0^\eta g(x-\alpha(\eta-t),t)dt.$$

For $n \geq 0$,

$$v_{n+1}(x,\eta) := v_0(x,\eta) + \sum_{i=1}^4 J_i v_n(x,\eta). \tag{3.4}$$

Next, we define $\{w_n(x,\eta)\}_{n=0}^\infty$ by $w_0(x,\eta) := v_0(x,\eta)$ and $w_n(x,\eta) = v_n(x,\eta) - v_{n-1}(x,\eta)$ $(n \geq 1)$, and define $\{W_n(x,\eta,t)\}_{n=0}^\infty$ by

$$W_n(x,\eta,t) := w_n(x-\alpha(\eta-t),t). \tag{3.5}$$

Definition 3.1 (1) For $\lambda \geq 0$ and $\rho > 0$, $U_\rho[0,\lambda]$ denotes the ρ-neighborhood of $[0,\lambda]$ in $\mathbf{C}$.

(2) For $\eta \in \mathbf{C}$, we define the function $G_\eta(\tau)$ by

$$G_\eta(\tau) = \tau e^{i\arg(\eta)}, \quad \tau \in \mathbf{C},$$

and define G_η and G_η^ρ as follows:

$$\begin{aligned}
G_\eta &:= \{G_\eta(R) \in \mathbf{C};\ 0 \leq R \leq |\eta|\},\\
G_\eta^\rho &:= \{G_\eta(\tau) \in \mathbf{C};\ \tau \in U_\rho[0,|\eta|]\}.
\end{aligned}$$

We remark that G_η is the segment from 0 to η and that G_η^ρ is the ρ-neighborhood of G_η.

Now we can take $r_0 > 0$ and $\kappa_0 > 0$ such that

$$\{x-\alpha\zeta;\ |x| \leq r_0,\ \zeta \in E_+(\theta,\kappa_0)\} \subset E_+(\theta+\pi+\arg(\alpha),\kappa). \tag{3.6}$$

So let us define $\gamma(x,\zeta,\varepsilon)$ by

$$\gamma(x,\zeta,\varepsilon) := \gamma(x-\alpha\zeta,\varepsilon). \tag{3.7}$$

Then it follows from the assumption (A2) and (3.6) that $\gamma(x,\zeta,\varepsilon)$ is holomorphic on $\{x \in \mathbf{C};\ |x| \le r_0\} \times E_+(\theta,\kappa_0) \times \{\varepsilon \in \mathbf{C};\ |\varepsilon| \le c\}$. Moreover it holds that

$$M_0 := \sup_{\{x\in\mathbf{C};\ |x|\le r_0\}\times E_+(\theta,\kappa_0)\times\{\varepsilon\in\mathbf{C};\ |\varepsilon|\le c\}} |\gamma(x,\zeta,\varepsilon)| < \infty. \tag{3.8}$$

Next let us define $\mathcal{B}(\gamma)(x,\zeta,\eta)$ by

$$\mathcal{B}(\gamma)(x,\zeta,\eta) := \mathcal{B}(\gamma)(x-\alpha\zeta,\eta) \ \left(= \sum_{n=1}^{\infty} \gamma_n(x-\alpha\zeta)\frac{\eta^n}{n!}\right). \tag{3.9}$$

Then it follows from (3.8) and Cauchy's integral formula that $\mathcal{B}(\gamma)(x,\zeta,\eta)$ is holomorphic on $\{x \in \mathbf{C};\ |x| \le r_0\} \times E_+(\theta,\kappa_0) \times \mathbf{C}$ and that there exist some positive constants M_1 and δ_0 such that

$$\begin{cases} \displaystyle\sup_{\{x\in\mathbf{C};\ |x|\le r_0\}\times E_+(\theta,\kappa_0)} \left|\frac{1}{\alpha}\mathcal{B}(\gamma)_\eta(x,\zeta,\eta)\right| \le M_1 e^{\delta_0|\eta|}, & \eta\in\mathbf{C}, \\ \displaystyle\sup_{\{x\in\mathbf{C};\ |x|\le r_0\}\times E_+(\theta,\kappa_0)} \left|\frac{1}{\alpha}\mathcal{B}(\gamma)_{\eta\eta}(x,\zeta,\eta)\right| \le M_1 e^{\delta_0|\eta|}, & \eta\in\mathbf{C}, \\ \displaystyle\sup_{\{x\in\mathbf{C};\ |x|\le r_0\}\times E_+(\theta,\kappa_0')} \left|\frac{1}{\alpha}\frac{d}{d\zeta}\mathcal{B}(\gamma)_\eta(x,\zeta,\eta)\right| \le M_1 e^{\delta_0|\eta|}, & \eta\in\mathbf{C}, \end{cases} \tag{3.10}$$

where $\kappa_0' = \kappa_0/2$.

Under these preparations let us take a monotonically decreasing positive sequence $\{\rho_n\}_{n=0}^{\infty}$ satisfying

$$\widetilde{\kappa} := \kappa_0' - \sum_{n=0}^{\infty} \rho_n > 0. \tag{3.11}$$

Then we obtain the following lemma:

Lemma 3.1 *$W_n(x,\eta,t)$ is continued analytically to $\{(x,\eta,t);\ |x| \le r_0,\ \eta \in E_+(\theta,\kappa_0' - \sum_{j=0}^{n}\rho_j),\ t \in G_\eta^{\rho_n}\}$. Moreover on $\{(x,\eta,t);\ |x| \le r_0,\ \eta \in E_+(\theta,\kappa_0' - \sum_{j=0}^{n}\rho_j),\ t \in G_\eta\}$ we have the following estimate: For some positive constant C_1,*

$$|W_n(x,\eta,G_\eta(R))| \le C_1 e^{\delta_1|\eta|}(2M_1)^n \sum_{k=n}^{2n} \binom{n}{k-n}\frac{R^k}{k!}, \tag{3.12}$$

where $\delta_1 = \max\{\delta|\alpha|, \delta_0\}$.

If we admit Lemma 3.1, Theorem 1.2 is proved as follows: It follows from Lemma 3.1 that $w_n(x,\eta)$ $(= W_n(x,\eta,\eta))$ is continued analytically to $B(r_0)\times E_+(\theta,\kappa_0'-\sum_{j=0}^n \rho_j)$ with the estimate

$$\begin{aligned} |w_n(x,\eta)| &= |W_n(x,\eta,G_\eta(|\eta|))| \\ &\le C_1 e^{\delta_1|\eta|}(2M_1)^n \sum_{k=n}^{2n} \binom{n}{k-n}\frac{|\eta|^k}{k!}. \end{aligned}$$

Hence on $B(r_0)\times E_+(\theta,\widetilde{\kappa})$ we obtain

$$\begin{aligned} \sum_{n=0}^{\infty}|w_n(x,\eta)| &\le C_1 e^{\delta_1|\eta|}\sum_{n=0}^{\infty}(2M_1)^n \sum_{k=n}^{2n}\binom{n}{k-n}\frac{|\eta|^k}{k!} \\ &\le \widetilde{C}e^{\widetilde{\delta}|\eta|}, \end{aligned}$$

for some positive constants $\widetilde{C}$ and $\widetilde{\delta}$.

This shows that $v_n(x,\eta)$ $(=\sum_{k=0}^n w_k(x,\eta))$ converges to the solution $V(x,\eta)$ of (3.3) uniformly on $B(r_0)\times E_+(\theta,\widetilde{\kappa})$. Therefore $V(x,\eta)$ is an analytic continuation of $v(x,\eta)$ and it holds that

$$\max_{|x|\le r_0}|V(x,\eta)|\le \widetilde{C}e^{\widetilde{\delta}|\eta|},\quad \eta\in E_+(\theta,\widetilde{\kappa}).$$

It follows from the above argument that $v(x,\eta)$ satisfies the condition (ii) in Theorem 2.1. This completes the proof of Theorem 1.2. ∎

Therefore it is sufficient to prove Lemma 3.1.

Proof of Lemma 3.1. It is proved by the induction. First we consider the case $n=0$. $W_0(x,\eta,t)$ has the following form:

$$\begin{aligned} W_0(x,\eta,t) &= f(x-\alpha\eta,0)+\int_0^t g(x-\alpha(\eta-s),s)ds \\ &=: I_1(x,\eta,t)+I_2(x,\eta,t). \end{aligned}$$

Before proving the lemma for W_0, we remark the following: It follows from the assumption (A1) and Cauchy's integral formula that $g(x,\eta)$ is holomorphic on $E_+(\theta+\pi+\arg(\alpha),\kappa)\times\mathbf{C}$ with the estimate

$$(3.13)\qquad |g(x,\eta)|\le C'e^{\delta|x|}e^{\delta'|\eta|},\quad (x,\eta)\in E_+(\theta+\pi+\arg(\alpha),\kappa)\times\mathbf{C},$$

for some positive constants C' and δ'.

Let us prove that $I_1(x,\eta,t)$ and $I_2(x,\eta,t)$ are well-defined on $\{(x,\eta,t);\ |x|\leq r_0,\ \eta\in E_+(\theta,\kappa_0'-\rho_0),\ t\in G_\eta^{\rho_0}\}$. Let $|x|\leq r_0$, $\eta\in E_+(\theta,\kappa_0'-\rho_0)$, $t\in G_\eta^{\rho_0}$, and let us write $t\in G_\eta^{\rho_0}$ as $t=G_\eta(\tau)$ ($\tau\in U_{\rho_0}[0,|\eta|]$).

On the well-definedness of $I_1(x,\eta,G_\eta(\tau))$: It is clear from the assumption (A1) and (3.6).

On the well-definedness of $I_2(x,\eta,G_\eta(\tau))$: In the integral expression of $I_2(x,\eta,G_\eta(\tau))$, by taking an integral path as

$$s(\sigma)=\sigma e^{i\arg(\eta)}\quad(\sigma\in[0,\tau]),\tag{3.14}$$

where $[0,\tau]$ is a segment from 0 to τ, it holds that $\eta-s(\sigma)\in E_+(\theta,\kappa_0')$ $(\subset E_+(\theta,\kappa_0))$. Hence it follows from (3.6) and the above remark that $I_2(x,\eta,G_\eta(\tau))$ is well-defined.

Therefore $W_0(x,\eta,t)$ is well-defined on $\{(x,\eta,t);\ |x|\leq r_0,\ \eta\in E_+(\theta,\kappa_0'-\rho_0),\ t\in G_\eta^{\rho_0}\}$. Moreover on $\{(x,\eta,t);\ |x|\leq r_0,\ \eta\in E_+(\theta,\kappa_0'-\rho_0), t\in G_\eta\}$ we have the following representation:

$$\begin{aligned}W_0(x,\eta,G_\eta(R)) &= f(x-\alpha\eta,0)\\ &\quad+\int_0^R g(x-\alpha(|\eta|-R_1)e^{i\arg(\eta)},R_1e^{i\arg(\eta)})e^{i\arg(\eta)}dR_1\\ &=: \mathcal{I}_1(x,\eta,R)+\mathcal{I}_2(x,\eta,R).\end{aligned}$$

Let us estimate each $\mathcal{I}_1(x,\eta,R)$ and $\mathcal{I}_2(x,\eta,R)$.

On $\mathcal{I}_1(x,\eta,R)$: It follows from (1.11) that

$$\begin{aligned}|\mathcal{I}_1(x,\eta,R)| &= |f(x-\alpha\eta,0)|\leq Ce^{\delta|x-\alpha\eta|}\\ &\leq C''e^{\delta|\alpha||\eta|},\end{aligned}$$

where $C''=Ce^{\delta r_0}$.

On $\mathcal{I}_2(x,\eta,R)$: It follows from (3.13) that

$$|g(x-\alpha(|\eta|-R_1)e^{i\arg(\eta)},R_1e^{i\arg(\eta)})|\leq C'''e^{\delta|\alpha||\eta|}e^{-\delta|\alpha|R_1}e^{\delta'R_1}=C'''e^{\delta|\alpha||\eta|}e^{-(\delta|\alpha|-\delta')R_1},$$

where $C'''=C'e^{\delta r_0}$. Here we may take $\delta>0$ so large that $\delta'':=\delta|\alpha|-\delta'>0$. Hence we obtain

$$|\mathcal{I}_2(x,\eta,R)|\leq C'''e^{\delta|\alpha||\eta|}\int_0^R e^{-\delta''R_1}dR_1\leq\frac{C'''}{\delta''}e^{\delta|\alpha||\eta|}.$$

By the above argument, we have

$$|W_0(x,\eta,G_\eta(R))|\leq C_1e^{\delta|\alpha||\eta|}\leq C_1e^{\delta_1|\eta|},$$

where $C_1 = C'' + C'''/\delta''$. Therefore the case $n = 0$ is proved.

Next, we assume that the claim of the lemma is proved up to n and prove it for $n+1$. By (3.4) and (3.5) we have the following relation between W_n and W_{n+1}:

$$W_{n+1}(x,\eta,t) = \sum_{i=1}^{4} \mathcal{J}_i W_n(x,\eta,t), \tag{3.15}$$

where

$$\begin{aligned}
\mathcal{J}_1 W_n(x,\eta,t) &= J_1 w_n(x-\alpha(\eta-t),t) \\
&= -\frac{1}{\alpha}\int_0^t \mathcal{B}(\gamma)_\eta(x,\eta-t,t-s)W_n(x,\eta-t+s,s)ds, \\
\mathcal{J}_2 W_n(x,\eta,t) &= J_2 w_n(x-\alpha(\eta-t),t) \\
&= \frac{1}{\alpha}\int_0^t \mathcal{B}(\gamma)_\eta(x,\eta-s,0)W_n(x,\eta,s)ds, \\
\mathcal{J}_3 W_n(x,\eta,t) &= J_3 w_n(x-\alpha(\eta-t),t) \\
&= \frac{1}{\alpha}\int_0^t\int_0^s \mathcal{B}(\gamma)_{\eta\eta}(x,\eta-s,s-y)W_n(x,\eta-s+y,y)dyds, \\
\mathcal{J}_4 W_n(x,\eta,t) &= J_4 w_n(x-\alpha(\eta-t),t) \\
&= -\frac{1}{\alpha}\int_0^t\int_0^s \frac{d}{d\zeta}\mathcal{B}(\gamma)_\eta(x,\zeta,s-y)\Big|_{\zeta=\eta-s} \cdot W_n(x,\eta-s+y,y)dyds.
\end{aligned}$$

Let us prove that each $\mathcal{J}_i W_n(x,\eta,t)$ ($i=1, 2, 3, 4$) is well-defined on $\{(x,\eta,t);\ |x| \le r_0,\ \eta \in E_+(\theta,\kappa_0' - \sum_{j=0}^{n+1}\rho_j),\ t \in G_\eta^{\rho_{n+1}}\}$ by taking suitable integral paths. Let us write $t \in G_\eta^{\rho_{n+1}}$ as $t = G_\eta(\tau)$ ($\tau \in U_{\rho_{n+1}}[0,|\eta|]$).

On $\mathcal{J}_1 W_n(x,\eta,G_\eta(\tau))$: Let us take an integral path as (3.14). Then we have $\eta - G_\eta(\tau) + s(\sigma) \in E_+(\theta,\kappa_0' - \sum_{j=0}^{n}\rho_j)$ and $s(\sigma) \in G^{\rho_n}_{\eta-G_\eta(\tau)+s(\sigma)}$. Hence $W_n(x,\eta-G_\eta(\tau)+s(\sigma),s(\sigma))$ is well-defined. It is clear that $\mathcal{B}(\gamma)_\eta(x,\eta-G_\eta(\tau),G_\eta(\tau)-s(\sigma))$ is well-defined. Therefore $\mathcal{J}_1 W_n(x,\eta,G_\eta(\tau))$ is well-defined.

On $\mathcal{J}_2 W_n(x,\eta,G_\eta(\tau))$: Let us take an integral path as (3.14). Then we have $\eta \in E_+(\theta,\kappa_0' - \sum_{j=0}^{n}\rho_j)$ and $s(\sigma) \in G_\eta^{\rho_n}$. Hence $W_n(x,\eta,s(\sigma))$ is well-defined. It is clear that $\mathcal{B}(\gamma)_\eta(x,\eta-s(\sigma),0)$ is well-defined. Therefore $\mathcal{J}_2 W_n(x,\eta,G_\eta(\tau))$ is well-defined.

On $\mathcal{J}_3 W_n(x,\eta,G_\eta(\tau))$ and $\mathcal{J}_4 W_n(x,\eta,G_\eta(\tau))$: We only state the integral paths. The suitable integral paths are (3.14) and

$$y(\lambda) = \lambda e^{i\arg(\eta)}, \quad (\lambda \in [0,\sigma]), \tag{3.16}$$

for both $\mathcal{J}_3 W_n(x,\eta,G_\eta(\tau))$ and $\mathcal{J}_4 W_n(x,\eta,G_\eta(\tau))$.

By taking the above integral paths, we see that each $\mathcal{J}_i W_n(x,\eta,t)$ ($i=1, 2, 3, 4$) is well-defined (therefore $W_{n+1}(x,\eta,t)$ is well-defined) on $\{(x,\eta,t);\ |x| \le r_0,\ \eta \in E_+(\theta,\kappa_0' -$

$\sum_{j=0}^{n+1}\rho_j)$, $t\in G_\eta^{\rho_{n+1}}\}$. Moreover on $\{(x,\eta,t);\ |x|\le r_0,\ \eta\in E_+(\theta,{\kappa_0}'-\sum_{j=0}^{n+1}\rho_j),\ t\in G_\eta\}$ we have the following representations:

$$\begin{aligned}
\mathcal{J}_1W_n(x,\eta,G_\eta(R)) &= -\frac{1}{\alpha}\int_0^R \mathcal{B}(\gamma)_\eta(x,(|\eta|-R)e^{i\arg(\eta)},(R-R_1)e^{i\arg(\eta)})\\
&\qquad\qquad \times \mathcal{W}_n(x,\eta,R,R_1)e^{i\arg(\eta)}dR_1,\\
\mathcal{J}_2W_n(x,\eta,G_\eta(R)) &= \frac{1}{\alpha}\int_0^R \mathcal{B}(\gamma)_\eta(x,(|\eta|-R_1)e^{i\arg(\eta)},0)\mathcal{W}_n(x,\eta,R_1,R_1)e^{i\arg(\eta)}dR_1,\\
\mathcal{J}_3W_n(x,\eta,G_\eta(R)) &= \frac{1}{\alpha}\int_0^R\int_0^{R_1} \mathcal{B}(\gamma)_{\eta\eta}(x,(|\eta|-R_1)e^{i\arg(\eta)},(R_1-R_2)e^{i\arg(\eta)})\\
&\qquad\qquad \times \mathcal{W}_n(x,\eta,R_1,R_2)\{e^{i\arg(\eta)}\}^2dR_2dR_1,\\
\mathcal{J}_4W_n(x,\eta,G_\eta(R)) &= -\frac{1}{\alpha}\int_0^R\int_0^{R_1} \frac{d}{d\zeta}\mathcal{B}(\gamma)_\eta(x,\eta,(R_1-R_2)e^{i\arg(\eta)})\Big|_{\zeta=(|\eta|-R_1)e^{i\arg(\eta)}}\\
&\qquad\qquad \times \mathcal{W}_n(x,\eta,R_1,R_2)\{e^{i\arg(\eta)}\}^2dR_2dR_1,
\end{aligned}$$

where

$$\mathcal{W}_n(x,\eta,\mu,\nu) = W_n(x,(|\eta|-\mu+\nu)e^{i\arg(\eta)},G_{(|\eta|-\mu+\nu)e^{i\arg(\eta)}}(\nu)). \tag{3.17}$$

Let us estimate each $\mathcal{J}_iW_n(x,\eta,G_\eta(R))$.

On $\mathcal{J}_1W_n(x,\eta,G_\eta(R))$: It follows from the assumption of the induction that

$$|\mathcal{W}_n(x,\eta,R,R_1)| \le C_1e^{\delta_1|\eta|}e^{-\delta_1R}e^{\delta_1R_1}(2M_1)^n\sum_{k=n}^{2n}\binom{n}{k-n}\frac{{R_1}^k}{k!}. \tag{3.18}$$

Hence (3.10) and $\delta_0\le\delta_1$ imply that

$$\begin{aligned}
|\mathcal{J}_1W_n(x,\eta,G_\eta(R))| &\le C_1e^{\delta_1|\eta|}M_1(2M_1)^n\sum_{k=n}^{2n}\binom{n}{k-n}\int_0^R\frac{{R_1}^k}{k!}dR_1\\
&= C_1e^{\delta_1|\eta|}M_1(2M_1)^n\sum_{k=n}^{2n}\binom{n}{k-n}\frac{R^{k+1}}{(k+1)!}.
\end{aligned}$$

On $\mathcal{J}_2W_n(x,\eta,G_\eta(R))$: Let us consider R_1 instead of R in (3.18). Then we have

$$|\mathcal{W}_n(x,\eta,R_1,R_1)| \le C_1e^{\delta_1|\eta|}(2M_1)^n\sum_{k=n}^{2n}\binom{n}{k-n}\frac{{R_1}^k}{k!}.$$

Therefore by (3.10), it holds that

$$|\mathcal{J}_2W_n(x,\eta,G_\eta(R))| \le C_1e^{\delta_1|\eta|}M_1(2M_1)^n\sum_{k=n}^{2n}\binom{n}{k-n}\frac{R^{k+1}}{(k+1)!}.$$

By the above argument it holds that

$$
\begin{aligned}
(3.19)\qquad & |\mathcal{J}_1 W_n(x,\eta,G_\eta(R))| + |\mathcal{J}_2 W_n(x,\eta,G_\eta(R))| \\
\le\ & C_1 e^{\delta_1|\eta|}(2M_1)^{n+1}\sum_{k=n}^{2n}\binom{n}{k-n}\frac{R^{k+1}}{(k+1)!} \\
=\ & C_1 e^{\delta_1|\eta|}(2M_1)^{n+1}\sum_{k=n+1}^{2n+1}\binom{n}{k-(n+1)}\frac{R^k}{k!}.
\end{aligned}
$$

On $\mathcal{J}_3 W_n(x,\eta,G_\eta(R))$: It follows from the assumption of the induction that

$$
|\mathcal{W}_n(x,\eta,R_1,R_2)| \le C_1 e^{\delta_1|\eta|}e^{-\delta_1 R_1}e^{\delta_1 R_2}(2M_1)^n\sum_{k=n}^{2n}\binom{n}{k-n}\frac{{R_2}^k}{k!}.
$$

Hence (3.10) implies that

$$
\begin{aligned}
|\mathcal{J}_3 W_n(x,\eta,G_\eta(R))| \ &\le\ C_1 e^{\delta_1|\eta|}M_1(2M_1)^n\sum_{k=n}^{2n}\binom{n}{k-n}\int_0^R\int_0^{R_1}\frac{{R_2}^k}{k!}dR_2dR_1 \\
&=\ C_1 e^{\delta_1|\eta|}M_1(2M_1)^n\sum_{k=n}^{2n}\binom{n}{k-n}\frac{R^{k+2}}{(k+2)!}.
\end{aligned}
$$

On $\mathcal{J}_4 W_n(x,\eta,G_\eta(R))$: Similarly to the calculation for $\mathcal{J}_3 W_n(x,\eta G_\eta(R))$, we have

$$
|\mathcal{J}_4 W_n(x,\eta,G_\eta(R))| \le C_1 e^{\delta_1|\eta|}M_1(2M_1)^n\sum_{k=n}^{2n}\binom{n}{k-n}\frac{R^{k+2}}{(k+2)!}.
$$

By the above argument it holds that

$$
\begin{aligned}
(3.20)\qquad & |\mathcal{J}_3 W_n(x,\eta,G_\eta(R))| + |\mathcal{J}_4 W_n(x,\eta,G_\eta(R))| \\
\le\ & C_1 e^{\delta_1|\eta|}(2M_1)^{n+1}\sum_{k=n}^{2n}\binom{n}{k-n}\frac{R^{k+2}}{(k+2)!} \\
=\ & C_1 e^{\delta_1|\eta|}(2M_1)^{n+1}\sum_{k=n+2}^{2(n+1)}\binom{n}{k-(n+1)-1}\frac{R^k}{k!}.
\end{aligned}
$$

Therefore it follows from (3.19) and (3.20) that

$$
\begin{aligned}
& |W_{n+1}(x,\eta,G_\eta(R))| \\
\le\ & \sum_{i=1}^{4}|\mathcal{J}_i W_n(x,\eta,G_\eta(R))| \\
\le\ & C_1 e^{\delta_1|\eta|}(2M_1)^{n+1} \\
& \times\left\{\frac{R^{n+1}}{(n+1)!} + \sum_{k=n+2}^{2n+1}\left\{\binom{n}{k-(n+1)} + \binom{n}{k-(n+1)-1}\right\}\frac{R^k}{k!} + \frac{R^{2(n+1)}}{\{2(n+1)\}!}\right\} \\
=\ & C_1 e^{\delta_1|\eta|}(2M_1)^{n+1}\sum_{k=n+1}^{2(n+1)}\binom{n+1}{k-(n+1)}\frac{R^k}{k!},
\end{aligned}
$$

which implies the lemma for $n+1$. The proof is completed. ∎

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