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Formal solutions of the complex heat equation in higher spatial dimensions

Werner Balser* and Stéphane Malek†
Abteilung Angewandte Analysis
Universität Ulm
89069 Ulm, Germany
balser@mathematik.uni-ulm.de

Abstract

We present a result on summability of power series in one variable, whose coefficients are holomorphic functions of several other complex variables. This result then is applied to the Cauchy problem for the heat equation in several spatial variables.

1 Introduction

In very recent articles, formal power series solutions of partial differential equations ***in two variables*** have been investigated: Some authors determined their Gevrey order, while others have been concerned with their (multi-)summability properties. Without claim of completeness, we here mention, in alphabetical order, *W. Balser* [1,3,4], *Balser and Kostov* [5], *Balser and Miyake* [6], *Chen, Luo, and Zhang* [7], *Gérard and Tahara* [8], *M. Hibino* [9–13], *K. Ichinobe* [14], *Lutz, Miyake, and Schäfke* [15], *M. Miyake* [17–20], *Miyake and Hashimoto* [21], *Miyake and Yoshino* [22–24], *S. Ōuchi* [25–28], and *Pliś and Ziemian* [29].

A first attempt to generalize results from [3] to the case of more than two variables has been made by *S. Malek* [16]. He considered a general PDE with constant coefficients, but required several technical assumptions in order to be able to adapt the proofs from [3] to this situation. In this paper we shall study the heat equation in several spatial dimensions, but follow a different approach: First, we shall generalize a lemma from [5] to the case of power series in more than two variables. Then we shall apply this result and briefly indicate the chances as well as the technical difficulties arising in cases of more general PDE.

2 Summability of series with variable coefficients

In this and later sections we shall be concerned with holomorphic functions in several complex variables, and it shall make sense to seperate these variables into two groups, denoted as $z = (z_1, \ldots, z_n)$ resp. $w = (w_1, \ldots, w_m)$, with non-negative integers n and m. While the case of $n = 0$ shall not be of interest here, it makes sense to allow that $m = 0$, in which case we should interprete functions of z and w as being ***independent of*** $w_1, \ldots, w_m$.

Let $(x_j(z,w))_{j\geq 0}$ be a given sequence of functions that are holomorphic in a polydisc $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$ about the origin of $\mathbb{C}^n \times \mathbb{C}^m$, and let $k > 0$ and $d \in \mathbb{R}$ be given. Then the formal power series

$$\hat{x}(t,z,w) = \sum_{j=0}^{\infty} \frac{t^j}{j!} x_j(z,w) \tag{2.1}$$

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is said *to be k-summable in the direction d*, if the following two conditions hold:

(a) There exist $\rho, \rho_1 \in \mathbb{R}$ such that the series

$$y(t,z,w) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1+s_+ j)}\, x_j(z,w), \qquad s_+ = 1+1/k,$$

is absolutely convergent for $\|(z,w)\|_\infty = \sup\{|z_1|,\dots,|z_n|,|w_1|,\dots,|w_m|\} \le \rho_1$ and $|t| < \rho$.

(b) There exists $\delta > 0$ such that, for all (z,w) as above, the function $y(t,z,w)$ can be analytically continued with respect to t into the sector $S_{d,\delta} = \{t \in \mathbb{C} : 2|d - \arg(t)| < \delta\}$. Moreover, for all $\delta_1 < \delta$ there exist $C > 0$ and $K > 0$ such that

$$\sup_{\|(z,w)\|_\infty \le \rho_1} |y(t,z,w)| \le C\, e^{K|t|^k} \qquad \forall\, t \in S_{d,\delta_1}.$$

Functions satisfying such an estimate in every such subsector S_{d,δ_1} of $S_{d,\delta}$ shall be said to be of *exponential growth in $S_{d,\delta}$ at most of order k.*

This definition of k-summability is slightly modified to better suit the situation of formal solutions of PDE. From the general theory of moment summability presented in [2, Section 6.5] one can deduce equivalence of this and the standard definition of *J.-P. Ramis* [30,31]. However, observe that with the definition given here, the k-sum $x(t,z,w)$ of the formal series $\hat{x}(t,z,w)$ is not obtained as the Laplace transform of index k, with respect to t, of the function $y(t,z,w)$; instead, one has to use *J. Ecalle's acceleration operator* corresponding to the indices 1 and $1/s_+$ – this, however, shall not be of importance here.

As the main tool for this article, we shall prove a lemma that rephrases k-summability of formal series of the form (2.1) in terms of infinitely many formal power series whose coefficients are independent of the variables $z = (z_1,\dots,z_n)$. To formulate this result, we shall use the following notation: By $\nu = (\nu_1,\dots,\nu_n)$ we always denote a *multi-index*; i. e., the entries ν_j are non-negative integers. We shall write $|\nu| = \nu_1 + \dots + \nu_n$ for *the length of* ν, and $\partial_z^\nu = \partial_{z_1}^{\nu_1} \dots \partial_{z_n}^{\nu_n}$ for the operator of partial differentiation of orders $\nu_1,\dots,\nu_n$ with respect to the variables $z_1,\dots,z_n$, respectively. In addition, we set $\nu! = \nu_1! \cdot \dots \cdot \nu_n!$

Lemma 1 *Let $k > 0$, $d \in \mathbb{R}$, and $\hat{x}(t,z,w)$ as in (2.1) be given. Then the following statements are equivalent:*

(a) *The formal series $\hat{x}(t,z,w)$ is k-summable in the direction d.*

(b) *There exist $\rho, \rho_1, \delta > 0$, such that for $s_+ = 1 + 1/k$ and every multi-index ν the series*

$$y_\nu(t,w) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1+s_+ j)}\, x_{j,\nu}(w), \qquad x_{j,\nu}(w) = \partial_z^\nu x_j(z,w)|_{z=0}, \tag{2.2}$$

converge for $|t| < \rho$ and $\|w\| \le \rho_1$, and the functions $y_\nu(t,w)$, for every such w, can be holomorphically continued with respect to t into the sector $S_{d,\delta}$. Finally, for every $\delta_1 < \delta$, there exist constants $C, K > 0$, independent of ν and w, so that

$$\sup_{\|w\| \le \rho_1} |y_\nu(t,w)| \le C^{|\nu|}\, \nu!\, e^{K|t|^k} \qquad \forall\, t \in S_{d,\delta_1}.$$

(c) *For every multi-index ν, the formal series*

$$\hat{x}_\nu(t,w) = \partial_{z_1}^\nu \hat{x}(t,z,w)|_{z=0} = \sum_{j=0}^{\infty} \frac{t^j}{j!}\, x_{j,\nu}(w)$$

all are k-summable in the direction d. Moreover, there exist a sectorial region G with bisecting direction d and opening greater than π/k and a polydisc $\mathcal{D}$ about the origin of $\mathbb{C}^m$ which both are

independent of ν, so that the sums $x_\nu(t,w)$ of $\hat{x}_\nu(t,w)$ all are holomorphic in $G \times \mathcal{D}$, and for every closed subsector $\bar{S} \in G$ there exist constants $C, K > 0$, independent of ν, such that

$$\sup_{t \in \bar{S}, w \in \mathcal{D}} |\partial_t^\ell x_\nu(t,w)| \leq C K^{|\nu|+\ell} \nu! \, \ell! \, \Gamma(1+\ell/k) \tag{2.3}$$

for all multi-indices ν and all non-negative integers ℓ.

Proof: For the special case of $n = 1$ and $m = 0$, a proof has been given in [4], and one can use the same approach for the general case. For this reason, we shall restrict ourselves and only present the main ideas: Assume that (a) holds, and let $y(t,z,w)$ be as in (2.2). Then $y_\nu(t,w)$ can be represented by the standard multi-dimensional Cauchy formula for partial derivatives. Estimating this formula in a standard manner then shows (b). For the converse implication, use the standard multi-dimensional Taylor expansion of $y(t,z,w)$ with respect to $z = (z_1, \ldots, z_n)$. Analogously one can prove equivalence of (a) and (c), using the sums $x(t,z,w)$ and $x_\nu(t,w)$ instead of $y(t,z,w)$ and $y_\nu(t,w)$. □

3 The heat equation in several spatial dimensions

In the following sections we shall apply Lemma 1 to the Cauchy problem for the heat equation in several spatial variables, for which we shall use the following convenient notation: For z and w as in the introduction, let $\phi(z,w)$ be a given function, holomorphic in a polydisc $\mathcal{D}$ about the origin of $\mathbb{C}^n \times \mathbb{C}^m$. Abbreviating

$$\Delta_z = \sum_{j=1}^{n} \partial_{z_j}^2, \qquad \Delta_w = \sum_{k=1}^{m} \partial_{w_k}^2,$$

we consider the Cauchy problem for the heat equation in $n+m$ spatial dimensions, written as

$$\partial_t u = (\Delta_z + \Delta_w) u, \qquad u(0,z,w) = \phi(z,w). \tag{3.1}$$

This problem has a unique formal power series solution $\hat{u}(t,z)$ which can be written as

$$\hat{u}(t,z) = \sum_{j=0}^{\infty} \frac{t^j}{j!} u_j(z,w), \quad u_j(z,w) = (\Delta_z + \Delta_w)^j \phi(z,w) = j! \sum_{\substack{\mu,\ell \geq 0 \\ \mu+\ell=j}} \frac{\Delta_z^\mu \Delta_w^\ell}{\mu! \, \ell!} \phi(z,w). \tag{3.2}$$

For $n = 1$ and $m = 0$, or in other words, for one spatial dimension, this formal series has been investigated in detail in [15] and [1]: In general, its Gevrey order is equal to $s = 1$, but for entire functions $s < 1$ may occur as well. Moreover, it is shown in [15] that the series is 1-summable in a direction d if, and only if, the initial condition can be holomorphically continued into the union of two sectors with bisecting directions $d/2$ and $\pi + d/2$ and is of exponential growth at most of order 2 there. An analogous result has been obtained in [1] for the case of k-summability, with $k > 1$, however, in this situation the condition for k-summability in a direction d cannot be formulated in terms of the initial condition but involves its Laplace transform of a corresponding order. Nothing was known so far about the summability of (3.2) in the case of several spatial dimensions, since this case is not covered by the results obtained in [16]. Here, we shall prove results quite analogous to those in the one-dimensional situation, except that the conditions we obtain are less easy to verify.

Remark 1: Note that in (3.1) the essential quantity is the number of spatial variables $n+m$, and it is up to us to decide how to subdivide this number into n and m. It shall turn out to be convenient to choose $m = 0$ when discussing matters where all spatial variables are of equal importance, while for the question of summability we shall take $n = 1$. □

Since the initial condition $\phi(z,w)$ is assumed to be holomorphic in a polydisc about the origin which we shall, for simplicity of notation, assume to be the Cartesian product of discs of equal radius denoted by $r > 0$, we see that the same holds for the coefficients $x_j(z,w)$, for all $j \geq 0$. Expanding these functions with respect to $z = (z_1, \ldots, z_n)$, we have

$$\phi(z,w) = \sum_\nu \frac{z^\nu}{\nu!} \phi_\nu(w), \quad u_j(z,w) = \sum_\nu \frac{z^\nu}{\nu!} u_{j\nu}(w) \qquad \|(z,w)\|_\infty < r, \tag{3.3}$$

where summation extends over all multi-indices ν in dimension n. Observing $x_0(z,w) = \phi(z,w)$ and $x_{j+1}(z,w) = (\Delta_z + \Delta_w)\, x_j(z,w)$, we find the following relations for the coefficients of these series, for all multi-indices ν:

$$u_{0\nu}(w) \;=\; \phi_\nu(w)\,, \quad u_{j+1,\nu}(w) \;=\; \Delta_w u_{j,\nu} \;+\; \sum_{k=1}^{n} u_{j,\nu+2e_k}(w) \qquad \forall\, \|w\|_\infty < r\,,\; j \geq 0\,, \tag{3.4}$$

with e_k denoting the kth unit vector in dimension n.

4 Gevrey estimates

The notion of Gevrey estimates that is discussed in this section is symmetric with respect to all spatial variables, and for this reason we shall without loss of generality restrict to the case of $m = 0$; if this were not so, we could set $z_{n+k} = w_k$ for $1 \leq k \leq m$ and then replace n by $n+m$.

Let $s \geq 0$ be given. Due to the form of the formal solution $\hat{u}(t,z)$, we set $s_+ = s+1$ and say that such a series is (at most) *of Gevrey order* s, provided that we can find constants $\rho, C, K > 0$ such that

$$|u_j(z)| \;\leq\; C\, K^j\, \Gamma(1 + s_+ j) \qquad \forall\, j \geq 0\,, \quad \|z\|_\infty \leq \rho\,. \tag{4.1}$$

Note that this definition, when the functions $x_j(z)$ all are constants, coincides with the standard definition of the Gevrey order of power series. Moreover, observe that a series is of Gevrey order $s = 0$ if, and only if, it converges (for sufficiently small $|t| > 0$). As we shall show now, the Gevrey order of (3.2) is independent of the spatial dimension:

Lemma 2 *For $m = 0$ and arbitrary $\phi(z)$, holomorphic in a polydisc $\mathcal{D} \subset \mathbb{C}^n$ about the origin, the series* (3.2) *is of Gevrey order* 1.

Proof: In the case $m = 0$, all functions $u_{j\nu}(w)$ and $\phi_\nu(w)$, defined by (3.3), are constants which we shall denote as $u_{j\nu}$ and ϕ_ν. We set

$$c_{j\ell} \;=\; \sum_{|\nu|=\ell} \frac{|u_{j\nu}|}{\nu!} \qquad \forall\, j,\ell \geq 0\,.$$

From (3.4) we conclude that

$$c_{j+1,\ell} \;\leq\; \sum_{k=1}^{n} \sum_{|\nu|=\ell} \frac{|u_{j,\nu+2e_k}|}{(\nu+2e_k)!}\, (\nu_k+1)\,(\nu_k+2) \;\leq\; (\ell+1)\,(\ell+2)\, c_{j,\ell+2} \qquad \forall\, j,\ell \geq 0\,.$$

Cauchy's formula in several dimensions shows that $|c_{0\nu}| = |\phi_\nu| \leq C\, K^{|\nu|}$ for every multi-index ν, with sufficiently large $C, K > 0$. Using this, one can show by induction with respect to j the estimate $c_{j\nu} \leq C\, K^{j+\nu}\, (\ell+2j)!/\ell!$, from which follows that the series $\sum_{j,\ell} c_{j\ell}\, \rho^\ell\, x^j/(2j)!$ converges for sufficiently small $x, \rho > 0$. This and the fact that

$$|u_j(z)| \;\leq\; \sum_{\ell=0}^{\infty} \sum_{|\nu|=\ell} \frac{|u_{j\nu}|}{\nu!}\, |z_1|^{\nu_1} \cdot \ldots \cdot |z_n|^{\nu_n} \;\leq\; \sum_{\ell=0}^{\infty} \rho^\ell\, c_{j\nu} \qquad \forall\, \|z\|_\infty \leq \rho$$

complete the proof. □

While the Gevrey order of $\hat{u}(t,z)$ is never larger than 1, it may well be smaller, and in some cases the series even may converge:

Lemma 3 *Let $m = 0$ and $0 \leq s < 1$, and assume that the initial condition $\phi(z)$ is entire and, for some $C, K > 0$, satisfies*

$$|\phi(z)| \;\leq\; C \exp\left(K\, \rho^{2/(1-s)}\right) \qquad \forall\, \rho > 0\,, \quad \|z\|_\infty \leq \rho\,. \tag{4.2}$$

Then $\hat{u}(t,z)$ is of Gevrey order s, and in particular converges for $s = 0$.

Proof: Observe that Cauchy's formula for the coefficients of a power series (in several variables) implies that $|\phi_\nu| \leq \rho^{-|\nu|}\, C\, \exp(K\, \rho^{2/(1-s)})$ for every $\rho > 0$ and z as in (4.2). Taking ρ such that the right hand side becomes minimal, one then obtains, with $C, K > 0$ not necessarily the same as above:

$$|\phi_\nu| \;\leq\; \frac{C\, K^{|\nu|}}{\Gamma(1 + (1-s)|\nu|/2)}$$

for all multi-indices ν. Proceeding exactly as in the proof of the previous lemma, using this improved estimate for the coefficients of $\phi(z)$, one can complete the proof. □

Remark 2: Observe that the proofs of both lemmata can be generalized to give the same result for equations where Δ_z is replaced by $\sum_j a_j\, \partial_{z_j}^2$, with arbitrary non-zero constants a_j, or even more general ones. □

5 Summability of the formal solution

In Section 2 we showed that summability of a series of the form (2.1) is equivalent to that of the series $\hat{x}_\nu(t,w)$ plus an estimate of the form (2.3) for their sums. To discuss summability of the formal solution of the heat equation (3.1), we shall take $n = 1$ and arbitrary $m \geq 0$, and define $\hat{u}_\nu(t,w)$ as in (3.3), observing that for $n = 1$ multi-indices ν are just integer numbers ≥ 0. In this situation we prove the following result:

Theorem 1 *For $\hat{u}(t,z,w)$ as in (3.2), with $n = 1$ and arbitrary $m \geq 0$, we choose $d \in \mathbb{R}$, $k \geq 1$, and set $s_+ = 1 + 1/k$. Then the following statements are equivalent:*

(a) *The formal solution $\hat{u}(t,z,w)$ is k-summable in the direction d.*

(b) *There exist $\rho, \rho_1, \delta > 0$, such that for $\nu = 0$ and $\nu = 1$ the series*

$$v_\nu(t,w) \;=\; \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1 + s_+\, j)}\; u_{j,\nu}(w)\,, \qquad u_{j,\nu}(w) = \partial_z^\nu u_j(z,w)|_{z=0}\,, \tag{5.1}$$

converge for $|t| < \rho$ and $\|w\|_\infty \leq \rho_1$, and the functions $v_\nu(t,w)$, for every such w, can be holomorphically continued with respect to t into the sector $S_{d,\delta}$ and is of exponential order at most k there.

(c) *For $\nu = 0$ and $\nu = 1$, the formal series*

$$\hat{u}_\nu(t,w) \;=\; \partial_{z_1}^\nu \hat{u}(t,z,w)|_{z=0} \;=\; \sum_{j=0}^{\infty} \frac{t^j}{j!}\; u_{j,\nu}(w)$$

both are k-summable in the direction d.

Proof: If (a) holds, then Lemma 1 can be applied and shows that (b) and (c) hold as well. Moreover, by definition of k-summability we see that (b) is equivalent to (c). This leaves to show, e. g., that (c) implies (a). To do so, observe that for $n = 1$ the relation (3.4) becomes

$$u_{0\nu}(w) \;=\; \phi_\nu(w)\,, \quad u_{j+1,\nu}(w) \;=\; \Delta_w u_{j,\nu} \;+\; u_{j,\nu+2}(w) \qquad \forall\, \|w\|_\infty < r\,,\; \nu, j \geq 0\,. \tag{5.2}$$

This shows that $\hat{u}_{\nu+2}(t,w) = (\partial_t - \Delta_w)\, \hat{u}_\nu(t,w)$ for $\nu \geq 0$, and from this and the general theory of k-summability we conclude that all $\hat{u}_\nu(t,w)$ are k-summable in the direction d. Moreover, if $u_\nu(t,w)$ are their sums, then they satisfy

$$u_{\nu+2}(t,w) \;=\; (\partial_t \;-\; \Delta_w)\; u_\nu(t,w) \qquad \forall\, \nu \geq 0\,, \quad (t,w) \in G \times \mathcal{D}\,,$$

with a sectorial region G of opening larger than π/k and bisecting direction d, and a suitably small polydisc $\mathcal{D}$. Observe that this relation also guarantees that G does not depend upon ν. Expanding

$$u_\nu(t,w) = \sum_\mu \frac{w^\mu}{\mu!}\, u_{\nu\mu}(t)\,,$$

with summation over all multi-indices μ in dimension m, we obtain through differentiation with respect to t (ℓ times) the relation

$$u^{(\ell)}_{\nu+2,\mu}(t) = u^{(\ell+1)}_{\nu,\mu}(t) - \sum_{k=1}^{m} u^{(\ell)}_{\nu,\mu+2e_k}(t)$$

for all $t \in G$, all multi-indices μ, and $\nu \geq 0$. Choosing a closed subsector $\overline{S}$ of G, we set

$$u_{\ell\nu j} = \sup_{t\in\overline{S}} \sum_{|\mu|=j} \frac{|u^{(\ell)}_{\nu\mu}(t)|}{\mu!}$$

and obtain $u_{\ell,\nu+2,j} \leq u_{\ell+1,\nu,j} + (j+1)\,(j+2)\,u_{\ell,\nu,j+2}$ for all $\ell,\nu,j \geq 0$. By induction with respect to ν, this implies

$$u_{\ell,2\nu,j} \leq \sum_{\kappa=0}^{\nu} \frac{(j+2\kappa)!}{j!}\, u_{\ell+\nu-\kappa,0,j+2\kappa}\,, \qquad u_{\ell,2\nu+1,j} \leq \sum_{\kappa=0}^{\nu} \frac{(j+2\kappa)!}{j!}\, u_{\ell+\nu-\kappa,1,j+2\kappa}$$

for all $\ell,\nu,j \geq 0$. The assumption of k-summability of $\hat{u}_0(t,w)$, $\hat{u}_1(t,w)$ implies, with help of Lemma 1, that $C,K>0$ exist for which $|u^{(\ell)}_{\nu,\mu}(t)| \leq C\,K^{\nu+|\mu|}\,\mu!\,\Gamma(1+s_+\ell)$ for $t \in \overline{S}$, $\nu=0$ and $\nu=1$, and all μ,ℓ. Using this, one can complete the proof, very much along the line of the proofs of Lemmas 2 and 3. □

We can improve this result by setting

$$\tilde{v}(t,w) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1+s_+j/2)}\, \tilde{u}_j(w)\,, \qquad \tilde{u}_{2j}(w) = u_{j0}(w) \quad \tilde{u}_{2j+1}(w) = u_{j1}(w)\,. \tag{5.3}$$

In terms of this *auxiliary function*, we can show:

Theorem 2 *For $\hat{u}(t,z,w)$ as in (3.2), with $n=1$ and arbitrary $m \geq 0$, we choose $d \in \mathbb{R}$, $k \geq 1$, and set $s_+ = 1+1/k$. Then the formal solution $\hat{u}(t,z,w)$ is k-summable in the direction d if, and only if, there exist $\rho_1,\rho_2,\delta>0$ so that the series(5.3) converges for $|t|<\rho_1$ and $|w|<\rho_2$, and the function $\tilde{v}(t,w)$, for fixed w, can be holomorphically continued into the two sectors $S_{d/2,\delta}$ and $S_{\pi+d/2,\delta}$, and is of exponential growth at most of order $2k$ there.*

Proof: The function $\tilde{v}(t,w)$ has the properties stated if, and only if, the same properties hold for its odd and even parts, and according to the definition of summability this is equivalent to $(2k)$-summability in the directions $d/2$ and $\pi+d/2$ of the series

$$\sum_{j=0}^{\infty} \frac{t^{2j}}{\Gamma(1+s_+j)}\, u_{j0}(w)\,, \qquad \sum_{j=0}^{\infty} \frac{t^{2j+1}}{\Gamma(1+s_+(1/2+j))}\, u_{j1}(w)\,.$$

The general theory then implies that this is equivalent to condition (c) of Theorem 1. □

Remark 3: Using (3.2) for the case of $n=1$, one can show that

$$u_{j0}(w) = j! \sum_{\substack{\mu,\ell\geq 0\\ \mu+\ell=j}} \frac{\Delta_w^\ell\, \phi_{2\mu}(w)}{\mu!\,\ell!}\,, \qquad u_{j1}(w) = j! \sum_{\substack{\mu,\ell\geq 0\\ \mu+\ell=j}} \frac{\Delta_w^\ell\, \phi_{2\mu+1}(w)}{\mu!\,\ell!} \qquad \forall\, j \geq 0\,. \tag{5.4}$$

Hence for $n=1, m=0$, we find $u_{j0}=\phi_{2j}$, $u_{j1}=\phi_{2j+1}$, so $\tilde{v}(t) = \sum_j t^j\,\phi_j/\Gamma(1+s_+j/2)$, which is equal to $\phi(t)$ for $k=1$, i. e., $s_+=2$. So in these cases, Theorem 2 coincides with the results obtained in [15] for $k=1$, resp. in [1] for $k>1$.

In the general case we can, in principle, compute the auxiliary function $\tilde{v}(t,w)$ in terms of the initial condition $\phi(z,w)$, and then verify whether or not the conditions for k-summability of $\hat{u}(t,z,w)$ given in Theorem 2 are satisfied. Vice versa, it it also possible to start with a function $\tilde{v}(t,w)$ that satisfies these conditions, and from its coefficients $\tilde{u}_j(w)$ find the functions $\phi_\nu(w)$, for $\nu \geq 0$, using the relations (5.4). Doing so, one can (theoretically) find examples of initial conditions $\phi(z,w)$ leading to k-summable series $\hat{u}(t,z,w)$. Unfortunately, the authors have not been able (except for the case of $m=0$ and $n=1$) to determine explicitely those cases of $\phi(z,w)$ for which k-summability holds. □

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