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Energy Exchange and Excitation of Internal Modes in Near Separatrix Soliton Collisions

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Solitary wave collisions are of interest in a diverse variety of physical settings. We discuss the near separatrix soliton collisions in a number of integrable, Hamiltonian systems under weak and moderate perturbations. In the weak perturbation regime, the radiationless energy exchange reported in our recent works can take place under the conditions of attractive soliton interactions and of the number of free soliton parameters being larger than the number of invariant properties. In the moderate perturbation regime, the soliton internal modes can be excited for a particular sign of perturbation parameter and they can strongly enhance the energy exchange between solitons to the extent of complete annihilation of some of them.

1. Introduction

Solitons are the exact solutions to the integrable nonlinear equations. The reason why solitons are so stable is the infinite number of conserved quantities for such equations. Dynamical properties of the system are severely restricted by the existence of an infinite number of conservation laws. However, the integrable equations describe roughly idealized physical systems and realistic applications demand the inclusion of various perturbations. In the literature there exist quite a lot of data on the soliton collisions in various nearly integrable and non-integrable models [1-8]. Collisions between intrinsic localised modes have also been studied [9]. It has been demonstrated that the result of soliton collision, even in the regime of weak perturbation, may differ drastically from the prediction obtained from the integrable limit [10-13].

In this paper we continue investigation of the phenomena related to the collisions between solitons. We formulate necessary conditions to observe a strong energy exchange in the weakly perturbed integrable systems. In the case of weak perturbation, the energy exchange is the only possible manifestation of inelasticity of collision. We demonstrate that, in the case of moderate perturbation, the soliton internal modes can be excited and they may strongly affect the outcome of the soliton interactions. It is well known that the unusual effects observed in soliton collisions can be attributed to the existence of the separatrix solutions [7,8,14]. We divide the separatrix solutions into two classes, the separatrices in the space of parameters defining the energy of solitons and the separatrices in the space of parameters that do not affect the soliton energies. It is then demonstrated that the existence of the separatrices of the second kind implies the probabilistic nature of the soliton collisions in the perturbed systems.

2. Three-soliton collisions in SGE

The integrable sine-Gordon equation (SGE)

$$u_{tt} - u_{xx} + \sin u = 0, \qquad (1)$$

has the following discrete analogue

$$\frac{d^2 u_n}{dt^2} - \frac{1}{h^2}(u_{n-1} - 2u_n + u_{n+1}) + \sin u_n = 0, \qquad (2)$$

where h is the lattice spacing and h^2 will be used as a measure of discreteness. When $h^2 \to 0$, discrete equations (2) reduces to the continuum limit (1). $h^2 \sim 1$, $h^2 \sim 0.1$, and $h^2 \sim 0.01$ are the cases of strong, weak, and extremely weak discreteness. The physical meaning of this classification will become clear later.

Here we describe the possible outcome of collision between three kinks/antikinks in SGE in the regime of extremely weak discreteness ($h = 0.04$).

One particular three-soliton SGE solution is defined by nine parameters. Three of them influence the total energy of the system. In the case of three-kink solution these are the velocities of the kinks v_j, $j = 1,2,3$. Three other parameters define the positions of solitons $(x_0)_j$ at $t = 0$ (before the collisions) and three more define the topological charges of solitons, $q = 1$ for kink (K) and $q = -1$ for antikink ($\bar{\text{K}}$).

Energy E and momentum P of one SGE kink are defined by its velocity v as follows

$$E = 8\delta, \quad P = 8v\delta, \quad \delta^{-1} = \sqrt{1-v^2} . \qquad (3)$$

The SGE has separatrix solutions of two different kinds. One is the separatrix in the space of parameters that define the total energy of the system. For example, there exists a two-soliton separatrix solution with the energy equal to 16 (total momentum is assumed to be equal to zero). This solution is an intermediate one between the kink-antikink solution with the energy of two kinks greater than 16 and the breather solution with the energy less than 16. Some three-soliton separatrix solutions of this kind are given in [14]. The second type of separatrix solutions can be found in the space of parameters that do not affect the total energy [14].

We number the kinks in a way that at $t = 0$ (before the collisions) their positions are related as $(x_0)_1 < (x_0)_2 < (x_0)_3$ and momenta as $P_1 > P_2 > P_3$. Because of Lorentz invariance, we have only two independent momenta, say, P_1 and P_2. Here we vary only one of them, P_1, setting for the others $P_2 = 0$ and $P_3 = -P_1$, i.e., we restrict ourselves to symmetric collisions. Consideration of non-symmetric collisions does not bring any new important physical effects.

Parameters of kinks such as topological charges, q_j, or initial positions, $(x_0)_j$, do not affect energy and momentum of the system and thus, there is no physical meaning to *a priori* discriminate any set of these parameters.

Three solitons can pass through each other in two successive two-soliton collisions or in a three-soliton collision. In a weakly perturbed SGE, two-soliton collisions must be almost elastic because the energy exchange between them is forbidden by the energy and momentum conservation. For this reason we are interested in three-soliton collisions, which can be achieved by proper choice of the initial coordinate of, say, the middle kink, $(x_0)_2$. For the symmetric collisions it is convenient to set $(x_0)_1 = -(x_0)_3$ so that the three-soliton collisions are expected when $(x_0)_2$ is nearly zero.

Thus, we have the following parameters: momentum, P_1, the initial coordinate of the middle kink, $(x_0)_2$, which defines the collision phase; and finally, the topological charges of the kinks. There are eight variants to assign the charges to three kinks. Taking into account the symmetry, the eight variants are divided into three groups of topologically different collisions: $K\bar{K}K=\bar{K}K\bar{K}$, $KKK=\bar{K}\bar{K}\bar{K}$, and $KK\bar{K}=\bar{K}KK=K\bar{K}\bar{K}=\bar{K}\bar{K}K$. We will refer to the groups referring to their first members.

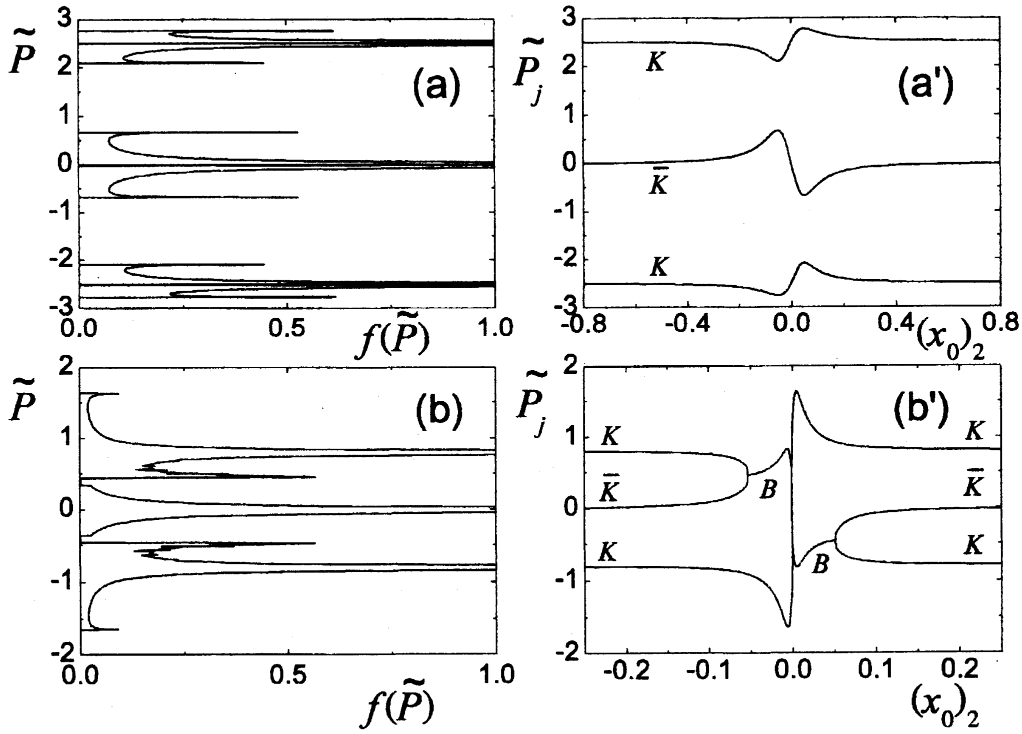

Fig. 1. Attractive three-soliton collisions, $K\bar{K}K$. Right panels show the momenta of particles after collision $\tilde{P}_i$ as the functions of collision phase, $(x_0)_2$, and left panels show the corresponding PDF. Collision with a high velocity, $P_1 = 2.5$, in (a,a') results only in quantitative change of kink parameters while collision with a small velocity, $P_1 = 0.8$, in (b,b') may result in fusion of a kink-antikink pair in a breather.

In the following we present the numerical results in the following way. We plot the soliton momenta after collision $\tilde{P}_i$ as the functions of $(x_0)_2$ for two different magnitudes of P_1. We assume that the collision is inelastic if $|(\tilde{P}_1 - P_1)/P_1| > \varepsilon$ and we set $\varepsilon = 0.01$. For *inelastic* collisions we plot the probability density function (PDF) $f_{\tilde{P}}(\tilde{P})$, such that $\int f_{\tilde{P}}(\tilde{P})d\tilde{P}_K = 1$. The PDF represents the result of inelastic collisions.

First we note that, KKK and KK$\bar{\text{K}}$ collisions are always elastic regardless P_1 and $(x_0)_2$ and only K$\bar{\text{K}}$K collisions can be inelastic. This is because only in this case the collision is of *attractive* type, when soliton cores of all three kinks can merge and the radiationless energy exchange between solitons can happen. Thus, if the probabilities for kinks to have positive or negative charge are equal, then energy exchange between three colliding kinks can be expected only in two cases from eight.

Let us focus on the attractive three-soliton collisions, K$\bar{\text{K}}$K. In Fig. 1 we show that for different magnitudes of P_1 there are possible two qualitatively different scenarios of three-kink collisions. When P_1 is sufficiently large ($> P_1^*$), only quantitative change in the system is possible [see Fig. 1 (a,a'), where $P_1 = 2.5$]. In this case, kink momenta after collision $\tilde{P}_i$ are different from the pre-collision momenta. Note that the right panels of Fig. 1 show the momenta of particles after collision $\tilde{P}_i$ while the left panels show the corresponding PDF. The threshold value of momentum P_1^* increases with increase in perturbation parameter, h^2. Note that inelastic collisions are observed in the vicinity of $(x_0)_2 = 0$, that is, when all three kinks participate in the collision. For $P_1 < P_1^*$, kink and antikink can merge in a breather [see Fig. 1 (b,b'), where $P_1 = 0.8$]. Here and later we assume that the two kinks constituting breather have equal momenta, that is why the two lines in (b') merge when K$\bar{\text{K}}$ pair merge in a breather (B). We note that the result of collision is extremely sensitive to the collision phase, $(x_0)_2$, in the vicinity of $(x_0)_2 = 0$, especially for small collision velocities, as in (b').

3. Three particle model

To give a clear explanation to the peculiarities of three-kink collisions described in the preceding section, let us consider the dynamics of three point-wise particles in one-dimensional space. Particles have masses $m = 8$, which is the rest mass of SGE kink, and they carry topological charges $q_j = \pm 1$. Particles with $q = 1$ and $q = -1$ will be called kinks and antikinks by analogy with SGE solitons. We assume that particles i and j having coordinates x_i and x_j interact via potential

$$U_{ij}(r_{ij}) = 16 + q_i q_j \frac{16}{\cosh(r_{ij})}, \qquad r_{ij} = x_j - x_i, \tag{4}$$

which in a crude approximation simulates the interaction between two SGE kinks/antikinks. Without the loss in generality we assume that total momentum in the system is equal to zero, $m(\dot{x}_1 + \dot{x}_2 + \dot{x}_3) = 0$. Introducing new variables $x_2 - x_1 \to \sqrt{3}\alpha + \beta$, $x_3 - x_1 \to 2\beta$, $t \to \sqrt{2m}\,t$, the three particle motion can be presented by the Hamiltonian of a unit-mass particle moving in the two-dimensional potential:

$$H = \frac{1}{2}\left(\dot{\alpha}^2 + \dot{\beta}^2\right) + U_{12}\left(\sqrt{3}\alpha + \beta\right) + U_{13}(2\beta) + U_{23}\left(\sqrt{3}\alpha - \beta\right). \tag{5}$$

Now we solve numerically the equations of motion for three particles and present the three-particle dynamics in the (α,β)-plane.

In Fig. 2, we compare the KKK, $KK\bar{K}$, and $K\bar{K}K$ collisions. For the three cases, the scattering potential in Eq. (5) is different because the charges of particles are different. When solitons move in (x,t) space toward the collision point, the representative particle moves in the (α,β)-plane along $\alpha=0$ toward the origin from the positive side. In (a), the like particles repel each other and, in (a'), particle hits the potential barrier and goes back. In (b), the particles collide in two successive two-soliton collisions. In this case, particle in (b') passes the two potential troughs one after another and then moves away from the origin in the direction symmetrically equivalent to the direction it came from. Cores of all three particles merge in the collision in (c) and the representative particle in (c') moves along the ridge of the scattering potential, passing the origin. This kind of motion is motion along the separatrix and, unlike the motion in (a') and (b'), it is very sensitive to small deviations from $(x_0)_2=0$.

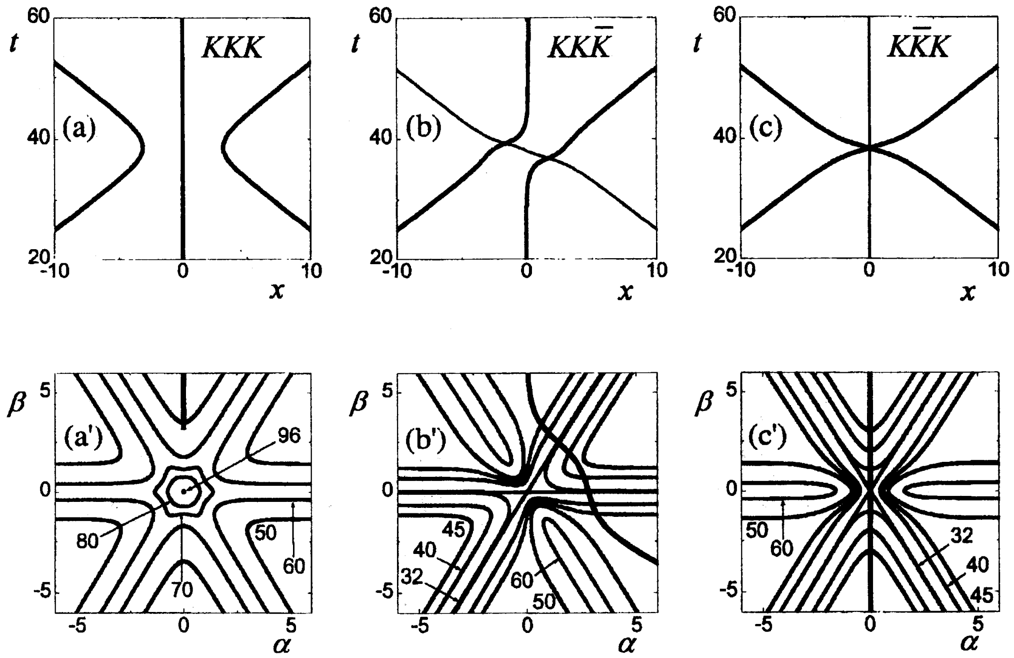

Fig. 2. Comparison of (a,a') KKK, (b,b') $KK\bar{K}$, and (c,c') $K\bar{K}K$ symmetric collisions for $(x_0)_1=-(x_0)_3=-25$, $(x_0)_2=0$ and $(\dot{x}_0)_1=-(\dot{x}_0)_3=0.6$, $(\dot{x}_0)_2=0$. Top panels show the three-particle dynamics in the (x,t) space while bottom panels show the corresponding dynamics of a particle in the (α,β)-plane.

The sensitivity of the result of near-separatrix collision to small deviations from $(x_0)_2=0$ is demonstrated by Fig. 3, where we set $(x_0)_2=1.2$ in (a,a'), $(x_0)_2=0.2$ in (b,b'), and $(x_0)_2=0.123323$ in (c,c'). In Fig. 3 (a,a'), the deviation from the separatrix is rather large and only quantitative change in the particle parameters can be seen. In (b,b'), collision is near-separatrix and here kink and antikink merge in a breather. Taking into account the time reversibility in the Hamiltonian systems, this picture can be also regarded as an illustration of the break-up of a breather colliding with a kink. Collision in (c,c') illustrates the origin of the fractal soliton scattering [4,10,11]. Note that the representative particle can oscillate in the scattering potential moving along $\beta=0$ line. This trajectory (periodic orbit) is obviously unstable and in the presence of any perturbation the particle will exponentially deviate from it. In (c,c'), we choose $(x_0)_2$ in a way that particle is sent almost along this trajectory and before it leaves the origin it makes a few oscillations in the saddle shape potential. Note that every time when particle passes the origin in (α,β)-plane all three particles in (x,t)-plane collide at one point. The time the particle spends near the origin of (α,β)-plane is the lifetime of the three-soliton bound state. When the scattering potential has periodic orbits, the probability p to observe a bound state with the lifetime T decreases algebraically, $p \sim T^{-\gamma}$ [11,15]. We also note that the collisions presented in Fig. 3 result in strong symmetry breaking, i.e., after the collision, particles do not recover their pre-collision velocities, though, the total momentum and energy are conserved exactly.

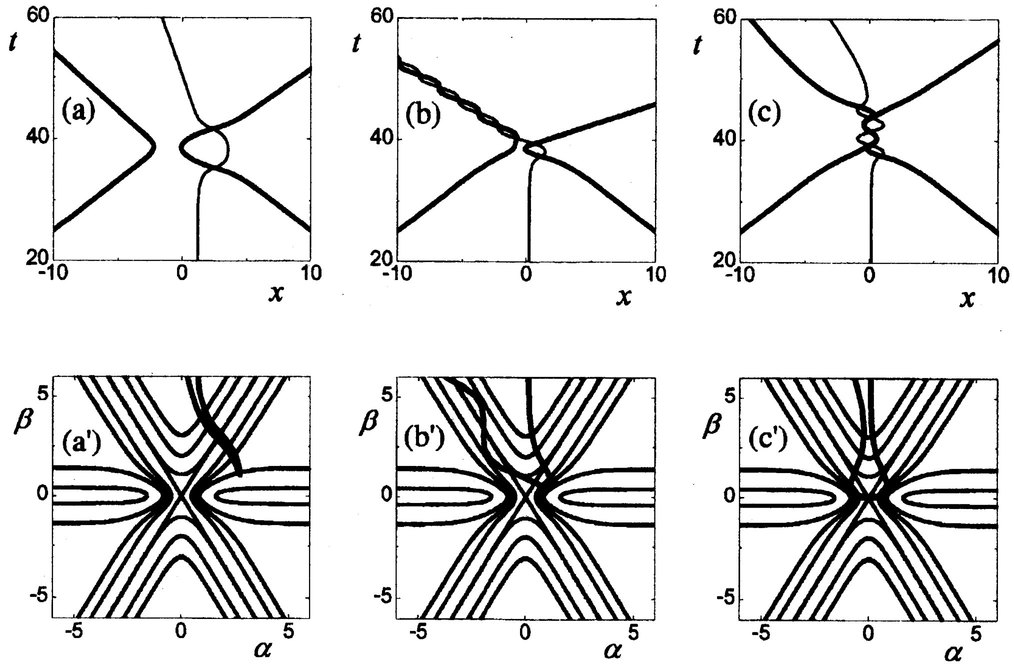

Fig. 3. The sensitivity of the result of near-separatrix collision to a small deviations from $(x_0)_2=0$ demonstrated by setting (a,a') $(x_0)_2=1.2$, (b,b') $(x_0)_2=0.2$, and (c,c') $(x_0)_2=0.123323$. In (a,a') only quantitative change in the system can be seen after collision. In the near-separatrix collision shown in (b,b'), kink and antikink merge in a breather. (c,c') illustrates the origin of the fractal soliton scattering.

3. Near-separatrix excitation of internal modes

The role of internal modes in near-separatrix collisions will be demonstrated for the nonlinear Schrödinger equation with quintic perturbation:

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = \varepsilon|\psi|^4\psi . \quad (6)$$

For extremely weak discreteness, $|\varepsilon| \sim 0.01$, the only manifestation of perturbation is the energy exchange between colliding solitons. For a moderate perturbation, $|\varepsilon| \sim 0.1$, the soliton internal modes can be excited for $\varepsilon < 0$ and new physical effects can be observed. We set $\varepsilon = -0.15$ and study the collisions between two symmetric solitons with initial velocities $v_1 = -v_2 = 0.15$ and amplitudes $A_1 = A_2 = 1$ for different collision phase, $-\pi < \varphi \leq \pi$. The separatrix collision corresponds to $\varphi = 0$. In Fig. 4 we show the amplitudes of solitons as the functions of time. Solitons collide at about $t = 60$. Collision in (a) at $\varphi = 1$ is rather far from the separatrix and the inelasticity of collision is small. At $\varphi = 0.5$ in (b) the energy exchange between solitons is already large but the internal modes are not excited yet. In (c), the collision is already close to the separatrix, $\varphi = 0.18$, and not only the energy exchange between solitons but also the excitation of the soliton internal mode become very pronounced. Collision in (d) at $\varphi = 0.01$ is very close to the separatrix and one of the solitons annihilates completely. The energy of this soliton is first given to the internal mode of the second soliton and then the energy of the internal mode gradually transforms into the energy of the remaining soliton that is why the lower envelop in (d) increases.

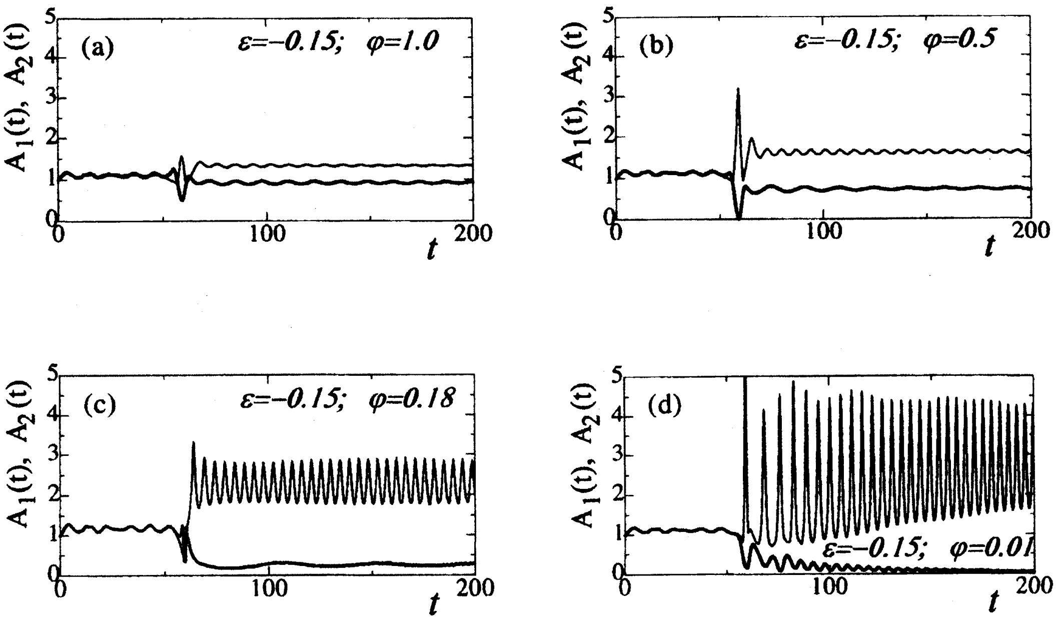

Fig. 4. Amplitudes of the two colliding solitons as the functions of time. Only collision phase φ is different for the four collisions presented in (a)-(d). Separatrix solution corresponds to $\varphi = 0$ so that moving from (a) to (d) the collision becomes closer to the separatrix. In (a), the inelasticity of collision is rather small, in (b) the energy exchange between solitons becomes large, in (c) the energy exchange is accompanied by the excitation of a large internal mode, and in (d) one of the solitons disappears giving its energy to excite a very large internal mode on the remaining soliton.

4. Discussion and conclusions

With the use of the SGE and NLSE as examples, we have formulated the two necessary conditions for the radiationless energy exchange in a nearly integrable system and consequently, for the probabilistic nature of their interaction. The conditions are: the energy exchange should not be forbidden by the conservation laws existing in the perturbed system and the collision should be of attractive type. In the weakly discrete SGE these conditions are satisfied when at least three kinks participate in the collision (because each kink has one parameter and there are two constraints from the energy and momentum conservation laws) and only when kinks meet each other in the spatial order $K\bar{K}K$ (condition of the attractive collision).

For example, in Korteweg-de Vries (KdV) equation, collisions are not probabilistic because soliton interactions are always mutually repulsive. In the NLSE, in-phase solitons attract, while out-of-phase solitons repel each other. One soliton has two parameters (amplitude and phase) and for many practically important perturbations there are two conserved quantities, the Hamiltonian and norm of the solution. Thus, the energy exchange between two nearly in-phase NLSE solitons is possible in the presence of a weak perturbation and the probabilistic character of soliton interactions should be respected.

In the limit of a weak perturbation, language of the probability theory naturally enters the soliton physics. For example, one can talk about probability to observe an inelastic collision or about the lifetime of a multi-soliton bound state.

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