数理解析研究所講究録 1369 巻 2004 年 85-120

COMPUTABLE STARTING CONDITIONS FOR THE EXISTENCE OF NON-UNIFORMLY HYPERBOLIC SYSTEMS

HIROKI TAKAHASI
GRADUATE SCHOOL OF SCIENCE, KYOTO UNIVERSITY

1. INTRODUCTION

We are interested in dynamical phenomena which are persistent under small perturbations of the system. Here, the meaning of persistence should be interpreted from the viewpoint of measure theory, and a positive Lyapunov exponent in one-dimensional system is our primary concern. Namely, we address the question when

$$|\{a\in\Omega:\liminf_{n\to\infty}\frac{1}{n}\log|Df_a^n(c_0)|>0\}|>0$$

is satisfied for a given parameterized family of unimodal maps $\{f_a\}_{a\in\Omega}$. There are numerous results concerning this subject. [BC85,91], [Tsu93b], [Lu99], [Yoc99], [Sen] give alternative proofs of the so-called Jakobson theorem [Ja81] on the quadratic family $Q_a\colon x\to 1-ax^2$. [TTY92], [Tsu93a],[MelStr93] extend these arguments to broader classes of families satisfying certain conditions. However, these conditions are in general hard to be verified for a given family $\{f_a\}_{a\in\Omega}$, i.e. not computable in practice, and hence are serious obstacle to application of these theorems. We intend to improve this point. We shall introduce computable (in principle, and hopefully in practice) starting conditions that guarantee the persistence of chaotic dynamics. This is a joint work with Stefano Luzzatto.

2. DEFINITIONS, NOTATIONS, AND PROPOSITIONS

To formulate our result, we introduce several definitions, notations, and propositions.

- **Unimodal map:** an interval map $f\colon[-1,1]\to[-1,1]$ is called *unimodal* if 0 is the unique critical point of f, i.e. $Df(0)=0$. A C^2 family of unimodal maps $\{f_a\}_{a\in\Omega}$ is a parameterized family of unimodal maps such that $(a,x)\to f_ax$ is C^2. We use the following notation, $c_i(a):=f_a^{i+1}(0)$.
- **Collet-Eckmann condition** [CE83]: We say a unimodal map f satisfies $(CE)_{n,\nu}$ if we have $|Df^k(c_0)|\geq e^{\nu k}$ for any $k\leq n$.
- **Essential return, Bounded recurrence:** [1] We say n is *not* an essential return for f_a if there exists $i<n$ such that

$$-\log|c_i(a)|\geq 2\sum_{\substack{i+1\leq j\leq n\\ c_j(a)\in(-\delta,\delta)}}-\log|c_j(a)|.$$

Otherwise n is called an *essential return.*

We say f_a satisfies $(BR)_{n,\alpha}$ if the following is true for all $k\leq n$:

$$\sum_{\substack{0\leq j\leq k\\ j:\text{essential return}}}-\log|c_j(a)|\leq\alpha k.$$

[1] We have temporarily and partly borrowed these formulations from [Tsu93b].

- **Cantor structure:** We say a nested sequence $\{E^{(i)}\}_{i=0}^{n-1}$ of closed subsets of $\mathbb{R}$ has (N,β) - *Cantor structure* of length n if the following is true:
  (i) $|E^{(0)}|>0$.
  (ii) $E^{(0)}=E^{(1)}=\cdots,E^{(N-1)}\supsetneq E^{(N)}\supset\cdots$.
  (iii) $|E^{(k)}|-|E^{(k+1)}|\le e^{-\beta k}|E^{(0)}|$.
  Notice that $|\bigcap_{0\le i\le n-1}E^{(i)}|>|E^{(0)}|(1-\sum_{i=N}^{n-2}e^{-\beta i})>0$ if we have $1-\sum_{i=N}^{\infty}e^{-\beta i}>0$.

- **Proposition A(n):** If f_a satisfies $(CE)_{n,\nu}$ and $(BR)_{n,\alpha}$, then it also satisfies $(CE)_{n+1,\nu}$.
- **Proposition B(n):** If f_a satisfies $(CE)_{n,\nu}$, then we have

$$D_1\le\frac{|\partial_a c_{n+1}(a)|}{|Df_a^{n+1}(c_0(a))|}\le D_2.$$

- **Proposition C(n):** $\{\Omega^{(i)}\}_{i=0}^{n}$ has the (N,β) - Cantor structure of length $n+1$.

- **(HYP):** There exist $\lambda>0$ and $\delta>0$ such that we have $|Df_a^n z|\ge e^{\lambda n}$ for any $a\in\Omega$, $n\ge1$ and $z\in I$ such that $z,f_a z,\cdots,f^{n-1}z\notin(-\delta,\delta)$.
- **(START):** (i) N is the smallest integer such that $\{c_n(a);a\in\Omega\}\cap(-\delta^\iota,\delta^\iota)\ne\emptyset$.
  (ii) $|\{c_N(a);a\in\Omega\}|\ge\delta^\iota$.
  (iii) $1-\left|\sum_{i=1}^{N}\frac{1}{(f_a^i)'(c_0)}\right|>0\quad\forall a\in\Omega$.
  (iv) $1-2\delta^{1-\iota}<e^{-\beta N}$, $0<\iota<1$.

3. Result

Main theorem. *Suppose (HYP) holds for given $\{f_a\}_{a\in\Omega}$, a C^2 family of unimodal maps. There exists a finite set of inequalities $\{*\}:=\{(START),(\mathcal{A}),(\mathcal{B}),(\mathcal{C})\}$ involving $\{f_a\}_{a\in\Omega}$ and $(\delta,\lambda,N,\alpha,\beta,\iota,\nu,D_1,D_2)$ such that the following flowchart does not stop forever provided that $\{*\}$ are satisfied.*

Corollary. *Suppose $\{f_a\}_{a\in\Omega}$ satisfies (HYP) and $\{*\}$. Then*

$$|\{a\in\Omega:\liminf_{n\to\infty}\frac{1}{n}\log|Df_a^n(c_0)|\ge\nu\}|\ge|\bigcap_{n=0}^{\infty}\Omega^{(n)}|>0.$$

If $a\in\bigcap_{n=0}^{\infty}\Omega^{(n)}$, then f_a has no periodic attractor. There exists a set $A\subset I$ of positive Lebesgue measure such that

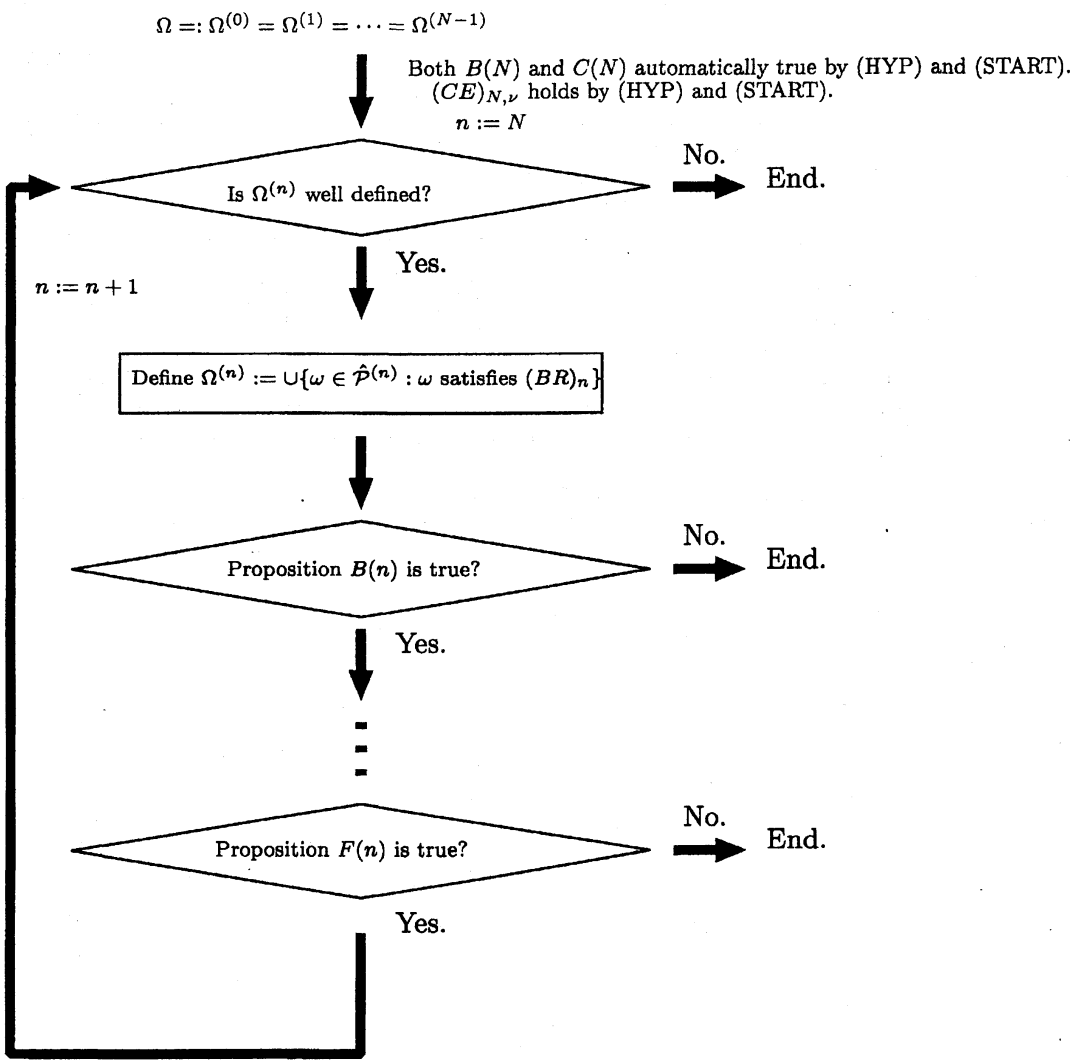
$$\liminf_{n\to\infty}\frac{1}{n}\log|Df_a^n(z)|>0\quad\textit{for any } z\in A.$$

We remark that (HYP) is very crucial in our argument. This means that as far as derivative growth along the critical orbit is concerned, we can restrict ourselves to take care of the time when it falls inside $(-\delta,\delta)$. It is reasonable to assume (HYP) at this moment due to ongoing work by Kokubu et al. which will give a test algorithm in order to examine if a given $\{f_a\}_{a\in\Omega}$ satisfies (HYP).

We believe that if we assume certain additional computable inequalities, f_a will be shown to be *non-uniformly expanding*, i.e. there exists $\lambda_e>0$ such that

$$\liminf_{n\to\infty}\frac{1}{n}\log|Df_a^n(z)|>\lambda_e\quad\text{for a.e. } z\in I.$$

In particular, f_a will admit an absolutely continuous invariant probability measure if f_a is C^3.



4. Proof of the main theorem.

Due to the structure of the above flowchart, it suffices to show the next three:

Lemma 1. *(HYP), $(START)$, and $(\mathcal{A})$ imply Proposition $A(n)$ for any $n \in \mathbb{N}$.*

Lemma 2. *(HYP), $(START)$, $(\mathcal{A})$, and $(\mathcal{B})$ imply Proposition $B(n)$ for any $n \in \mathbb{N}$.*

Lemma 3. *(HYP), $(START)$, $(\mathcal{A})$, $(\mathcal{B})$, $(\mathcal{C})$, $A(n-1)$, and $B(n)$ imply Proposition $C(n)$ for any $n \in \mathbb{N}$.*

We shall concentrate on the proof of Lemma 1, in which we will exploit the key notion of *binding* introduced in [BC85,91].

HIROKI TAKAHASI GRADUATE SCHOOL OF SCIENCE, KYOTO UNIVERSITY

References

[BC85] M. Benedicks and L. Carleson - On iterations of $1 - ax^2$ on $(-1, 1)$, *Ann. of Math.* **122** (1985), 1-25.

[BC91] M. Benedicks and L. Carleson - The dynamics of the Hénon map, *Ann. of Math.* **133** (1991), 73-169.

[CE83] P. Collet and J. P. Eckmann - Positive Lyapunov exponents and absolute continuity for maps of the interval, *Ergod. Th. and Dyn. Sys.* **3** (1983), 13-46.

[Ja81] M. Jakobson - Absolutely continuous invariant measures for one-parameter families of one-dimensional maps, *Comm. Math. Phys.* **81(1)** (1981), 39-88.

[Lu99] S. Luzzatto - Bounded recurrence of critical points and Jacobson theorem, *London Math, Soc, Lecture Note.* Ser **274** (1999). 173-210.

[MelStr93] W. de Melo and S. van Strien - *One-Dimensional Dynamics*, Springer, 1993.

[Sen] S. Senti - Dimension of weakly expanding points for quadratic maps, *To appear, in Bulletin de la Societe Mathematique de France.*

[TTY92] P. Thieullen, C. Tresser, and L-S. Young - Exposant de Lyapunov positif dans des familles a un parametre d applications unimodales, *C. R. Acad. Sci. paris Sér. I. Math* **315** (1992),no.1, 69-72.

[Tsu93a] M. Tsujii - Positive Lyapunov exponents in families of one-dimensional dynamical systems, *Invent Math.* **111** (1993), 113-137.

[Tsu93b] M. Tsujii - A proof of Benedicks-Carleson-Jacobson theorem, *Tokyo J. Math.* **16** (1993), 295-310.

[Yoc99] J-C. Yoccoz - Dynamique des polynômes quadratiques, *Panor. Synthéses.* **8** (1999), 187-222.

Appendix A
Abundance of stochastic dynamics for one-dimensional mappings

Hiroki Takahasi
Graduate School of Science, Kyoto University

March 16, 2004

Abstract

We give a detailed proof of the Jacobson theorem by making substantial modifications of the argument recently developed by Stefano Luzzatto.

1 Introduction

[1]In the study of dynamical systems, persistence of an invariant measure is an important problem. More specifically, let $f_\mu : N \to N$ be a map from a compact interval N to itself which is parameterized by $\mu \in A \subset \mathbb{R}$. One is interested in whether the set of parameter values corresponding to maps which carry an absolutely continuous invariant probability measure—a.c.i.p.—has positive Lebesgue measure.

A breakthrough in this direction is due to M. Jacobson [Ja] on the logistic family $f_a(x) = x^2 - a$.

Theorem (Jacobson). *There exists a parameter set with positive Lebesgue measure for which the corresponding map f_a admits an absolutely continuous invariant probability measure. In addition, $a = 2$ is a density point of such parameters.*

The central part of the proof given in his paper is an inductive construction, for a positive measure set of parameter values, of an induced Markov map which implies the existence of an a.c.i.p. Since this pioneering work, the subject of persistence of an a.c.i.p. in one-dimensional families has been under intense research, and there are numerous alternative proofs or generalizations of the Jacobson theorem available.

M. Benedicks & L. Carleson [BC85], [BC91] gave an alternative proof which involves inductive parameter selection, aimed at attaining the Collet-Eckmann condition (*CE*), an exponential growth condition of the derivative along the critical orbit [CE], for the remaining large parameter set.

On the other hand, J. Guckenheimer [Gu] and J-C. Yoccoz [Yoc91], [Yoc99] did not ask for (*CE*). The proof of Yoccoz is similar in flavor to Jacobson's.

Contrary to these, M. Tsujii [Tsu93b] took a completely different approach. He abandoned the use of an inductive argument. Instead, he estimated the Lebesgue measure of "bad sets" for which the corresponding maps violate (*CE*). Further, [Tsu93a] generalized the Jacobson theorem to multimodal families with non-degenerate critical points.

The primary reason why vast attention has been given to just one theorem is that necessary arguments are complicated and hence proofs cannot be simple, in spite of the great importance of the statement.

[1]This paper was submitted as a master thesis of the author.

Among those and other approaches, we would like to focus on the alternative recently given by S. Luzzatto [Lu]. His philosophy resembles Benedicks & Carleson approach, in the sense that it aims at attaining (*CE*) for a large set of parameter values by inductive parameter selection. However, Luzzatto's construction is both cleaner and more intuitive than the original work of Benedicks & Carleson.

One key difference is the simplification of imposed conditions which selected parameters are required to satisfy. To attain sufficient growth of the derivative along the orbit of the critical point, we need to impose some conditions on selected parameters. In [BC91], they require two conditions (*BA*) and (*FA*), which makes the inductive process considerably complicated. On the other hand, Luzzatto imposes just a single condition (*BR*), which effectively combines the previous two conditions.

Upon reading Luzzatto's proof, however, the author was unable to reconstruct some of the arguments not explicitly given in his paper. This read him to construct substantial modifications of some portions of the proof.

The present paper provides these modifications of Luzzatto's argument and establishes a consistent proof. This attempt will hopefully help to clarify several works on Hénon family [BC91], [WY01], and reformulate their arguments in terms of Luzzatto's approach.

The organization is as follows. §2 gives a statement of Luzzatto's formulation of the Jacobson theorem. §3 explains significance of revisiting one-dimensional argument in terms of a future perspective. In §4, we briefly explain delicate issues in Luzzatto's argument as well as strategies for overcoming them. From §5 to §10, we basically follow Luzzatto's argument, but making substantial modifications. The entire proof is essentially divided into two parts. In the first part from §5 to §7, we carry out inductive parameter selection to obtain good parameter values satisfying $BR(\alpha,\delta)$. In the second part from §8 to §10, we show that this parameter set has positive Lebesgue measure.

2 Statement of the result

We deal with the logistic family

$$f_a(x) = x^2 - a.$$

In what follows, we will introduce some system constants $0 < \hat{\lambda} < \log 2$, $\alpha > 0$, $\iota > 0$, $\kappa > 0$, $\delta > 0$ and $\epsilon > 0$, chosen in this order. For the parameter interval $\Omega_\epsilon := [2-\epsilon, 2]$ and each $j \in \mathbb{N}$, define the map $c_j : \Omega_\epsilon \to [-2,2]$ by $c_j(a) := f_a^{j+1}(0)$ and let $\Delta := (-\delta, \delta)$.

Definition. $a \in \Omega_\epsilon$ satisfies *the bounded recurrence condition* $BR(\alpha,\delta)_n$ if

$$\sum_{\substack{i=0 \\ c_i(a)\in\Delta}}^{k} \log |c_i(a)|^{-1} \le \alpha k$$

holds for all $0 \le k \le n$. For convenience we also allow to say f_a satisfies $BR(\alpha,\delta)_n$.

Theorem (Luzzatto). *Define*

$$\Omega_\epsilon^* := \{a \in \Omega_\epsilon : f_a \textit{ satisfies } BR(\alpha,\delta)_n \textit{ for all } n \ge 0\}.$$

Then, for arbitrarily small $\alpha > 0$, there exists $\delta > 0$ such that

$$\lim_{\epsilon\to 0} \frac{|\Omega_\epsilon^*|}{|\Omega_\epsilon|} = 1.$$

The Jacobson Theorem follows from this theorem, since $BR(\alpha,\delta)$ implies the Collet-Eckmann condition as shall be seen later.

3 Historical developments surrounding the Jacobson theorem

One of the main branches in the theory of dynamical systems is to classify generic diffeomorphisms. In this direction, S. Smale conjectured in the early sixties that in any dimension, the class of uniformly hyperbolic systems exhausts topologically almost all possibilities. But it turned out to be false as proven by S. Newhouse [Ne70], J. Palis & M. Viana [PV] with C^2-topology, and M. Shub [S], R. Mañé [M], C. Bonatti & L. J. Díaz [BD] in any dimension greater than 2 with C^1-topology. Therefore, it becomes important to study the complement of uniformly hyperbolic systems. Here, by uniformly hyperbolic systems, we mean a diffeomorphism whose non-wandering set admits an invariant splitting of the tangent bundle into uniformly expanding and contracting directions.

One of the known mechanisms which destroy hyperbolicity is the presence of folding where stable and unstable directions are mixed, or roughly speaking, homoclinic tangencies, a counterpart of critical points in unimodal or multimodal maps.

In spite of the presence of the above mechanism, systems may support some degree of hyperbolicity in terms of Lyapunov exponents and Oseledec decomposition. This broader notion is called ***nonuniform hyperbolicity***. In particular, the existence of a ***strange attractor***—a nonuniformly hyperbolic set attracting "many" orbits —implies ***sensitive dependence on initial conditions*** in observable region, and hence an observable chaotic behavior. Such systems are most likely meager in topological sense, due to C^2–Newhouse phenomenon. This means that measure theoretical persistence with respect to generic arcs of diffeomorphisms should be discussed. In the famous case of Hénon families, many systems were shown to have a strange attractor [BC91], [WY01]. However, as can be imagined from their works, it is very hard in general to show this sort of persistence for given nonhyperbolic systems.

Note that the techniques developed in [BC91], [WY01] are in many respects based on one-dimensional arguments concerning the Jacobson theorem. This means one cannot comprehend their results without having one-dimensional techniques at one's disposal.

4 Delicate issues to be considered

We mainly consider two delicate issues in Luzzatto's argument. One is related to the inductive construction of the nested sequence of parameter sets $\{\Omega^{(n)}\}_{n\geq 0}$ and the other concerns measure estimate of their intersection. For the sake of a precise description, some technical terms shall be used prior to their definitions. In particular, the reader should be referred to Lemma 5.3, 5.4 and §6.1, §6.2.

4.1 Return and escape, binding, bounded distortion

Let $\omega^{(\nu_i)} \in \mathcal{P}^{(\nu_i)}$, $\nu_{i-1} < \nu_i$ be two consecutive (essential) returns or (essential or substantial) escapes of $\omega^{(\nu_i)}$. By the inductive construction, there exists a parameter interval $\omega^{(\nu_{i-1})} \in \mathcal{P}^{(\nu_{i-1})}$ containing $\omega^{(\nu_i)}$. In other words, $\omega^{(\nu_i)}$ is obtained by deleting bad parameters from $\omega^{(\nu_{i-1})}$ which violate $BR(\alpha,\delta)_{\nu_i}$. To conclude $|\bigcap_{n\geq 0}\Omega^{(n)}| > 0$, it is crucial to estimate the ratio $|\omega^{(\nu_i)}|/|\omega^{(\nu_{i-1})}|$. In general, the length of a parameter interval at n-th inductive step gets smaller and smaller as the induction proceeds, and hence we need a bounded distortion argument concerning the map $c_j : \Omega_\varepsilon \to [-2,2]$. That is to say, the estimate of the above ratio is reduced to considering the quantity

$$\frac{|c_j(\omega^{(\nu_i)})|}{|c_j(\omega^{(\nu_{i-1})})|}$$

for some appropriate $j \in \mathbb{N}$. By the construction, one can easily see that if ν_{i-1} is either an essential return or an essential escape, then $c_{\nu_{i-1}}(\omega^{(\nu_{i-1})})$ occupies an element of the partition $\mathcal{I}^+$. Hence we can easily estimate the length $|c_{\nu_{i-1}}(\omega^{(\nu_{i-1})})|$. However, this is not enough. $|c_{\nu_{i-1}}(\omega^{(\nu_{i-1})})|$ is too small to estimate the ratio.

In the case where ν_{i-1} is an essential return, Luzzatto has overcome this "small denominator problem" by showing that the bounded distortion property holds until the end of a binding period [Lu; Lemma 5.2], and by deriving a uniform expansion property during the period [Lu; Lemma 4.3]. Now, a binding period p_{i-1} is associated to the essential return ν_{i-1}, and some derivative growth during the period contributes to uniform expansion of the size of the image via $c_{\nu_{i-1}+p_{i-1}+1}$, which is much greater than $|c_{\nu_{i-1}}(\omega^{(\nu_{i-1})})|$. Namely, we have

$$|c_{\nu_{i-1}+p_{i-1}+1}(\omega^{(\nu_{i-1})})| \geq |c_{\nu_{i-1}}(\omega^{(\nu_{i-1})})|^{8\beta} \gg |c_{\nu_{i-1}}(\omega^{(\nu_{i-1})})|,$$

where $\beta = \alpha/\lambda \ll 1$.

On the other hand, if ν_{i-1} is an essential escape, the same argument does not work in the context of Luzzatto's argument, since a binding period of essential escapes was not defined.

In order to fix this problem, we have defined a binding period of essential escapes and modified the bounded distortion argument [Lemma 9.1] so that it can deal with essential escapes. What we want to concude is the following:

Proposition. *Let $\omega \in \mathcal{P}^{(\nu)}$, ν an essential escape, and p be the corresponding binding period. Then, there exists a constant $D = D(\delta)$ such that*

$$\frac{|c_k'(a)|}{|c_k'(b)|} \leq D$$

for any $a, b \in \omega$ and $0 \leq k \leq \nu + p + 1$. In addition, D stays bounded as $\delta \to 0$.

There is no obstruction to defining a binding period of essential escapes, because the notion of *binding* or a binding period, a replication process of the critical orbit introduced in [BC85], is purely topological, and both *return* and *escape* are topologically equivalent in the sense that at these times the orbit of the critical point comes close to the critical point.

There is, however, a serious obstruction to extending the bounded distortion argument to essential escapes. To illustrate this, let $\nu' < \nu$ the last free return before ν and p' be its binding period. We need to find a proper upper bound of the quantity

$$\sum_{j=\nu'+p'+1}^{\nu} \frac{|c_j(\omega)|}{\inf_{a\in\omega} |c_j(a)|}.$$

Suppose that $c_\nu(\omega)$ is very close to the boundary of $\Delta^+ = (-\delta^\iota, \delta^\iota)$, namely, $|c_\nu(\omega)| \sim (e^{-r_{\delta^+}} - e^{-(r_{\delta^+}+1)})/r_{\delta^+}^2$. Then, an upper bound of the numerator is given by

$$|c_j(\omega)| \leq e^{\hat{\lambda}(\nu-j)}|c_\nu(\omega)| \sim e^{\hat{\lambda}(\nu-j)}(e^{-r_{\delta^+}} - e^{-(r_{\delta^+}+1)})/r_{\delta^+}^2.$$

One can easily see that the right hand side has the order higher than δ, since δ is taken after ι is specified. On the other hand, since ν' is a return, $c_j(\omega)$ may come close to the boundary of Δ for some $\nu' + p' + 1 \leq j \leq \nu - 1$, and hence the denominator is not compatible with the numerator as δ tends to 0, which leads to failure of the argument.

This problem is overcome by specifying the above j as an *inessential escape* with its binding period, and accordingly decomposing the above sum into pieces to estimate them one by one. More specifically, let $\mu_1 < \mu_2, \cdots, < \mu_u$ be all inessential escapes between $\nu'+p'$ and ν and ρ_k ($k = 1, \cdots, u$) be the corresponding binding periods. The sum is decomposed as follows:

$$\sum_{j=\nu'+p'+1}^{\nu} \frac{|c_j(\omega)|}{\inf_{a\in\omega}|c_j(a)|} = \sum_{j=\nu'+p'+1}^{\mu_1+p_1} \frac{|c_j(\omega)|}{\inf_{a\in\omega}|c_j(a)|} + \sum_{k=1}^{u-1}\sum_{j=\mu_k+p_k+1}^{\mu_{k+1}+p_{k+1}} \frac{|c_j(\omega)|}{\inf_{a\in\omega}|c_j(a)|} + \sum_{j=\mu_u+p_u+1}^{\nu} \frac{|c_j(\omega)|}{\inf_{a\in\omega}|c_j(a)|},$$

which enables more detailed analysis to obtain a proper distortion constant.

However, we need to consider how other parts of the entire argument in [Lu] are affected by these considerations. For example, there is a chance that what Luzzatto regarded as an inessential return turns out to be a bound return associated with the previous essential or inessential escape (we have observed that such cases do not happen in reality [Sublemma 7.1.3.]). In all, it is necessary to examine how several types of these recurrent times are distributed in the history of a time sequence. This shall be thoroughly discussed in §6,§7.

For convenience, we make it a rule to refer to both essential and inessential escapes as *escapes*, in order to make clear the difference from substantial escapes.

These crucial arguments, together with other minor modifications, will allow us to deal with escapes and returns similarly when estimating the Lebesgue measure of parameter sets. It seems difficult to find another way to deal with escapes. Finally, we stress that substantial escapes must be treated differently.

4.2 Extension of the period during which *BR* holds

Suppose that f_a satisfies $BR(\alpha,\delta)_k$ and $c_k(a) \in (-2\delta^\iota, 2\delta^\iota)$. After the recurrence, the orbit keeps track of its initial piece during the binding period. Hence, it is expected that f_a satisfies $BR(\alpha,\delta)_{k+p}$. This is, however, not true. Nevertheless, we can ensure that the period during which *BR* holds is properly extended, in order to proceed the inductive argument. This is formulated in Lemma 5.4. The difficulty for proving the lemma is to find a way to cope with the situation in which two bound orbits fall separately, one inside the neighborhood Δ and the other outside Δ. This can be manipulated by introducing the *regularity of bound returns* and weakening the condition *BR*. More specifically, we treat both $BR(\alpha,\delta)_n$ and $BR(5\alpha,\delta)_n$ from situation to situation.

We remark that a similar argument, suggested by a comment made by Luzzatto [Lu; Sublemma 5.1.3] works. He argued that one can avoid the above problems, by slightly modifying the definition of *BR*, namely one should shrink the critical neighborhood Δ as the induction proceeds. However, even if this modification were valid, it does not work in higher dimensional cases. For instance, consider the Hénon family

$$H_{a,b}(x,y) = (1 - ax^2 + y, bx).$$

In order to have an analogy with one-dimensional argument, one must shrink the dissipation $b > 0$ as much as necessary, keeping the size of a neighborhood of critical regions. These arguments are seen in [BC91] and [WY01].

5 Preliminary lemmas

Let β_a denote one of the fixed points of f_a bigger than the other. Put

$$K_a := \bigcap_{n\geq 0} f_a^{-n}([-\beta_a,\beta_a]),$$

which equals $[-\beta_a,\beta_a]$ if and only if $a \in [-1/4, 2]$.

5.1 Hyperbolic behavior

Lemma 5.1. *For all $0 < \hat{\lambda} < \log 2$ and $\delta > 0$ small, there exist constants $\epsilon > 0$ and $C_\delta > 0$ such that the following hold for any $a \in \Omega_\epsilon$. If $x \in K_a$ satisfies $x, f_a(x), \cdots, f_a^{n-1}(x) \notin \Delta$, then*

$$|(f_a^n)'(x)| \geq C_\delta e^{\hat{\lambda} n} \tag{1}$$

In addition, if $|f_a^n(x)| \leq |x|$, then

$$|(f_a^n)'(x)| \geq e^{\hat{\lambda} n}. \tag{2}$$

Let $\iota > 0$ be such that $\iota < \frac{4\alpha}{\hat{\lambda} - 2\alpha} < 1$. If $|x|$, $|f_a^n(x)| \leq 2\delta^\iota$, then we have

$$|(f_a^n)'(x)| \geq \frac{1}{2} e^{\hat{\lambda} n}. \tag{3}$$

Proof. Let g_2 be a continuous map from $[-1, 1]$ to itself defined by

$$g_2(\theta) = \operatorname{sgn}(\theta) 2\theta - 1.$$

Then f_2 is conjugate to g_2 via a homeomorphism

$$h : [-1, 1] \to [-2, 2] : h(\theta) = 2 \sin \frac{\pi\theta}{2},$$

i.e. $g_2 = h^{-1} \circ f_2 \circ h$. Let $g_a = h^{-1} \circ f_a \circ h|_{h^{-1}(K_a)}$. Then by the chain rule

$$|(f_a^n)'(x)| = |(g_a^n)'(h^{-1}(x))| \cdot \frac{|h'(g_a^n(h^{-1}(x)))|}{|h'(h^{-1}(x))|}.$$

Now we estimate the first term. Define

$$\mathcal{D}(\epsilon, \delta) := \bigcup_{a \in \Omega_\epsilon} a \times h^{-1}(K_a \setminus \Delta)$$

and let $G(a, \theta)$ be a C^2 map from $\mathcal{D}(\epsilon, \delta)$ to itself defined by $G(a, \theta) = g_a(\theta)$. For each $\theta \in h^{-1}(K_a \setminus \Delta)$, we use the mean value theorem to obtain

$$\left| \frac{\partial G(2, \theta)}{\partial \theta} - \frac{\partial G(a, \theta)}{\partial \theta} \right| = |g_2'(\theta) - g_a'(\theta)| \leq \sup_{(a, \theta) \in \mathcal{D}(\epsilon, \delta)} |\partial_a \partial_\theta g_a(\theta)| \cdot \epsilon < \epsilon M,$$

where $M > 0$ is some constant. Hence, for any given $0 < \hat{\lambda} < \log 2$, we can find ϵ such that $\log(2 - \epsilon M) \geq \hat{\lambda}$. For such ϵ and arbitrary $a \in \Omega_\epsilon$, we have $|(g_a)'(\theta)| \geq 2 - \epsilon M$. On the other hand, the assumption that $x, \cdots, f_a^{n-1}(x) \notin \Delta$ means $h^{-1}(x), \cdots, g^{n-1}(h^{-1}(x)) \in h^{-1}(K_a \setminus \Delta)$. This fact and the chain rule give $|(g_a^n)'(\theta)| \geq e^{\hat{\lambda} n}$.

Next we estimate the second term. By the fact that h' is an even function, $h'(\theta) > 0$ on $(-1, 1)$, $h''(0) = 0$ and $h'''(\theta) < 0$ on $(-1, 1)$, we immediately get (2). Concerning (3), let δ be sufficiently small so that $|h'(h^{-1}(x)) - h'(h^{-1}(y))| < \pi/4$ if $|x - y| < 2\delta^\iota$, and $h'(h^{-1}(x)) \geq \pi/2$ if $|x| \leq \delta^\iota$. Then $|f_a^n(x)|, |x| \leq 2\delta^\iota$ implies $|h'(h^{-1}(f_a^n(x))) - h'(h^{-1}(x))| < \pi/4$. By the triangle inequality we have

$$\frac{|h'(g_a^n(h^{-1}(x)))|}{|h'(h^{-1}(x))|} > \frac{1}{2}.$$

It remains to show (1). One can easily see that there is nothing to prove if the orbit stays in the region $\{|x| \geq e^{\hat{\lambda}}/2\}$. Suppose that $|f_a^i(x)| < e^{\hat{\lambda}}/2$ for some $i \leq n$. Then we clearly have $|f_a^n(x)| \leq 2 - \delta^2$. Therefore, by the properties of h as above, we can conclude

$$\frac{|h'(g_a^n(h^{-1}(x)))|}{|h'(h^{-1}(x))|} \geq \frac{|h'(h^{-1}(2-\delta^2)|}{|h'(h^{-1}(0))|} = \cos\frac{\pi}{2}h^{-1}(2-\delta^2).$$

As a consequence, we may set

$$C_\delta := \min\left\{\cos\frac{\pi}{2}h^{-1}(\delta^2 - 2), 1/2\right\},$$

which is equal to $\cos\frac{\pi}{2}h^{-1}(\delta^2 - 2)$ for small δ. □

The proof is very specific to the real quadratic family, but a similar conclusion holds for maps whose critical point is non-recurrent. See [DV].

Corollary 5.1.1. *For all sufficiently small $\epsilon > 0$, $a \in \Omega_\epsilon$ and $k \geq 1$ such that f_a satisfies $BR(\alpha, \delta)_k$ we have*

$$|(f_a^{k+1})'(c_0(a))| \geq e^{\lambda(k+1)}$$

where $\lambda := \hat{\lambda} - 2\alpha$.

Proof. Let $N(\epsilon) \in \mathbf{N}$ be large so that we have $C_\delta(3.5/e^{\hat{\lambda}})^{N(\epsilon)} \geq 1$, and $|(f_a^i)'(c_0(a))| \geq (3.5)^i$ for any $i \leq N(\epsilon)$, $a \in \Omega_\epsilon$. Let $0 < N(\epsilon) < \nu_1 < \cdots < \nu_s \leq k$ be the sequence of times such that $c_{\nu_i}(a) \in \Delta$. By the chain rule

$$(f_a^{k+1})'(c_0(a)) = (f_a^N)'(c_0(a))(f_a^{\nu_1 - N})'(c_N(a))(f_a^{\nu_2-\nu_1})'(c_{\nu_1}(a))\cdots$$
$$\cdots(f_a^{\nu_s - \nu_{s-1}})'(c_{\nu_s - 1}(a))(f_a^{k+1-\nu_s})'(c_{\nu_s}(a)).$$

Letting $\nu_0 := N(\epsilon)$ we have

$$|(f_a^{\nu_i - \nu_{i-1}})'(c_{\nu_{i-1}}(a))| \geq e^{\hat{\lambda}(\nu_i - (\nu_{i-1}+1))}|f_a'(c_{\nu_{i-1}}(a))|$$

for $i = 1, \cdots, s$, by (2) of Lemma 5.1. Concerning the last remaining part, we use (1) of Lemma 5.1 to obtain

$$|(f_a^{k+1-\nu_s})'(c_{\nu_s}(a))| \geq C_\delta e^{\hat{\lambda}(k+1-(\nu_s+1))}|f_a'(c_{\nu_s}(a))|.$$

Putting these together yields

$$|(f_a^{k+1})'(c_0(a))| \geq C_\delta(3.5)^N e^{\hat{\lambda}(k+1-N-s-1)}\prod_{j=0}^{s}|f_a'(c_{\nu_j}(a))|$$
$$\geq e^{\hat{\lambda}(k+1)}e^{-s\hat{\lambda}}e^{-\alpha k} \geq e^{\hat{\lambda}(k+1)}e^{-2\alpha k} \geq e^{(\hat{\lambda}-2\alpha)(k+1)},$$

where we have used the following:

$$s\hat{\lambda} < s\log\delta^{-1} < \sum_{i=1}^{s}\log|c_{\nu_i}|^{-1} \leq \alpha k.$$

□

Corollary 5.1.2. *For the system constants including ϵ, we have*

$$|(f_a^{k+1})'(c_0(a))| \geq e^{(\hat{\lambda}-10\alpha)(k+1)},$$

provided f_a satisfies $BR(5\alpha, \delta)_k$.

5.2 Similarity between critical curves evolution and phase space dynamics

Lemma 5.2. *For all $a \in \Omega_\epsilon$ and all $k \geq 1$ such that f_a satisfies $BR(5\alpha, \delta)_k$ we have*

$$\frac{1}{2} \leq \frac{|c'_{k+1}(a)|}{|(f_a^{k+1})'(c_0(a))|} \leq 2.$$

Proof of Lemma 5.2. For each $1 \leq i \leq k+1$, define a map $F_i : \Omega_\epsilon \times K_a \to K_a$ by a recursive formula $F_1(a,x) = f_a(x)$ and $F_i(a,x) = F_1(a, f_a^{i-1}(x))$. Letting $x = c_0(a)$ we have

$$c'_i(a) = \partial_a F_i(a, c_0(a)) = \partial_a F_1(a, f_a^{i-1}(c_0(a))) = -1 + f'_a(c_{i-1}(a))c'_{i-1}(a).$$

Applying this equality recursively for $i = 1, \cdots, k+1$, we have

$$-c'_{k+1}(a) = 1 + f'_a(c_k(a)) + f'_a(c_k(a))f'_a(c_{k-1}(a)) + \cdots$$
$$\cdots + f'_a(c_k(a))f'_a(c_{k-1}(a)) \cdots f'_a(c_1(a))f'_a(c_0(a)).$$

By the Corollary 5.1.2, it is possible to divide both sides by $(f_a^{k+1})'(c_0(a)) \neq 0$ and we obtain

$$-\frac{c'_{k+1}(a)}{(f_a^{k+1})'(c_0(a))} = 1 + \sum_{i=1}^{k+1} \frac{1}{(f_a^i)'(c_0(a))}.$$

Recall that we have chosen a large number $N(\epsilon)$ satisfying $(f_a^i)'(c_0(a)) \leq -(3.5)^i$ for any $i \leq N(\epsilon)$. Therefore we have $N(\epsilon) < k+1$ and, if necessary, we can make $N(\epsilon)$ larger by letting ϵ small so that

$$\sum_{i=N(\epsilon)+1}^{\infty} e^{-(\hat{\lambda}-4\alpha)i} \leq \frac{1}{10}.$$

Then

$$\frac{|c'_{k+1}(a)|}{|(f_a^{k+1})'(c_0(a))|} \geq 1 - \sum_{i=1}^{N(\epsilon)} \frac{1}{(f_a^i)'(c_0(a))} - \sum_{i=N(\epsilon)+1}^{k+1} \frac{1}{(f_a^i)'(c_0(a))}.$$

Applying Corollary 5.1.2, the right hand side of the above can be estimated from below by

$$1 - \sum_{i=1}^{N(\epsilon)} \frac{1}{(f_a^i)'(c_0(a))} - \sum_{i=N(\epsilon)+1}^{k+1} \frac{1}{(f_a^i)'(c_0(a))} \geq 1 - \sum_{i=1}^{N(\epsilon)} 3.5^{-i} - \sum_{i=N(\epsilon)+1}^{k+1} e^{-\lambda}$$
$$\geq 1 - \sum_{i=1}^{\infty} 3.5^{-i} - \sum_{i=N(\epsilon)+1}^{\infty} e^{-\lambda} \geq 1 - \frac{2}{5} - \frac{1}{10} \geq \frac{1}{2}.$$

An upper bound is easily obtained by

$$\frac{|c'_{k+1}(a)|}{|(f_a^{k+1})'(c_0(a))|} \leq 1 + \sum_{i=1}^{\infty} e^{-\lambda i} < 2.$$

□

Corollary 5.2.1. *Let $\omega \subset \Omega_\epsilon$ be an interval such that any $a \in \omega$ satisfies $BR(5\alpha, \delta)_k$. Then for all $1 \leq i \leq j \leq k+1$ there exists $\xi \in \omega$ such that*

$$\frac{1}{4}|(f_\xi^{j-i})'(c_i(\xi))| \leq \frac{|\omega_j|}{|\omega_i|} \leq 4|(f_\xi^{j-i})'(c_i(\xi))|.$$

Proof. This is an immediate consequence of the previous lemma and the mean value theorem. By Lemma 5.2, the map c_i is a diffeomorphism on ω. Hence, we can consider the inverse c_i^{-1}, and by the mean value theorem, there exists some $\xi_i \in \omega_i$ such that

$$|\omega_j| = |(c_j c_i^{-1})'(\xi_i)||\omega_i|.$$

Letting $\xi := c_i^{-1}(\xi_i)$ and by the chain rule we have

$$\frac{|\omega_j|}{|\omega_i|} = \frac{|c_j'(\xi)|}{|c_i'(\xi)|}.$$

Applying Lemma 5.2 again and the chain rule gives the conclusion. □

Corollary 5.2.2. *Suppose the system constants $\hat{\lambda}, \alpha, \iota, \delta$ have been specified. One can choose $\epsilon > 0$ in such a way that $|c_k(a)| \geq e^{-\alpha k}$ holds for any $a \in \Omega_\epsilon$ satisfying $BR(\alpha, \delta)_k$.*

Proof. Let $M(\delta)$ be the minimum integer such that $e^{-\alpha M(\delta)} < \delta$. In other words, $M(\delta)$ is the first time when f_a satisfying $BR(\alpha, \delta)_{M(\delta)}$ can have a return to Δ. According to this $M(\delta)$, choose ϵ so that

$$2 - 2 \cdot 4^i \epsilon \geq e^{-\alpha i} \text{ for } j = 0, \cdots, M(\delta) - 1.$$

One can check that this is always possible for arbitrarily large $M(\delta)$. If $i \geq M(\delta)$ and f_a satisfies $BR(\alpha, \delta)_i$, it is easy to see $|c_i(a)| \geq e^{-\alpha i}$. Consider the case $i < M(\delta)$. By the mean value theorem

$$|f_a^{i+1}(0) - f_2^{i+1}(0)| = |c_i(a) - c_i(0)| \leq \epsilon \sup_{a \in \Omega_\epsilon} |c_i'(a)|.$$

By Lemma 5.2, it holds that

$$\epsilon \sup_{a \in \Omega_\epsilon} |c_i'(a)| \leq 2\epsilon \sup_{a \in \Omega_\epsilon} |(f_a^i)'(c_0(a))| < 2 \cdot 4^i \epsilon,$$

and therefore we obtain

$$|c_i(a)| \geq |c_i(0)| - 2 \cdot 4^i \epsilon \geq 2 - 2 \cdot 4^i \epsilon \geq e^{-\alpha i}.$$

□

5.3 Binding

The next lemma introduces the notion of binding. This notion and Lemma 5.1 are key ingredients to ensure derivative growth along the orbit of the critical point. The derivative grows exponentially as long as the orbit stays outside Δ. Once the orbit falls inside Δ, the derivative may become very small. However, loss of the derivative is to some extent compensated by shadowing some initial piece of the orbit during which the exponential growth has already been guaranteed.

Lemma 5.3. *Suppose that $c_k(a) \in (-2\delta^\iota, 2\delta^\iota)$, and f_a satisfies $BR(\alpha, \delta)_k$. Introducing new system constant $0 < \kappa < 1$, we can specify some integer in the following way:*

$$p(a, k) := \min\{i \in \mathbf{N} : |\gamma_i| \geq \kappa e^{-2\alpha i}\}.$$

Here, $\gamma := [0; c_k(a)]$, $\gamma_j := f_a^{j+1}(\gamma)$ and we denote by $[0; c_k(a)]$ the interval whose two endpoints are 0 and $c_k(a)$. Then $p = p(a, k)$ has the following properties:

$$\log |c_k(a)|^{-1} \leq p \leq \frac{2}{\lambda} \log |c_k(a)|^{-1}, \tag{4}$$

$$|(f_a^{p+1})'(c_k(a))| \geq |c_k(a)|^{5\beta - 1}, \tag{5}$$

$$|(f_a^{p+1})'(c_k(a))| \geq e^{\frac{\lambda(p+1)}{6}}, \tag{6}$$

where $\beta := \alpha/\lambda$.

We call $p(a,k)$ the binding period associated to the recurrence $c_k(a)$. A proof requires the following distortion lemma during the binding period.

Sublemma 5.3.1. *Suppose that $c_k(a) \in (-2\delta^\iota, 2\delta^\iota)$ and that f_a satisfies $BR(\alpha,\delta)_k$. Then, for all y_0, $z_0 \in \gamma_0$ and $0 \le i \le \hat{p}+1$, we have*

$$\frac{|(f_a^i)'(z_0)|}{|(f_a^i)'(y_0)|} \le \exp\left(\frac{1}{(1-e^{-\alpha})^2}\right) =: D_\alpha,$$

where $\hat{p} := \min\{p-1, k\}$.

Proof. The chain rule gives

$$\frac{|(f_a^i)'(z_0)|}{|(f_a^i)'(y_0)|} \le \prod_{j=0}^{i-1} \frac{|f_a'(z_j)|}{|f_a'(y_j)|} = \prod_{j=0}^{i-1}\left|1+\frac{f_a'(z_j)-f_a'(y_j)}{f_a'(y_j)}\right|.$$

On the other hand, by the mean value theorem, $|f_a'(z_j)-f_a'(y_j)| \le 2|\gamma_j|$. Therefore we have

$$\frac{|(f_a^i)'(z_0)|}{|(f_a^i)'(y_0)|} \le \exp\left(\log\prod_{j=0}^{i-1}\left|1+\frac{f_a'(z_j)-f_a'(y_j)}{f_a'(y_j)}\right|\right)$$

$$\le \exp\left(\sum_{j=0}^{i-1}\log\left(1+\frac{|\gamma_j|}{|y_j|}\right)\right) \le \exp\left(\sum_{j=0}^{i-1}\frac{|\gamma_j|}{|y_j|}\right).$$

It suffices to prove $\sum_{j=0}^{i-1}\frac{|\gamma_j|}{|y_j|} \le (1-e^{-\alpha})^{-2}$. On the other hand, by the definition of the binding period, we have $|\gamma_j| \le \kappa e^{-2\alpha j} < e^{-2\alpha j}$. Hence we have the conclusion if $|y_j| \ge (1-e^{-\alpha})e^{-\alpha j}$. The last inequality easily follows from Corollary 5.2.2, because $|\gamma_j| \ge |c_j - y_j| \ge |c_j| - |y_j|$ and $|y_j| \ge |c_j| - |\gamma_j| \ge e^{-\alpha j} - e^{-2\alpha j} \ge e^{-\alpha j}(1-e^{-\alpha})$. □

Proof of Lemma 5.3.

(4)

By the the mean value theorem, there exists $\xi \in \gamma_0$ such that

$$\kappa e^{-2\alpha\hat{p}} \ge |\gamma_{\hat{p}}| = |(f_a^{\hat{p}})'(c_0)| \cdot \frac{|(f_a^{\hat{p}})'(\xi)|}{|(f_a^{\hat{p}})'(c_0)|}|\gamma_0| \ge e^{\lambda\hat{p}} c_k(a)^2 D_\alpha^{-1}.$$

Here, the first inequality follows from the definition of the binding period. The second is by virtue of Corollary 5.1.1 and the distortion estimate of Sublemma 5.3.1. Taking the logarithm we get

$$\hat{p} \le \frac{2\log|c_k(a)|^{-1}}{\lambda+2\alpha} + \log D_\alpha + \log\kappa \le \frac{2\log|c_k(a)|^{-1}}{\lambda} - 1,$$

where the second inequality is true if δ is taken sufficiently small. More specifically, it holds as long as $-2\log\delta^\iota(\lambda^{-1}-\hat{\lambda}^{-1}) \ge \log D_\alpha + \log\kappa - 1$. Finally we obtain

$$\hat{p} \le \frac{2\log|c_k(a)|^{-1}}{\lambda} - 1 < \frac{2}{\lambda}\alpha k \ll k.$$

For the lower estimate, note that $p = \hat{p}+1$ by the above inequality. By the relation $|\gamma_p| \ge \kappa e^{-2\alpha p}$, $|f_a'(z)| \le 4$ and the mean value theorem we get

$$4^p c_k(a)^2 D_\alpha \ge |(f_a^p)'(c_0)| D_\alpha |\gamma_0| \ge |\gamma_p| \ge \kappa e^{-2\alpha p}.$$

Hence we have

$$p \ge \frac{2\log|c_k(a)|^{-1} - \log D_\alpha + \log\kappa}{\log 4 + 2\alpha} > \log|c_k(a)|^{-1},$$

where the last inequality is true as long as $-\log 2\delta^\iota \ge \frac{\log 4+2\alpha}{2-\log 4-2\alpha}\cdot\log\frac{D_\alpha}{\kappa}$.

(5)

For the above $\xi \in \gamma_0$, we have

$$|(f_a^{p+1})'(c_k(a))| = |(f_a)'(c_k(a))| \frac{|(f_a^p)'(\xi)|}{|(f_a^p)'(\xi)|} |(f_a^p)'(c_{k+1}(a))|$$

$$\geq 2|c_k(a)| \frac{1}{D_\alpha} \frac{|\gamma_p|}{|\gamma_0|} \geq \frac{2\kappa e^{-2\alpha p}}{|c_k(a)| D_\alpha} \geq \frac{2\kappa}{D_\alpha} |c_k(a)|^{\frac{4\alpha}{\lambda}-1},$$

where the last inequality holds because $e^{-2\alpha p} \geq e^{\frac{4\alpha}{\lambda} \log |c_k(a)|^{-1}} = |c_k(a)|^{\frac{4\alpha}{\lambda}}$. Recall that $p \leq \frac{2}{\lambda} \log |c_k(a)|^{-1}$. Therefore, we obtain the formula as long as δ is sufficiently small in such a way that $\frac{2\kappa}{D_\alpha} \geq (2\delta^\iota)^{\frac{\alpha}{\lambda}}$.

(6)

Use $|c_k(a)|^{-1} \geq \lambda p/2$ to get

$$|(f_a^{p+1})'(c_k(a))| \geq \frac{2\kappa e^{-2\alpha p}}{|c_k(a)| D_\alpha} \geq \frac{2\kappa e^{\lambda p/2 - 2\alpha p}}{D_\alpha} \geq e^{\frac{p+1}{6}\lambda},$$

where the last inequality holds if $-(\frac{\lambda}{3} - 2\alpha) \log 2\delta^\iota \geq \log D_\alpha - \log 2\kappa + \frac{\lambda}{6}$. □

5.4 Extention of the period during which BR holds

Lemma 5.4. *Suppose that $c_k(a) \in (-2\delta^\iota, 2\delta^\iota)$, f_a satisfies $BR(\alpha, \delta)_k$ and p be the corresponding binding period. Then f_a satisfies up to $BR(5\alpha, \delta)_{k+p}$.*

This lemma is very crucial for our inductive argument. During the binding period, the critical orbit duplicates its initial piece. Namely, $c_\zeta(a)$ and $c_{\zeta-k-1}(a)$ are very close to each other for $\zeta \in [k, k+p]$. Thus we are liable to argue that $c_\zeta(a) \in \Delta$ if and only if $c_{\zeta-k-1}(a) \in \Delta$, and as a result, the total sum of bound return depths is essentially almost the same as the sum of return depths up to p, which implies $BR(\alpha, \delta)_{k+p}$. However, this argument is wrong. Indeed, we have the case where $c_\zeta(a) \in \Delta$, but $c_{\zeta-k-1}(a) \notin \Delta$. A way to overcome this problem is to show that this kind of unfavorable situation does not occur so frequently and when it occurs, the corresponding two bound orbits fall near the boundary of Δ. In other words, it takes more than $\mathcal{O}(\log \delta^{-1})$ times of iteration to go from one unfavorable situation to the next one. If this is true, $\log \delta^{-1}$ multiplied by possible times of the unfavorable situation gives an upper bound of the total sum of the bound return depths in question. To illustrate this, let us make an additional classification of bound returns.

Definition. Let $a \in \Omega_\epsilon$ and k be as above. We say a bound return $\zeta \in [k+1, k+p]$ is *regular* if $c_{\zeta-k-1}(a) \in \Delta$. Otherwise we call it *irregular*.

By definition, irregular bound returns seem to be located near the boundary of Δ. This observation is justified by the following

Sublemma 5.4.1. *Let $a \in \Omega_\epsilon$, $c_k(a) \in (-2\delta^\iota, 2\delta^\iota)$, f_a satisfies $BR(\alpha, \delta)_k$ and $\zeta > k$ be the first bound return. Then we have $e^{-2\alpha(\zeta-k)} < \delta^2$. Therefore, any irregular bound return is located in the interval $[\delta - \delta^2, \delta]$ or $[-\delta, -\delta + \delta^2]$.*

Proof. This is never an immediate consequence of the simple definition of irregular bound returns, because δ is taken sufficiently small after $\kappa < 1$ has been fixed. We must analyze how small is the exponential term $e^{-2\alpha i}$ contributing to an error bound during bound state. For given α and appropriately chosen small δ, let ϵ shrink so that, for any $a \in \Omega_\epsilon$, a part of the critical orbit $f_a^i(0)$ $(i = 1, \cdots, -\log\delta/\alpha)$ stays in a neighborhood of 2. If the length of the binding period associated to $c_k(a)$ is smaller than $-\log\delta/\alpha$, there is no bound return by the definition. Otherwise, we clearly have $e^{-2\alpha(\zeta-k)} \le e^{-2\alpha(-\log\delta/\alpha)} = \delta^2$. □

Sublemma 5.4.2. *For any $a \in \Omega_\varepsilon$ and $x \in (-2\delta^\iota, 2\delta^\iota)$, let*

$$s(a,x) := \min\{k \ge 2 : f_a^k(x) \le 1\}.$$

Then we have $s(a,x) > \frac{-\log|x|}{\log 2}$.

This sublemma was inspired by Tsujii [Tsu93b; Lemma 3.1], although the direction of the inequality has been reversed. The critical orbit stays away from the critical point for a while after any recurrence. How long it stays far away from the critical point is essentially determined by the depth.

Proof. Note that $-f_a(x) > f_a^2(x) > \cdots > f_a^{s(a,x)-1}(x) > 1 \ge f_a^{s(a,x)}(x)$, and put $J = [f_a^2(x), -f_a(x)]$. Then it is easy to check that $|J| < 4x^2$. On the other hand, by the definition of $s(a,x)$ and using $f_a(x) = f_a(-x)$, we have $|f_a^{s(a,x)-2}(J)| \ge 1/2$. Therefore we obtain

$$(s(a,x)-2)\log 4 > \log\frac{|f_a^{s(a,x)-2}(J)|}{|J|} \ge -\log 8x^2,$$

which implies the inequality. □

Combining these two sublemmas yields the following.

Corollary 5.4.3. *The total number of possible irregular bound returns during $[k+1, k+p]$ is less than $1.5 \cdot \left[\frac{p\log 2}{\log\delta^{-1}}\right]$, where $[\ \ ]$ denotes the integer part.*

Proof of Lemma 5.4. We want to prove $\sum_{i=1}^{k+p} 1_\Delta(c_i(a))\log|c_i(a)|^{-1} < 5\alpha(k+p)$. By the assumption $BR(\alpha,\delta)_k$, this is equivalent to showing

$$\sum_{i=k+1}^{k+p} 1_\Delta(c_i(a))\log|c_i(a)|^{-1} < 5\alpha p.$$

Divide the sum into two parts according to regular or irregular bound returns:

$$\sum_{i=k+1}^{k+p} 1_\Delta(c_i(a))\log|c_i(a)|^{-1} = \sum_{\substack{k+1\le i\le k+p\\ k:\text{regular}}} 1_\Delta(c_i(a))\log|c_i(a)|^{-1} + \sum_{\substack{k+1\le i\le k+p\\ k:\text{irregular}}} 1_\Delta(c_i(a))\log|c_i(a)|^{-1}.$$

First, we estimate the regular part. Take $\kappa := \min\{\frac{\alpha}{2\Lambda}, \frac{1}{2}\}$, where $\Lambda := \sum_{j\ge 0} e^{-\alpha j}$. By the definition of the binding period, we have $|c_i(a) - c_{i-k-1}(a)| \le \kappa e^{-2\alpha(i-k-1)}$ for all $k+1 \le i \le k+p$. Using the triangle inequality and $\log(1+x) \le x$ for $x \ge 0$, we obtain

$$\begin{aligned}\log|c_i(a)|^{-1} &\le \log|c_{i-k-1}(a)|^{-1} + \log\left|1 - \frac{\kappa e^{-2\alpha(i-k-1)}}{|c_{i-k-1}(a)|}\right|^{-1}\\ &< \log|c_{i-k-1}(a)|^{-1} + \frac{\kappa e^{-2\alpha(i-k-1)}}{|c_{i-k-1}(a)| - \kappa e^{-2\alpha(i-k-1)}}.\end{aligned}$$

By Corollary 5.2.2, we have $|c_{i-k-1}(a)| \ge e^{-\alpha(i-k-1)}$, and as a result,

$$\log|c_i(a)|^{-1} \le \log|c_{i-k-1}(a)|^{-1} + 2\kappa e^{-\alpha(i-k-1)}.$$

Recall that only regular bound returns are now concerned. Hence we obtain

$$\sum_{\substack{k+1\le i\le k+p\\ k:\text{regular}}} 1_\Delta(c_i(a))\log|c_i(a)|^{-1} \le \sum_{i=0}^{p-1} 1_\Delta(c_i(a))\log|c_i(a)|^{-1} + 2\kappa\Lambda,$$

which is less than $\alpha(p-1)+\alpha = \alpha p$. For the irregular part, it follows that

$$\sum_{\substack{k+1\le i\le k+p\\ k:\text{irregular}}} 1_\Delta(c_i(a))\log|c_i(a)|^{-1} < 1.5\cdot p\log 2 < \frac{3\alpha k\log 2}{\lambda}$$

from Sublemma 5.4.1 and Corollary 5.4.3. Putting these together yields

$$\sum_{i=1}^{k+p} 1_\Delta(c_i(a))\log|c_i(a)|^{-1} < \alpha k + \frac{3\alpha k\log 2}{\lambda} + \alpha p < 5\alpha(k+p).$$

□

6 Getting the induction started

Now the system constants $\hat{\lambda}$, α, ι, κ have already been fixed. The subsequent argument is valid for any sufficiently small δ. Without loss of generality, we may assume $r_\delta := \log\delta^{-1} \in \mathbb{N}$. Let $r_{\delta^+} := [\iota\log\delta^{-1}]$ and $\Delta^+ := (-e^{-r_{\delta^+}}, e^{-r_{\delta^+}})$,where [ ] denotes the integer part. $0 < \iota \le \frac{4\alpha}{\hat{\lambda}-2\alpha} < 1$ and $\delta < 1$ imply $\Delta^+ \supset \Delta$. For $r \ge r_{\delta^+} > 0$, define $I_r := [e^{-r}, e^{-r+1}]$, $I_{-r} := -I_r$ and subdivide each $I_{\pm r}$ into r^2 intervals with equal length. They are denoted by $I_{\pm r,s}$, where $s \in [1, r^2]$. Define

$$\mathcal{I} := \{I_{\pm r,s} : r > r_\delta, 1 \le s \le r^2\}$$

and

$$\mathcal{I}^+ := \{I_{\pm r,s} : r > r_{\delta^+}, 1 \le s \le r^2\}.$$

Namely, $\mathcal{I}^+$ and $\mathcal{I}$ are partitions of Δ^+ and Δ respectively.

We are going to construct inductively a nested sequence of parameter sets $\Omega^{(\epsilon)} =: \Omega^{(0)} \supset \Omega^{(1)} \supset \Omega^{(2)} \supset \cdots$ and partitions $\mathcal{P}^{(n)}$ of $\Omega^{(n)}$ with the properties that

- any $a \in \Omega^{(n)}$ satisfies $BR(\alpha,\delta)_n$;
- any $\mathcal{P}^{(n)}$ has the bounded distortion property.

The procedure is carried out as follows. Suppose steps have been done up to $n-1$. Namely, we are given $\Omega^{(n-1)}$ and its partition $\mathcal{P}^{(n-1)}$ such that any $a \in \Omega^{(n-1)}$ satisfies $BR(\alpha,\delta)_{n-1}$. Then, we define a refinement $\hat{\mathcal{P}}^{(n)}$ of $\mathcal{P}^{(n-1)}$ via c_n, according to the partition $\mathcal{I}^+$, and from it discard bad elements with strong recurrence. Note that this refinement process is justified by Lemma 5.2, which states that c_n is a diffeomorphism, and hence especially one to one on each element $\omega \in \mathcal{P}^{(n-1)}$. The set of the remaining elements is denoted by $\mathcal{P}^{(n)}$. We put

$$\Omega^{(n)} := \bigcup_{\omega\in\mathcal{P}^{(n)}} \omega.$$

6.1 Initial step

For fixed ϵ, we call $n_0(\epsilon)$ the *first chopping time* if it is the smallest integer such that $c_{n_0}(\Omega_\epsilon)$ contains at least two elements of $\mathcal{I}^+$. We construct subdivision of $c_{n_0}(\Omega_\epsilon)$ according to the partition $\mathcal{I}^+$. Pull back via c_{n_0} of this subdivision induces the partition $\hat{\mathcal{P}}^{(n_0)}$ of Ω_ϵ. For simplicity, $c_{n_0}(\omega)$ is denoted by ω_{n_0}.

Definition. We say n_0 is
(A) an ***essential return*** of $\omega \in \hat{\mathcal{P}}^{(n_0)}$ if $\omega_{n_0} \cap \Delta \neq \emptyset$.
(B) an ***essential escape*** of $\omega \in \hat{\mathcal{P}}^{(n_0)}$ if $\omega_{n_0} \cap \Delta = \emptyset$ and $\omega_{n_0} \cap \Delta^+ \neq \emptyset$.
(C) a ***substantial escape*** of $\omega \in \hat{\mathcal{P}}^{(n_0)}$ if $\omega_{n_0} \cap \Delta^+ = \emptyset$.

Note that there still remains some ambiguity of the above subdivision, and hence we need to set some rules:

(i) surplus treatment: As far as we are concerned with the case inside Δ^+, subdivision is carried out in such a way that each subinterval produced by the subdivision of $c_{n_0}(\Omega_\epsilon)$ contains a unique element of $\mathcal{I}^+$.

(ii) boundary treatment: There is no longer the partition $\mathcal{I}^+$ defined outside Δ^+. To cope with the situation in which the image lies beyond the boundary of Δ^+, we obey the following rule. If the length of the connected component of $c_{n_0}(\Omega_\epsilon) \setminus \Delta^+$ does not exceed δ^ι, then this part is glued to the adjacent marginal element of $\mathcal{I}^+$. Otherwise, the component is regarded as one independent element of the subdivision.

In the cases (A) and (B), ω_{n_0} contains a unique subinterval of the form $I_{\pm r,s}$. We call this r the ***depth*** of ω.

If there is no fear of confusion, we also allow to refer to $\omega \in \hat{\mathcal{P}}^{(n_0)}$ as an essential return, essential escape, and so on.

We discard elements $\hat{\mathcal{P}}^{(n_0)}$ with strong recurrence. This is done in terms of the corresponding depth. Namely, elements with their depth greater than $\alpha n_0/16$ are discarded. Essential escapes are not thrown away as long as ϵ is so small that $\log \delta^{-1} < \alpha n_0(\epsilon)/16$.

For later use, the function which corresponds to each $\omega \in \hat{\mathcal{P}}^{(n_0)}$ its depth is denoted by $\mathcal{E}^{(n_0)}$. Put

$$\mathcal{P}^{(n_0)} := \{\omega \in \hat{\mathcal{P}}^{(n_0)} : \mathcal{E}^{(n_0)}(\omega) \leq \alpha n_0/16\}$$

and

$$\Omega^{(n_0)} = \bigcup_{\omega \in \mathcal{P}^{(n_0)}} \omega.$$

The binding periods are associated to both essential returns and essential escapes by the following formula

$$p = p(\omega, n_0) := \inf_{a \in \omega} p(a, n_0).$$

By definition, any $a \in \omega$ satisfies $BR(\alpha, \delta)_{n_0}$, and hence up to $BR(3\alpha, \delta)_{n_0+p}$ by Lemma 5.4.

6.2 General step

We shall explain how to proceed the inductive step.

Definition. Let $\omega \in \mathcal{P}^{(n_0)}$. We say $n > n_0$ is the ***chopping time*** if the following are true:

(i) ω_n contains at least two elements of the partition $\mathcal{I}^+$.
(ii) ω_n is not in a bound state.

Here, we say ω_k is in a ***bound state*** if $n_0 + 1 \leq k \leq n_0 + p(\omega, n_0)$. Such k as $\omega_k \cap \Delta \neq \emptyset$ is called a ***bound return***.

A *non-chopping time* means a time which is not a chopping time. At any non-chopping time, no parameter needs to be excluded. We say n is an *inessential return* of ω if n is a non-chopping time, ω_n not in bound state and $\omega_n \cap \Delta \neq \emptyset$. Similarly, we say n is an *inessential escape* of ω if n is a non-chopping time, ω_n not in bound state, $\omega_n \subset \Delta^+$ but $\omega_n \cap \Delta = \emptyset$. To both inessential returns and inessential escapes, we also associate the binding period by the above formula. Therefore the notion of a bound state and a bound return makes sense in these cases.

At any chopping time, ω_n is again subdivided according to the given algorithm as above and n is also called an essential return, an essential escape or a substantial escape accordingly. Among the subintervals arising from the subdivision at the chopping time, those with weak recurrence constitute $\mathcal{P}^{(n)}$ and $\Omega^{(n)}$.

The binding period is again associated to each essential return or essential escape in $\mathcal{P}^{(n)}$, and hence the notion of a bound state, a bound return and a chopping or a non-chopping time makes sense in the general case. Briefly, we have the following general expressions.

Definition. Let $\omega \in \hat{\mathcal{P}}^{(n)}$. A time n is called:

(A) an *essential return* if there exists $\omega' \in \mathcal{P}^{(n-1)}$ such that ω arises out of the chopping of $\omega' \in \mathcal{P}^{(n-1)}$ at n with $\omega_n \cap \Delta \neq \emptyset$.

(B) an *essential escape* if there exists $\omega' \in \mathcal{P}^{(n-1)}$ such that ω arises out of the chopping of $\omega' \in \mathcal{P}^{(n-1)}$ at n with $\omega_n \cap \Delta = \emptyset$ and $\omega_n \cap \Delta^+ \neq \emptyset$.

In both cases ω_n contains a unique subinterval of the form $I_{\pm r,s}$. We call the associated r the depth of ω. If we want to be more specific, we say an essential return depth and so on.

(C) a *substantial escape* if there exists $\omega' \in \mathcal{P}^{(n-1)}$ such that ω arises out of the chopping of $\omega' \in \mathcal{P}^{(n-1)}$ at n with $\omega_n \cap \Delta^+ = \emptyset$.

(D) an *inessential return* if $\omega \in \mathcal{P}^{(n-1)}$ (hence n is a non-chopping time of ω) and ω_n is not in bound state, $\omega_n \cap \Delta \neq \emptyset$.

(E) an *inessential escape* if $\omega \in \mathcal{P}^{(n-1)}$ (hence n is a non-chopping time of ω) and ω_n is not in bound state, $\omega_n \cap \Delta = \emptyset$ but $\omega_n \subset \Delta^+$.

In the last two cases we also define the depth r of ω to be $r := \max\{i \in \mathbb{N} : I_{\pm i} \cap \omega_n \neq \emptyset\}$. An essential and inessential return are called *a free return*.

6.3 Structure of a time history

Each element $\omega \in \hat{\mathcal{P}}^{(n)}$ is associated with the time history up to time n, which consists of several kinds of returns and escapes. This subsection gives a rough description of how returns and escapes are distributed in the time history.

Between two consecutive escapes there is a sequence of essential returns. Moreover, there are some inessential returns in a row between two consecutive essential returns. It is possible to show that a return that can follow an essential or a substantial escape is an essential return. This fact is crucial for inductive verification of $BR(\alpha, \delta)_n$ for $\Omega^{(n)}$. A formal proof is given in Corollaries 7.1.1.1 and 7.1.1.2, and hence we sketch the proof for the time being. Let $\omega \in \mathcal{P}^{(n)}$ and n be an escape of ω. Then ω_n occupies at least one element of the partition $\mathcal{I}^+ \setminus \mathcal{I}$, which grows exponentially in size until the next return (by Corollary 5.2.1) to attain sufficient length extending across more than three contiguous partition elements of $\mathcal{I}$. This implies no possibility of an inessential return. This observation is not true in the case of inessential escapes. There is no particular rule governing an order relation between inessential escapes and returns. The next return of inessential escapes can be an inessential one. All we can say is that an inessential escape has no bound return.

As an immediate corollary of the description given above, it follows that inessential returns are forbidden between two consecutive escapes if there is no essential return between them. Let us summarize some crucial facts on a time history:

- a return that follows essential or substantial escapes is an essential one — [Corollaries 7.1.1.1, 7.1.1.2];
- no bound return follows any essential escapes [Sublemma 7.1.3];
- no bound return follows any inessential escapes [—].

7 Verification of $BR(\alpha,\delta)_n$

In this section we will verify that any $a \in \Omega^{(n)}$ satisfies $BR(\alpha,\delta)_n$, under the next

Inductive assumption: For all $0 \le k \le n-1$, any $a \in \Omega^{(k)}$ satisfies $BR(\alpha,\delta)_k$.

First, let us recall the inductive construction of $\Omega^{(n)}$ and the associated partition $\mathcal{P}^{(n)}$. Suppose steps have been done up to time $n-1$. Then, we define a refinement $\hat{\mathcal{P}}^{(n)}$ of $\mathcal{P}^{(n-1)}$ via c_n, and from it discard bad elements which have strong recurrence and possibly violate $BR(\alpha,\delta)_{n+1}$. This is done in terms of the total sum of essential return depths. Namely, the formal definition is

$$\mathcal{P}^{(n)} := \{\omega \in \hat{\mathcal{P}}^{(n)} : \mathcal{E}^{(n)}(\omega) \le \alpha n/16\},$$

and

$$\Omega^{(n)} := \bigcup_{\omega \in \mathcal{P}^{(n)}} \omega,$$

where

$$\mathcal{E}^{(n)} : \hat{\mathcal{P}}^{(n)} \to \mathbf{N}$$

is a function which corresponds to each $\omega \in \hat{\mathcal{P}}^{(n)}$ the total sum of essential return depths up to time n. Similarly, define $\mathcal{I}^{(n)}$, $\mathcal{B}^{(n)}$, $\mathcal{R}^{(n)}$ as functions which give the total sum of inessential return depths, bound return depths and all return depths of each $\omega \in \hat{\mathcal{P}}^{(n)}$ respectively. By definition

$$\mathcal{R}^{(n)} = \mathcal{E}^{(n)} + \mathcal{I}^{(n)} + \mathcal{B}^{(n)}.$$

For our purpose it suffices to prove the abundance of essential return depths.

Proposition 7. *Assume that any $a \in \Omega^{(k)}$ satisfies $BR(\alpha,\delta)_k$ for all $0 \le k \le n-1$. Then we have*

$$\mathcal{R}^{(k)}(\omega) \le 8\mathcal{E}^{(k)}(\omega)$$

for each $\omega \in \hat{\mathcal{P}}^{(n)}$ and $0 \le k \le n$. In particular, any $a \in \Omega^{(n)}$ satisfies $BR(\alpha,\delta)_n$.

That is to say, the value $\mathcal{E}^{(k)}(\omega)$ accounts for more than 1/8 of the value $\mathcal{R}^{(k)}(\omega)$. It is essential that this ratio is bounded away from zero.

7.1 Preliminaries on time histories

To prove the above proposition requires the following preliminaries.

Sublemma 7.1.1. *Suppose $\alpha/\lambda = \beta < 1/36$. Assume any $a \in \Omega^{(k)}$ satisfies $BR(\alpha,\delta)_k$ for all $0 \le k \le n-1$. Let $\omega \in \mathcal{P}^{(\nu)}$, $0 \le \nu \le n-1$ and suppose that ν is an essential return or an essential escape of ω with the depth r_0. Set $\mu_0 := \nu$ and let $\mu_0 < \mu_1 < \cdots < \mu_u \le n-1$*

be the maximal sequence of the inessential returns before the subsequent chopping time. Let $r_1, \cdots, r_u$ be the corresponding inessential return depths. Then

$$\sum_{i=1}^{u} r_i \leq \frac{1}{2} r_0.$$

In particular, letting ν be an essential escape, we get the following

Corollary 7.1.1.1. *The next return of any essential escape must be an essential return.*

In addition, a similar conclusion holds for substantial escapes.

Corollary 7.1.1.2. *The next return of any substantial escape must be an essential return.*

Proof. of Corollary 7.1.1.2. This is an immediate consequence of the expansion outside the critical neighborhood Δ (Lemma 5.1), and the definition of a substantial escape. Let $\omega \in \hat{\mathcal{P}}^{(\nu)}$ and ν be a substantial escape of ω. Assume that $\mu > \nu$ is an inessential return of ω. By Lemma 5.1 and Corollary 5.2.1, we have $|\omega_\mu| \geq |\omega_\nu|/4 \geq \delta^\iota/4$. However, the right hand side exceeds the length of the union of the two contiguous marginal elements of $\mathcal{I}$, which is a contradiction. □

Remark. As mentioned earlier, these corollaries are crucial in verifying $BR(\alpha, \delta)_n$. They mean that we can take into account all inessential returns by considering only all essential returns. Recall that, logically, an inessential return can follow an essential return, an essential escape, or a substantial escape. But, we have guaranteed that the latter two possibilities cannot occur in reality.

Proof of Sublemma 7.1.1. By (5) of Lemma 5.3, we have

$$|(f_a^{p_i+1})'(c_{\mu_i}(a))| \geq |c_{\mu_i}(a)|^{5\beta-1} \geq e^{(1-6\beta)r_i}.$$

Now, clearly $|c_{\mu_{i+1}}(a)| \leq |c_{\mu_i+p_i+1}(a)|$, and therefore we can use (2) of Lemma 5.1 to obtain

$$|(f_a^{\mu_{i+1}-(\mu_i+p_i+1)})'(c_{\mu_i+p_i+1}(a))| \geq 1.$$

Putting these estimates together for $i = 0, \cdots, u-1$ and using the chain rule give

$$|(f_a^{\mu_u+p_u+1-\mu_0})'(c_{\mu_0}(a))| \geq \exp\Big((1-6\beta)\sum_{i=0}^{u} r_i\Big).$$

By the assumption $\mu_u \leq n-1$ and the Lemma 5.4, any $a \in \omega$ satisfies up to $BR(3\alpha, \delta)_{\mu_u+p_u}$. Thus we can use Corollary 5.2.1 to obtain

$$4 \geq |\omega_{\mu_u+p_u+1}| \geq \frac{1}{4}|\omega_{\mu_0}| \exp\Big((1-6\beta)\sum_{i=0}^{u} r_i\Big).$$

Shrink $\delta > 0$ in such a way that $e^{r_{\delta^+}/4}/r_{\delta^+} \geq 16$. Since μ_0 is either an essential return or an essential escape, we have $|\omega_{\mu_0}| \geq e^{-r_0}/r_0^2 \geq 16e^{-5r_0/4}$ and

$$16e^{-5r_0/4} \leq |\omega_{\mu_0}| \leq 16\exp\Big((6\beta-1)\sum_{i=0}^{u} r_i\Big).$$

Taking the logarithm we obtain

$$\sum_{i=0}^{u} r_i \leq \frac{5}{4(1-6\beta)} r_0 < \frac{3}{2} r_0,$$

by using $\beta < 1/36$. □

Sublemma 7.1.2. *Assume any $a \in \Omega^{(k)}$ satisfies $BR(\alpha,\delta)_k$ for all $0 \le k \le n-1$. Let $\omega \in \mathcal{P}^{(\xi)}$ and suppose that $\xi \in [0, n-1]$ is a return of ω with the return depth r. Let $p > 0$ be the binding period of ξ, $\xi < \zeta_1 < \cdots < \zeta_v \le \xi + p$ be all of the bound returns of ω, and $r_1, \cdots, r_v$ the corresponding bound return depths. Then*

$$\sum_{i=1}^{v} r_i \le \Big(10\alpha + \frac{4}{\lambda}\Big) r < 3r.$$

Proof. Let $\omega = [a_1, a_2]$. The image $c_{\zeta_i}(\omega)$ does not contain 0 because any $a \in \omega$ satisfies up to $BR(5\alpha,\delta)_{\xi+p}$ by Lemma 5.4. In addition, by Lemma 5.2, c_j is a diffeomorphism on ω for $j = 1, \cdots, \xi + p + 1$. Hence, either $c_{\zeta_i}(a_1)$ or $c_{\zeta_i}(a_2)$ gives the corresponding bound return depth. Which of the two actually gives the depth depends on the kneading sequence up to ζ_i. From the proof of Lemma 5.2, we have

$$\frac{c_k'(a)}{(f_a^k)'(c_0(a))} < 0,$$

if f_a satisfies $BR(5\alpha,\delta)_k$. Here, as is usual in the kneading theory, the symbol L denotes the left side position relative to the critical point 0. By the above inequality, $c_{\zeta_i}(a_1)$ gives the corresponding bound return depth if the kneading sequence up to ζ_i has even L and $c_{\zeta_i}(a) < 0$, or the kneading sequence up to ζ_i has odd L and $c_{\zeta_i}(a) > 0$ for $a \in \omega$. Namely we have

$$r_i < \log|c_{\zeta_i}(a_1))|^{-1} + 1 < 2\log|c_{\zeta_i}(a_1)|^{-1}$$

in this case. Otherwise, $c_{\zeta_i}(a_2)$ gives the corresponding bound return depth, and similarly we have $r_i < \log|c_{\zeta_i}(a_2))|^{-1} + 1 < 2\log|c_{\zeta_i}(a_2)|^{-1}$. For $i = 1, 2$, put

$$A_i := \{1 \le k \le v : c_{\zeta_k}(a_i) \text{ gives the corresponding bound return depth}\}.$$

Putting these together yields

$$\sum_{\substack{1\le i\le v\\ i:\text{regular}}} r_i < \sum_{i\in A_1} 2\log|c_{\zeta_i}(a_1)|^{-1} + \sum_{i\in A_2} 2\log|c_{\zeta_i}(a_2)|^{-1}.$$

Applying the same estimates in the previous proof of Lemma 5.4, we have

$$\sum_{i\in A_1} 2\log|c_{\zeta_i}(a_1)|^{-1} < \sum_{i\in A_1} 2\log|c_{\zeta_i-\xi-1}(a_1)|^{-1} + 2\kappa\Lambda,$$

and similarly

$$\sum_{i\in A_2} 2\log|c_{\zeta_i}(a_2)|^{-1} < \sum_{i\in A_2} 2\log|c_{\zeta_i-\xi-1}(a_2)|^{-1} + 2\kappa\Lambda.$$

Recall that we are concerned with only regular bound returns. Hence $c_{\zeta_i-\xi-1}(a_1) \in \Delta$ and by $BR(\alpha,\delta)_p$ we have

$$\sum_{\substack{1\le i\le v\\ i:\text{regular}}} r_i < 4\alpha p + 4\kappa\Lambda < 8\frac{\alpha r}{\lambda} + 4\kappa\Lambda.$$

If κ is chosen as in the proof of Lemma 5.4, this implies

$$\sum_{\substack{1\le i\le v\\ k:\text{regular}}} r_i < 8\frac{\alpha r}{\lambda} + 4\kappa\Lambda < 2\alpha + \frac{8\alpha r}{\lambda} < 10\alpha r.$$

Concerning the irregular bound returns, we trivially have

$$\sum_{\substack{1\le i\le v\\ i:\text{irregular}}} r_i \le \frac{4}{\lambda} r.$$

Putting these together we obtain the conclusion. □

We can show that as far as bound returns are concerned, we only need to consider essential or inessential returns.

Sublemma 7.1.3. *A bound return follows neither essential escapes nor inessential escapes.*

Proof. Let $\omega \in \mathcal{P}^{(k)}$, k be either an essential escape or an inessential escape of ω with the depth r and the binding period p. By the assumption $BR(\alpha,\delta)_k$, we have

$$\log |c_i(a)|^{-1} \leq \alpha i \leq \alpha p \leq \frac{2\alpha}{\lambda} \log |c_k(a)|^{-1},$$

for any $a \in \omega$ and $i = 1, 2, \cdots, p$, which in turn means $|c_i(a)| \geq |c_k(a)|^{\frac{2\alpha}{\lambda}}$. Therefore, we can conclude that $c_{k+1}(a), \cdots, c_{k+p}(a) \notin \Delta$, if δ is small enough to satisfy $\delta^{2\alpha/\lambda} - \delta^2 > \delta$. □

Sublemma 7.1.4. *Assume any $a \in \Omega^{(k)}$ satisfies $BR(\alpha,\delta)_k$ for all $0 \leq k \leq n-1$. Let $\omega \in \mathcal{P}^{(\nu)}$, $0 \leq \nu \leq n-1$ and suppose that ν is the last essential return of ω before n with the return depth r, and n is an inessential return with the return depth ρ. Then we have $\rho < 3r/2$.* □

Proof. By the same reasoning in Sublemma 7.1.1, we obtain $|\omega_n| \geq |\omega_\nu|/4$. Since ν is an essential return, we have $|\omega_\nu| \geq (e^{-r+1} - e^{-r})/r^2$. Thus ρ cannot exceed such $s > 0$ satisfying

$$\frac{|I_s|}{s^2} + \frac{|I_{s-1}|}{(s-1)^2} + \frac{|I_{s-2}|}{(s-2)^2} < \frac{1}{4}\frac{|I_r|}{r^2}.$$

Even more strictly, ρ cannot exceed such $s > 0$ satisfying

$$-\log 12 + s - 2 + 2\log(s-2) > r + \log r,$$

and therefore $\rho \leq 3r/2$. □

7.2 Proof of Proposition 7

We only need to prove the case $k = n$ by the inductive assumption. More precisely, we already have

$$\mathcal{R}^{(k)}(\omega) \leq 8\mathcal{E}^{(k)}(\omega)$$

for $k = 0, \cdots, n-1$ and $\omega \in \hat{\mathcal{P}}^{(n)}$. For $k = n$, the same inequality trivially follows if n is neither an inessential return nor a bound return. In the case where n is a bound return, we count on Sublemmas 7.1.1 and 7.1.2. Repeatedly applying Sublemma 7.1.2 yields

$$\mathcal{R}^{(n)}(\omega) = \mathcal{E}^{(n-1)}(\omega) + \mathcal{I}^{(n-1)}(\omega) + \mathcal{B}^{(n)}(\omega) \leq 4(\mathcal{E}^{(n-1)}(\omega) + \mathcal{I}^{(n-1)}(\omega)).$$

Then we successively apply Sublemma 7.1.1 to obtain

$$4(\mathcal{E}^{(n-1)}(\omega) + \mathcal{I}^{(n-1)}(\omega)) \leq 6\mathcal{E}^{(n-1)}(\omega) \leq 8\mathcal{E}^{(n)}(\omega).$$

If n is an inessential return, we count on Sublemma 7.1.4. Let ρ be the corresponding inessential return depth. Then we have

$$\begin{aligned}
\mathcal{R}^{(n)}(\omega) &= \mathcal{E}^{(n-1)}(\omega) + \mathcal{I}^{(n-1)}(\omega) + \mathcal{B}^{(n-1)}(\omega) + \rho \\
&\leq \frac{3}{2}\mathcal{E}^{(n-1)}(\omega) + \mathcal{B}^{(n-1)}(\omega) + \rho \\
&\leq \frac{3}{2}\mathcal{E}^{(n-1)}(\omega) + 3(\mathcal{E}^{(n-1)}(\omega) + \mathcal{I}^{(n-1)}(\omega)) + \frac{3}{2}\mathcal{E}^{(n-1)}(\omega),
\end{aligned}$$

which is less than

$$8\mathcal{E}^{(n-1)}(\omega) = 8\mathcal{E}^{(n)}(\omega).$$

It only remains to verify $BR(\alpha,\delta)_n$ for $a \in \Omega^{(n)}$. If $\nu \leq n$ is a return with the return depth r, we have $|c_\nu(a)| \geq e^{-(r+1)}$ by the construction. Therefore

$$\sum_{i=0}^{n} \log|c_i(a)|^{-1} \leq \mathcal{R}^{(n)}(a) + \frac{\alpha n}{\log \delta^{-1}} \leq 8\mathcal{E}^{(n)}(a) + \frac{\alpha n}{\log \delta^{-1}}$$
$$\leq \left(\frac{1}{2} + \frac{1}{\log \delta^{-1}}\right)\alpha n < \alpha n.$$

This completes the proof of Proposition 7. □

8 Combinatorial argument

The estimate of the Lebesgue measure of the set $\Omega_\epsilon^* = \bigcap_{n\geq 0} \Omega^{(n)}$ is a technical issue. This is because parameter values for which the corresponding f_a has a periodic attractor form an open dense subset [GS], [Lyu], and hence Ω_ϵ^* is nowhere dense. Therefore quite a delicate analysis is required. In order for our conclusion, combinatorial and analytic arguments need to work together. We remark that any subsequent part is irrelevant to the main induction step of the proof discussed in the previous sections.

First of all, let us introduce some combinatorial notations. Two sequences are associated to each $\omega \in \hat{\mathcal{P}}^{(n)}$. One is made up of all essential and substantial escapes of ω , $0 = \eta_0 < \eta_1 < \cdots < \eta_q \leq n$, and the other is the sequence of corresponding escaping components $\omega^{(\eta_i)} \in \mathcal{P}^{(\eta_i)} (0 \leq i \leq q)$. Letting $\omega^{(\eta_i)} = \omega$ for $q+1 \leq i \leq n$, we define a subset $\mathcal{Q}_n^{(i)}$ of $\bigcup_{k\leq n} \hat{\mathcal{P}}^{(k)}$ as

$$\mathcal{Q}_n^{(i)} = \{\omega^{(\eta_i)} : \omega \in \hat{\mathcal{P}}^{(n)}\}.$$

Intuitively, $\mathcal{Q}_n^{(i)}$ is the collection of the i-th escaping component of each $\omega \in \hat{\mathcal{P}}^{(n)}$. For $\omega \in \mathcal{Q}_n^{(i)}$, put

$$\mathcal{Q}_n^{(i+1)}(\omega) = \{\omega' \in \mathcal{Q}_n^{(i+1)} : \omega' \subset \omega\}$$

and

$$\mathcal{Q}_n^{(i+1)}(\omega, R) = \{\omega' \in \mathcal{Q}_n^{(i+1)}(\omega) : \Delta\mathcal{E}_n^{(i)}(\omega,\omega') = R\},$$

where the function $\Delta\mathcal{E}_n^{(i)}(\omega,\cdot) : \mathcal{Q}_n^{(i+1)}(\omega) \to \mathbf{N}$ is defined by

$$\Delta\mathcal{E}_n^{(i)}(\omega,\omega') = \mathcal{E}^{(\eta_{i+1})}(\omega') - \mathcal{E}^{(\eta_i)}(\omega).$$

We clearly have

$$\mathcal{Q}_n^{(i+1)}(\omega) = \bigsqcup_{R\geq 0} \mathcal{Q}_n^{(i+1)}(\omega, R).$$

Lemma 8.1. *For all* $0 \leq i \leq n-1$, $\omega \in \mathcal{Q}_n^{(i)}$ *and* $R > 0$, *we have*

$$\sharp\mathcal{Q}_n^{(i+1)}(\omega, R) \leq e^{\beta R}.$$

Proof. Define a set of pairs of integers

$$\mathcal{S}_R := \{(r_1,s_1),\cdots,(r_t,s_t) : t \geq 1, \sum_{i=1}^{t} |r_i| = R, |r_i| \geq \log\delta^{-1}, s_i \in [1, r_i^2]\}.$$

We can define a map $\mathcal{F} : \mathcal{Q}_n^{(i+1)}(\omega, R) \to \mathcal{S}_R$ as follows. Suppose $\omega \in \mathcal{P}^{(k)}$. For any $\omega' \in \mathcal{Q}_n^{(i+1)}(\omega, R)$, there exists $\eta_{i+1} \geq k$ such that $\omega' \in \mathcal{P}^{(\eta_{i+1})}$ and η_{i+1} is either an essential or a substantial escape of ω'. Take all essential returns that occur between k and η_{i+1}. The place where each essential return has taken place is specified by the partition $\mathcal{I}$, and hence by a pair of integers (r, s) with $|r| \geq \log \delta^{-1}$ and $s \in [1, r^2]$. $\mathcal{F}(\omega')$ is defined as the sequence of these pairs of integers. Then, we clearly have

$$\sharp \mathcal{Q}_n^{(i+1)}(\omega, R) \leq \sharp \mathcal{S}_R \cdot \sup_{x \in \mathcal{S}_R} \sharp \mathcal{F}^{-1}(x).$$

In fact, $\mathcal{F}$ is not injective. We estimate $\sharp \mathcal{S}_R$ and $\sup_{x \in \mathcal{S}_R} \sharp \mathcal{F}^{-1}(x)$ one by one.

Sublemma 8.1.1. *For $R > 0$ we have*

$$\sharp \mathcal{S}_R \leq e^{\beta R/2}.$$

Proof. Fix $t \leq R/r_\delta$. The number of sequences of natural numbers $r_1, r_2, \cdots, r_t$ with $r_i \geq r_\delta$ and $\sum_{i=1}^t r_i = R$ is less than

$$\binom{R+t-1}{t-1},$$

which is the way of combinations of taking $(t-1)$ balls from $(R+t-1)$ balls located in a row. By using the Stirling's formula

$$k! \sim \sqrt{2\pi k} k^k e^{-k} e^{-k}$$

as $k \to \infty$, we have

$$\binom{R+t-1}{t-1} = \frac{(R+t-1)!}{R!(t-1)!} < \frac{9\sqrt{2}}{8} \frac{(R+t-1)^{R+t-1}}{R^R(t-1)^{t-1}}$$

$$< 2\left(\frac{R+t-1}{R}\right)^R \left(\frac{R+t-1}{t-1}\right)^{t-1}.$$

The first factor is, by $s r_\delta \leq R$, less than

$$\left(\frac{R(1+1/r_\delta)}{R}\right)^R = \left(1 + \frac{1}{r_\delta}\right)^R = e^{R \log(1+\frac{1}{r_\delta})} < e^{\frac{R}{r_\delta}}.$$

For the second factor, we have

$$\left(\frac{R+t-1}{t-1}\right)^{t-1} = \left(\left(\frac{t-1}{R+t-1}\right)^{-\frac{t-1}{R}}\right)^R \leq \left(\left(\frac{t-1}{R}\right)^{\frac{t-1}{R}} \left(1 + \frac{t-1}{R}\right)^{\frac{t-1}{R}}\right)^R,$$

and hence we obtain

$$\left(\frac{R+t-1}{t-1}\right)^{t-1} \leq e^{\beta R/6}.$$

Putting these two inequalities together and recalling the definition of $\mathcal{S}_R$, we have

$$\sharp \mathcal{S}_R \leq \sum_{t=1}^{R/r_\delta} \left(2^t e^{\beta R/5} \prod_{i=1}^t r_i^2\right) < \sum_{t=1}^{R/r_\delta} e^{t \log 2} e^{\beta R/5} R^2 < e^{\beta R/2}$$

for sufficiently small $\delta > 0$. □

Sublemma 8.1.2. *For $R > 0$, we have*

$$\sup_{x \in \mathcal{S}_R} \sharp \mathcal{F}^{-1}(x) \le e^{\beta R/2}.$$

Proof. Let ν_1 be the chopping time of $\omega \in \mathcal{Q}_n^i$, and suppose $x = \{(r_1, s_1), \cdots, (r_m, s_m)\} \in \mathcal{S}_R$, $\mathcal{F}^{-1}(x) \neq \emptyset$. By the algorithm of the subdivision, if $\omega' \in \mathcal{F}^{-1}(x)$ then $c_{\nu_1}(\omega')$ is contained in the union of the three contiguous elements of the partition $\mathcal{I}$ centered on I_{r_1,s_1}, which is denoted by $\hat{I}_{r_1,s_1}$. By Lemma 5.2, c_{ν_1} is a diffeomorphism on ω, and hence

$$\omega^{(\nu_1)} := c_{\nu_i}^{-1}|_{\hat{I}_{r_1,s_1}}(\hat{I}_{r_1,s_1})$$

is an interval which contains $\mathcal{F}^{-1}(x) \neq \emptyset$. For $i = 2, \cdots, m$, let ν_i be the chopping time of $\omega^{(\nu_{i-1})}$ and define $\omega^{(\nu_i)} := c_{\nu_i}^{-1}|_{\hat{I}_{r_i,s_i}}(\hat{I}_{r_i,s_i})$. Then, by a recursive use of Lemma 5.2, one can see that each $\omega^{(\nu_i)}$ is an interval and all preimages $\mathcal{F}^{-1}(x)$ are contained in the set

$$\bigcap_{1 \le i \le m} \omega^{(\nu_i)} = \omega^{(\nu_m)}.$$

Here, the equality is because the sequence $\{\omega^{(\nu_i)}\}_{i=1}^m$ is nested. By the rules of the subdivision and Lemma 5.2, at the chopping time of $\omega^{(\nu_m)}$ there can arise a number of escaping components, whose total number is less than $2(r_\delta - r_{\delta+})r_\delta^2 + 2 < r_\delta^3 < e^{\beta R/2}$. □

9 Analytic Arguments

The aim of this section is to develop key analytic arguments. A bounded distortion argument and a uniform binding estimate are given.

9.1 Bounded Distortion Property

We are going to show that the partitions $\mathcal{P}^{(n_0)}, \mathcal{P}^{(n_0+1)}, \cdots$ have a nice distortion property. The precise statement is as follows.

Lemma 9.1. *Let $\omega \in \mathcal{P}^{(\nu)}$, ν be a free return, an essential escape, or an inessential escape of ω and p be the binding period. Then there exists a constant D such that*

$$\frac{|c_k'(a)|}{|c_k'(b)|} \le D$$

for any $a, b \in \omega$ and $0 \le k \le \nu+p+1$. Moreover, the distortion constant D remains bounded as δ tends to 0.

In terms of the probability theory, this lemma means that the conditional probability of the critical orbit $c_k(a)$ falling into some subinterval J of $c_k(\omega)$ is essentially proportional to the ratio $|J|/|c_k(\omega)|$. In other words, if we consider c_k as a random variable, its distribution is essentially constant bounded away from zero. On the other hand, we cannot expect that the same picture also holds in the case of substantial escapes, because in general the image of substantial escapes can spread out by the definition of the chopping. They have substantial length greater than δ^ι, which leads to nonuniform variation of the distribution. However, if we restrict ourselves to consider some small subinterval of a substantial escape, we can also have an analogue of the above lemma. Both types of distortion properties are indispensable for our ultimate conclusion.

Lemma 9.2. *Let $\omega \in \mathcal{P}^{(\nu)}$, ν be a substantial escape of ω and $l > \nu$ be the next chopping time of ω. Then there exists a constant $\overline{D} > D$ such that for any subinterval $\overline{\omega} \subset \omega$ with $\overline{\omega}_l \subset \Delta^+$, we have*

$$\frac{|c_k'(a)|}{|c_k'(b)|} \le \overline{D}$$

for any $a, b \in \overline{\omega}$ and $0 \le k \le l$. In addition, $\overline{D}$ also remains bounded as δ tends to 0.

9.2 Preliminaries and proofs of the distortion lemmas

We need to classify essential escapes into two classes as follows.

Definition. Let $\omega \in \hat{\mathcal{P}}^{(k)}$ and k be an essential escape of ω. k is said to be a *boundary essential escape* if $\omega_k \cap \partial\Delta^+ \neq \emptyset$, or an *interior essential escape* otherwise.

Note that, from the proof of Sublemma 5.3.1, it suffices to find a constant D such that

$$\sum_{j=0}^{\nu+p} D_j \le \log\left(\frac{D}{4}\right),$$

where

$$D_j := \frac{|\omega_j|}{\inf_{a\in\omega} |c_j(a)|}$$

and $\omega_j = c_j(\omega)$.

Sublemma 9.1.1. *For $\omega \in \mathcal{P}^{(\nu)}$, let ν be a free return, an essential escape, or an inessential escape of ω. Let $\sigma_1 < \sigma_2 < \cdots, < \sigma_q \le \nu$ be the maximal sequence made up of all returns, inessential escapes, and interior essential escapes of ω up to time ν. Denoting the corresponding binding periods and the depths by p_i and r_i, respectively, and letting $\sigma_0 + p_0 + 1 = 0$, we have*

$$\sum_{j=\sigma_i+p_i+1}^{\sigma_{i+1}+p_{i+1}} D_j \le D_\alpha^3 |\omega_{\sigma_{i+1}}| e^{r_{i+1}}$$

for all $i = 0, \cdots, q-1$.

Proof of Sublemma 9.1.1. We divide the sum into three parts:

$$\sum_{\sigma_i+p_i+1}^{\sigma_{i+1}+p_{i+1}} D_j = \sum_{j=\sigma_i+p_i+1}^{\sigma_{i+1}-1} D_j + D_{\sigma_{i+1}} + \sum_{j=\sigma_{i+1}+1}^{\sigma_{i+1}+p_{i+1}} D_j$$

and estimate one by one. For the first term, by Corollary 5.2.1, there exists some $a \in \omega_j$ such that $|(f_a^{\sigma_{i+1}-j})'(c_j(a))||\omega_j| \le 4|\omega_{\sigma_{i+1}}|$. With a possibility of $\omega_{\sigma_{i+1}} \cap \omega_j \neq \emptyset$ taken into account, (3) of Lemma 5.1 can be applied to yield $|\omega_j| \le 8e^{-\hat{\lambda}(\sigma_{i+1}-j)}|\omega_{\sigma_{i+1}}|$. On the other hand, $\inf_{a\in\omega} |c_j(a)| \ge \delta^{\iota} - 2 \cdot |I_{\delta^+}|/r_{\delta^+}^2 > \delta^{\iota}/2$ holds for all $\sigma_i + p_i + 1 \le j \le \sigma_{i+1} - 1$, because there is no return, interior essential escape, nor inessential escape during this time period (boundary essential escapes possibly exist). Therefore we obtain

$$D_j \le \frac{16e^{-\hat{\lambda}(\sigma_{i+1}-j)}|\omega_{\sigma_{i+1}}|}{\delta^{\iota}} \le 16e^{-\hat{\lambda}(\sigma_{i+1}-j)}|\omega_{\sigma_{i+1}}|e^{r_{i+1}},$$

and

$$\sum_{j=\sigma_i+p_i+1}^{\sigma_{i+1}-1} D_j \le 16\Theta|\omega_{\sigma_{i+1}}|e^{r_{i+1}},$$

where $\Theta = \sum_{j\geq 0} e^{-\hat{\lambda} i}$.

The estimate of the second term is trivial. From $|c_{\sigma_{i+1}}(a)| \geq e^{-(r_{i+1}+1)}$, we immediately get $D_{\sigma_{i+1}} \leq e|\omega_{\sigma_{i+1}}|e^{r_{i+1}}$.

Concerning the third term, Corollary 5.2.1 yields

$$|\omega_j| \leq 4|\omega_{\sigma_{i+1}}| \sup_{a\in\omega} |(f_a^{j-\sigma_{i+1}})'(c_{\sigma_{i+1}}(a))|.$$

We need to find a proper upper bound of this supremum. By the chain rule we have

$$|(f_a^{j-\sigma_{i+1}})'(c_{\sigma_{i+1}}(a))| = |(f_a^{j-\sigma_{i+1}-1})'(c_{\sigma_{i+1}+1}(a))| \cdot |f_a'(c_{\sigma_{i+1}}(a))|.$$

The first part of the right hand side can be estimated by the binding argument. Recall the following notations :

$$\gamma = \gamma(a) = [0; c_{\sigma_{i+1}}(a)]$$

$$\gamma_0 = f_a(\gamma), \gamma_j = f_a^j(\gamma_0).$$

By the definition of the binding period and the mean value theorem, we have

$$\kappa e^{-2\alpha(j-\sigma_{i+1}-1)} \geq |\gamma_{j-\sigma_{i+1}-1}| = |(f_a^{j-\sigma_{i+1}-1})'(x_0)||\gamma_0|$$

for some $x_0 \in \gamma_0$. Using Sublemma 5.3.1 yields

$$|(f_a^{j-\sigma_{i+1}-1})'(x_0)||\gamma_0| > D_\alpha^{-1}|(f_a^{j-\sigma_{i+1}-1})'(c_{\sigma_{i+1}+1}(a))|,$$

and therefore

$$|(f_a^{j-\sigma_{i+1}-1})'(c_{\sigma_{i+1}+1}(a))| < D_\alpha \kappa e^{-2\alpha(j-\sigma_{i+1}-1)}/|\gamma_0| \leq D_\alpha \kappa e^2 e^{-2\alpha(j-\sigma_{i+1}-1)} e^{2r_{i+1}},$$

because $|\gamma_0| = |c_{\sigma_{i+1}}(a)|^2 \geq (e^{-(r_{i+1}+1)})^2$. Substituting this into the equality of interest and using a trivial estimate $|f_a'(c_{\sigma_{i+1}}(a))| = 2|c_{\sigma_{i+1}}(a)| \leq 2e^{-r_{i+1}}e$, we have

$$\sup_{a\in\omega} |(f_a^{j-\sigma_{i+1}})'(c_{\sigma_{i+1}}(a))| \leq D_\alpha \kappa e^2 e^{-2\alpha(j-\sigma_{i+1}-1)} e^{2r_{i+1}} 2e^{-r_{i+1}} e,$$

which implies

$$|\omega_j| \leq 8D_\alpha \kappa e^3 e^{-2\alpha(j-\sigma_{i+1}-1)} |\omega_{\sigma_{i+1}}| e^{r_{i+1}}.$$

The remaining task is to estimate the denominator of D_j. Since there may be a bound return during the time period under consideration, a rather delicate estimate is necessary. In this part, the system constant κ plays a crucial role.

Claim.

$$|c_j(a)| \geq (1-\kappa)e^{-\alpha(j-\sigma_{i+1}-1)}.$$

Proof of the claim. By the definition of the binding period and the triangle inequality, we have $|c_j(a)| \geq |c_{j-\sigma_{i+1}-1}(a)| - \kappa e^{-2\alpha(j-\sigma_{i+1}-1)}$. Thus, it suffices to show $|c_i(a)| \geq e^{-\alpha i}$ for each $i \in \mathbf{N}$ such that f_a satisfies $BR(\alpha, \delta)_i$, which has already been proved in Corollary 5.2.2. □

As a consequence of this claim, we have

$$\sum_{j=\sigma_{i+1}+1}^{\sigma_{i+1}+p_{i+1}} D_j < \sum_{j=\sigma_{i+1}+1}^{\sigma_{i+1}+p_{i+1}} \frac{8D_\alpha \kappa e^3}{1-\kappa} e^{-\alpha(j-\sigma_{i+1}-1)} |\omega_{\sigma_{i+1}}| e^{r_{i+1}} < \frac{8D_\alpha \kappa e^3}{1-\kappa} \Xi |\omega_{\sigma_{i+1}}| e^{r_{i+1}},$$

where $\Xi = \sum_{i\geq 0} e^{-\alpha i} = \frac{1}{1-e^{-\alpha}} < D_\alpha$ for sufficiently small α. Finally, combining these three major estimates, we obtain

$$\sum_{j=\sigma_i+p_i+1}^{\sigma_{i+1}+p_{i+1}} D_j < \left(16\Theta + e + \frac{8D_\alpha \kappa e^3}{1-\kappa}\Xi\right)|\omega_{\sigma_{i+1}}|e^{r_{i+1}} < D_\alpha^3 |\omega_{\sigma_{i+1}}|e^{r_{i+1}},$$

for sufficiently small α (hence large D_α). □

Sublemma 9.1.2. *Let $\omega \in \mathcal{P}^{(k)}$ and $\sigma_1 < \cdots < \sigma_s \leq k$ be all returns, inessential escapes and interior essential escapes of ω up to k that have an equal depth r. Then*

$$\sum_{i=1}^{s} |\omega_{\sigma_i}| \leq \frac{10e^{-r}}{r^2}.$$

Proof of Sublemma 9.1.2. Let p_i denote the corresponding binding period of σ_i. Now ω satisfies $BR(\alpha,\delta)_k$ and $\sigma_s \leq k$. Therefore, we can use the binding argument discussed in the proof of Sublemma 7.1.1 to get

$$|(f_a^{\sigma_{i+1}-(\sigma_i+p_i+1)})'(c_{\sigma_i+p_i+1}(a))| \geq 1$$

for $i = 1, \cdots, s-1$. On the other hand, applying the chain rule gives

$$|(f_a^{\sigma_{i+1}-\sigma_i})'(c_{\sigma_i}(a))| = |(f_a^{\sigma_{i+1}-(\sigma_i+p_i+1)})'(c_{\sigma_i+p_i+1}(a))||(f_a^{p_i+1})'(c_{\sigma_i}(a))|.$$

By the above inequality and the (5) of Lemma 5.3, we have

$$\begin{aligned} |(f_a^{\sigma_{i+1}-\sigma_i})' c_{\sigma_i}(a)| &\geq |c_{\sigma_i}(a)|^{5\beta-1} \geq e^{(r+1)(1-5\beta)} \\ &\geq e^{(r_s+1)(1-5\beta)} \geq e^{-\log\delta^{\iota}(1-5\beta)} = \delta^{\iota(5\beta-1)}. \end{aligned}$$

Then, by Corollary 5.2.1, we obtain $|\omega_{\sigma_i}| \leq 4\delta^{\iota(1-5\beta)}|\omega_{\sigma_{i+1}}|$. Successive use of this inequality yields $|\omega_{\sigma_i}| \leq (4\delta^{\iota(1-5\beta)})^{s-i}|\omega_{\sigma_s}|$. Now, shrink δ sufficiently small so that $4\delta^{\iota(1-5\beta)} \leq 1/2$. Then we have

$$\sum_{i=1}^{s} |\omega_{\sigma_i}| \leq |\omega_{\sigma_s}| \sum_{i=1}^{\infty} (4\delta^{\iota(1-5\beta)})^i \leq |\omega_{\sigma_s}|.$$

Hence, it suffices to prove $|\omega_{\sigma_s}| \leq 10e^{-r}/r^2$, which is trivial. Recall that ω_{σ_s} possibly spreads across three contiguous partition elements of $\mathcal{I}^+$. □

Proof of Lemma 9.1. Let $\sigma_1 < \sigma_2 < \cdots < \sigma_q \leq \nu$ be the maximal sequence made up of all returns, inessential escapes and interior essential escapes up to time ν. The corresponding depth is denoted by r_i. For the moment, we postpone the exceptional case where $\sigma_q \neq \nu$, namely σ_q is a boundary essential escape. In the case where σ_q is not a boundary essential escape, we can directly apply Sublemmas 9.1.1 and 9.1.2 to obtain

$$\begin{aligned} \sum_{j=0}^{\nu+p} D_j &= \sum_{i=0}^{q-1} \sum_{j=\sigma_i+p_i+1}^{\sigma_{i+1}+p_{i+1}} D_j \leq D_\alpha^3 \sum_{i=0}^{q-1} |\omega_{\sigma_{i+1}}|e^{r_{i+1}} \\ &= D_\alpha^3 \sum_{r\geq r_{\delta+}} e^r \left(\sum_{i:r_{i+1}=r} |\omega_{\sigma_{i+1}}|\right) \leq \frac{10D_\alpha^3}{\iota r_\delta} < \frac{D_\alpha^4}{\iota r_\delta}. \end{aligned}$$

In the exceptional case where ν is a boundary essential escape, we need to estimate the remaining sum

$$\sum_{j=\sigma_q+p_q+1}^{\nu+p} D_j = \sum_{j=\sigma_q+p_q+1}^{\nu-1} D_j + D_\nu + \sum_{j=\nu+1}^{\nu+p} D_j.$$

Note that, between $\sigma_q + p_q + 1$ and $\nu - 1$, there is no return, inessential escape, nor interior essential escape, all but boundary essential escapes. Hence we have $\inf_{a\in\omega} |c_j(a)| \geq \delta^\iota - 2\cdot|I_{\delta^+}|/r_{\delta^+}{}^2 \geq \delta^\iota/2$. On the other hand, Corollary 5.2.1 and Lemma 5.1, (2), (3) are used to estimate the size of ω_j. Namely we have $\delta^\iota > |\omega_\nu| \geq \frac{1}{2}e^{\hat{\lambda}(\nu-j)}|\omega_j|$. As a consequence, we obtain

$$D_j \leq \frac{4\delta^\iota e^{-\hat{\lambda}(\nu-j)}}{\delta^\iota} = 4e^{-\hat{\lambda}(\nu-j)}.$$

For the second term, we clearly have $D_\nu = |\omega_\nu|/\inf_{a\in\omega} |c_\nu(a)| \leq 2$.

For the third term, we similarly have

$$\sum_{j=\nu+1}^{\nu+p} D_j \leq \frac{16 D_\alpha \kappa e^3 \Xi}{1-\kappa}.$$

As a whole, we obtain

$$\sum_{j=\sigma_q+p_q+1}^{\nu+p} D_j \leq \Theta + 2 + \frac{16 D_\alpha \kappa e^3 \Xi}{1-\kappa}.$$

Hence, we may take

$$D = 4\exp\left(\frac{D_\alpha^4}{\iota r_\delta} + \Theta + 2 + \frac{16 D_\alpha \kappa e^3 \Xi}{1-\kappa}\right).$$

□

Proof of Lemma 9.2. Let ν_q be the last free return, essential escape or inessential escape of ω before l, and p_q the corresponding binding period. By virtue of Lemma 9.1, we have

$$\sum_{j=0}^{\nu_q+p_q} D'_j \leq \sum_{j=0}^{\nu_q+p_q} D_j < \log\left(\frac{D}{4}\right),$$

where $D_j = |\omega_j|/\inf_{a\in\omega}|c_j(a)|$ and $D'_j = |\overline{\omega}_j|/\inf_{a\in\overline{\omega}}|c_j(a)|$. Hence, it suffices to find an upper bound of the remaining part $\sum_{\nu_q+p_q+1}^{l-1} D'_j$. By the choice of ν_q, there is no return, essential escape nor inessential escape (substantial escapes possibly exist). Hence we have

$$\inf_{a\in\overline{\omega}}|c_j(a)| \geq \inf_{a\in\omega}|c_j(a)| \geq \delta^\iota - 2\cdot|I_{r_{\delta^+}}|/r_{\delta^+}^2 > \delta^\iota/2.$$

Meanwhile, ω survives as an element of $\mathcal{P}^{(l-1)}$ due to absence of a chopping time between n and $l-1$, and as a result, ω and its subset $\overline{\omega}$ satisfy $BR(\alpha,\delta)_{l-1}$. Then, by Corollary 5.2.1, we have $|\overline{\omega}_l| \geq \frac{1}{4}e^{\hat{\lambda}(l-j)}|\overline{\omega}_j|$ and the assumption $\overline{\omega}_l \subset \Delta^+$ means $|\overline{\omega}_l| \leq 2\delta^\iota$, which implies

$$D_j \leq \frac{8\delta^\iota e^{-\hat{\lambda}(l-j)}}{\delta^\iota} = 8e^{-\hat{\lambda}(l-j)}.$$

Finally we obtain

$$\sum_{\nu_q+p_q+1}^{l-1} D'_j \leq 8\Theta.$$

We may set $\overline{D} = e^{8\Theta}D$. □

9.3 Uniform binding estimate

We are prepared to prove the following key ingredient on uniform parameter dependence of the derivative recovery. This is essentially the same as (4), (5) of Lemma 5.3. But, we need to take into account the variation of the binding period $p(a,k)$ as a varies in ω. This variation can be treated by the bounded distortion property given above.

Lemma 9.3. *Let $\omega \in \mathcal{P}^{(k)}$, k an essential return or an essential escape of ω, and let p be the corresponding binding period. Then*

$$|\omega_{k+p+1}| \geq |\omega_k|^{8\beta}.$$

Proof. Suppose that $a_0 \in \omega$ gives the minimum of $p(a,k)$. That is to say, we have $p = p(a_0,k) \leq p(a,k)$ for any $a \in \omega$.

Sublemma 9.3.1. *For any $a \in \omega$, we have*

$$|(f_a^{p+1})'(c_k(a))| \geq \frac{1}{16DD_\alpha^2}|(f_{a_0}^{p+1})'(c_k(a_0))|.$$

Proof of the claim. The chain rule gives

$$\frac{|(f_a^{p+1})'(c_k(a))|}{|(f_{a_0}^{p+1})'(c_k(a_0))|} = \frac{|(f_a^p)'(c_{k+1}(a))|}{|(f_{a_0}^p)'(c_{k+1}(a_0))|}\frac{|c_k(a)|}{|c_k(a_0)|}$$

for any $a \in \omega$, $c_0(a) = f_a(0) \in \gamma_0 = f_a([0;c_k(a)])$. Hence, we apply Sublemma 5.3.1 to get

$$|(f_a^p)'(c_{k+1}(a))| \geq D_\alpha^{-1}|(f_a^p)'(c_0(a))|$$

and

$$|(f_{a_0}^p)'(c_0(a_0))| \geq D_\alpha^{-1}|(f_{a_0}^p)'(c_{k+1}(a_0))|.$$

By Lemma 9.1, we have $|c_p'(a)| > D^{-1}|c_p'(a_0)|$ and by Lemma 5.2

$$|(f_a^p)'(c_0(a))| > \frac{1}{4}D^{-1}|(f_{a_0}^p)'(c_0(a_0))|.$$

Putting these three inequalities together, we obtain

$$\frac{|(f_a^p)'(c_{k+1}(a))|}{|(f_{a_0}^p)'(c_{k+1}(a_0))|} > \frac{1}{4DD_\alpha^2}.$$

The remaining term can be easily estimated by $|c_k(a)|/|c_k(a_0)| > 1/4$, since k is an essential return or an essential escape for ω. Hence we get the desired inequality. □

Returning to the proof of the lemma, recall that ω satisfies up to $BR(5\alpha,\delta)_{k+p}$ by Lemma 5.4. Then, applying Corollary 5.2.1 and the above sublemma, we have

$$|\omega_{k+p+1}| > \frac{1}{64DD_\alpha^2}|(f_{a_0}^{p+1})'(c_k(a_0))||\omega_k|.$$

Concerning the right hand side, the following holds.

$$\frac{1}{64DD_\alpha^2}|(f_{a_0}^{p+1})'(c_k(a_0))||\omega_k| > \frac{1}{64DD_\alpha^2}|\omega_k||c_k(a_0)|^{5\beta-1} \geq \frac{1}{64DD_\alpha^2}|\omega_k|e^{-(r+1)(5\beta-1)},$$

where the first inequality is due to (5) of Lemma 5.3 and the second by $|c_k(a_0)| \geq e^{-(r+1)}$.

Claim.

$$\frac{|\omega_k|}{e - e^{-1}} \le e^{-r}.$$

Proof of the claim. This is trivial when k is not a boundary essential escape, due to $\omega_k \subset [e^{-(r+1)}, e^{-(r-1)}]$. Otherwise we have $|\omega_k| \le \delta^{\iota} = e^{-r_{\delta}+} < e^{-r_{\delta}+}(e - e^{-1})$. □

By the above claim,

$$\begin{aligned}\frac{1}{64DD_{\alpha}^2}|\omega_k|e^{-(r+1)(5\beta-1)} &\ge \frac{1}{64DD_{\alpha}^2}|\omega_k|\left(\frac{|\omega_k|}{e-e^{-1}}\right)^{5\beta-1} e^{-5\beta+1}\\ &= \frac{1}{64DD_{\alpha}^2}|\omega_k|e^{-(r+1)(5\beta-1)} \ge \frac{1}{64DD_{\alpha}^2}|\omega_k|^{5\beta}\cdot(e^2-1)^{-5\beta+1}\\ &> |\omega_k|^{8\beta}.\end{aligned}$$

Thus it turns out that the target inequality $|\omega_{k+p+1}| > |\omega_k|^{8\beta}$ holds as long as δ is taken sufficiently small so that

$$\frac{(e^2-1)^{-5\beta+1}}{64DD_{\alpha}^2} \ge (e^{-r+1} - e^{-r-1})^{3\beta}.$$

It is possible to hold the last inequality because the distortion constant D stays bounded as δ tends to 0. □

10 Metric Estimate, Conclusion

This section combines the previously discussed analytic estimate with the combinatorial argument. In terms of the probability theory, we regard $e^{\mathcal{E}^{(n)}/2}$ as a random variable of the suitable probability space. For the conclusion, we need to estimate the conditional probability $|\Omega^{(n-1)}| - |\Omega^{(n)}|$. By definition, we have

$$|\Omega^{(n-1)}| - |\Omega^{(n)}| = \left| \bigcup \{\omega \in \hat{\mathcal{P}}^{(n)} : \mathcal{E}^{(n)}(\omega) \ge \alpha n/16\} \right| \le e^{-\alpha n/32} \int_{\Omega^{(n-1)}} e^{\mathcal{E}^{(n)}(a)/2} da,$$

where the inequality follows from

Tchebichev inequality. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and X be a random variable. Then for $x > 0$ we have*

$$\mu(|X| \ge x) \le \frac{E[|X|]}{x}.$$

Therefore we need to estimate the expectation $E[e^{\mathcal{E}^{(n)}/2}] = \int_{\Omega^{(n-1)}} e^{\mathcal{E}^{(n)}(a)/2} da$. In fact, this quantity is not too big, because members of such $\omega \in \hat{\mathcal{P}}^{(n)}$ that takes on big value of $\mathcal{E}^{(n)}(\omega)$ constitute small portions of $\Omega^{(n-1)}$. More precisely, elements of $\hat{\mathcal{P}}^{(n)}$ with strong recurrence (hence big $\mathcal{E}^{(n)}(\omega)$) are not too many (Lemma 8.1) and are not too large in size (Lemma 10.1). Indeed, the following holds.

Proposition 10.

$$\int_{\Omega^{(n-1)}} e^{\mathcal{E}^{(n)}(a)/2} da \le e^{3n/r_{\delta}} |\Omega_{\epsilon}|.$$

Suppose that this proposition is true. Then we have

$$|\Omega^{(n-1)}| - |\Omega^{(n)}| \le \exp(n(3/r_{\delta} - \alpha/32))|\Omega_{\epsilon}| \le e^{-\alpha n/40}|\Omega_{\epsilon}|,$$

which implies
$$|\Omega^{(n)}| \geq |\Omega^{(n-1)}| - e^{-\alpha n/40}|\Omega_\epsilon|.$$

To prove that $a = 2$ is the density point of Ω_ϵ, let us recall the definition of the first chopping time $n_0(\epsilon) \in \mathbb{N}$. By definition, $\Omega^{(0)} = \Omega^{(1)} = \cdots = \Omega^{(n_0-1)}$. Therefore we have

$$|\Omega^{(n)}| \geq \left(1 - \sum_{i=n_0-1}^{n} e^{-\alpha i/40}\right)|\Omega_\epsilon|,$$

and hence
$$|\Omega_\epsilon^*| = \lim_{n\to\infty} |\Omega^{(n)}| \geq \left(1 - \sum_{i=n_0-1}^{n} e^{-\alpha i/40}\right)|\Omega_\epsilon| > 0.$$

Note that $n_0(\epsilon) \to \infty$ as ϵ tends to zero, which shows the desired density result

$$\lim_{\epsilon\to 0} \frac{|\Omega_\epsilon^*|}{|\Omega_\epsilon|} = 1.$$

For the moment we postpone the proof of Proposition 10, since it requires an intricate analytic estimate to be developed below.

10.1 Preliminaries

Lemma 10.1. *For all $0 \leq i \leq n-1$, $\omega \in \mathcal{Q}_n^{(i)}$, $R \geq 0$ and $\tilde{\omega} \in \mathcal{Q}_n^{(i+1)}(\omega, R)$, we have*
$$|\tilde{\omega}| \leq e^{(9\beta-1)R}|\omega|.$$

Combining Lemmas 8.1 and 10.1 yields the following

Lemma 10.2. *For all $0 \leq i \leq n-1$, $\omega \in \mathcal{Q}_n^{(i)}$ and $R \geq 0$ we have*
$$\sum_{\omega' \in \mathcal{Q}_n^{(i+1)}(\omega,R)} |\omega'| \leq e^{(10\beta-1)R}|\omega|,$$

where $10\beta - 1 < 0$.

Proof of Lemma 10.1. Let $\omega = \omega^{(\nu_0)}$. There are two possibilities. Either there exists a nested sequence
$$\tilde{\omega} \subset \omega^{(\nu_s)} \subset \cdots \subset \omega^{(\nu_1)} \subset \omega^{(\nu_0)} = \omega$$
such that each $\omega^{(\nu_j)}$ $(j = 1, \cdots, s)$ is an essential return at ν_j, or there is not an essential return at all. In the second case, due to the property of time history, $\omega^{(\nu_0)}$ experiences no inessential return until the next chopping time [Corollaries 7.1.1.1, 7.1.1.2]. There are possibly some inessential escapes, but one already knows no bound return follows them [Sublemma 7.1.3]. As a result the inequality of the assertion trivially follows with $R = 0$. Thus we consider the other case. It suffices to show the next

Sublemma 10.1.1. *For $j = 0, \cdots, s-1$, we have*

$$\frac{|\omega^{(\nu_{j+1})}|}{|\omega^{(\nu_j)}|} \leq e^{-r_{j+1}+9\beta r_j}$$

where $r_0 := r_\delta$ and r_i is the depth at ν_i.

Proof of Sublemma 10.1.1. For the moment we postpone the exceptional case where $j = 0$ and ν_0 is a substantial escape of ω. By Lemma 5.4, $\omega^{(\nu_j)}$ satisfies up to $BR(5\alpha, \delta)_{\nu_j+p_j}$.

Then, Lemma 5.2 claims that $c_{\nu_j+p_j+1}$ is a diffeomorphism on $\omega^{(\nu_j)}$, and hence there exists some $a \in \omega^{(\nu_j)}$ such that

$$|\omega^{(\nu_j)}_{\nu_j+p_j+1}| = |c'_{\nu_j+p_j+1}(a)||\omega^{(\nu_j)}|.$$

On the other hand, $\omega^{(\nu_{j+1})}$ satisfies $BR(\alpha,\delta)_{\nu_j+p_j}$ because $\nu_{j+1} > \nu_j + p_j$. Thus for some $b \in \omega^{(\nu_{j+1})}$ we have

$$|\omega^{(\nu_{j+1})}_{\nu_j+p_j+1}| = |c'_{\nu_j+p_j+1}(b)||\omega^{(\nu_{j+1})}|.$$

Combining these equalities and by virtue of Lemma 9.1 we obtain

$$\frac{|\omega^{(\nu_{j+1})}|}{|\omega^{(\nu_j)}|} = \frac{|c'_{\nu_j+p_j+1}(a)|}{|c'_{\nu_j+p_j+1}(b)|} \cdot \frac{|\omega^{(\nu_{j+1})}_{\nu_j+p_j+1}|}{|\omega^{(\nu_j)}_{\nu_j+p_j+1}|} < D\frac{|\omega^{(\nu_{j+1})}_{\nu_j+p_j+1}|}{|\omega^{(\nu_j)}_{\nu_j+p_j+1}|}.$$

The upper estimate of the numerator of the right hand side in terms of $|\omega^{(\nu_{j+1})}_{\nu_{j+1}}|$ is not trivial, due to possible existence of a sequence of inessential returns $\mu_1, \mu_2, \cdots, \mu_m$ between $\nu_j + p_j$ and ν_{j+1}. Let q_i and ρ_i denote the corresponding binding period and return depth respectively. Then, we have

$$\frac{|\omega^{(\nu_{j+1})}_{\nu_{j+1}}|}{|\omega^{(\nu_{j+1})}_{\nu_j+p_j+1}|} = \frac{|\omega^{(\nu_{j+1})}_{\nu_{j+1}}|}{|\omega^{(\nu_{j+1})}_{\mu_m+q_m+1}|} \cdot \frac{|\omega^{(\nu_{j+1})}_{\mu_m+q_m+1}|}{|\omega^{(\nu_{j+1})}_{\mu_1}|} \cdot \frac{|\omega^{(\nu_{j+1})}_{\mu_1}|}{|\omega^{(\nu_{j+1})}_{\nu_j+p_j+1}|}$$

$$\geq e^{\hat{\lambda}(\nu_{j+1}-\mu_m-q_m-1)} \frac{\exp\left((1-6\beta)\sum_{i=0}^{u} \rho_i\right)}{256} \geq \frac{1}{256}.$$

For details, the reader should consult with the proof of Sublemma 7.1.1 using the binding estimate. In particular, we apply (3) of Lemma 5.1 instead of (2), in order to deal with possible overlaps between $\omega^{(\nu_{j+1})}$ and $\omega^{(\nu_{j+1})}_{\mu_m+q_m+1}$. On the other hand, Lemma 9.3 gives a lower estimate of the denominator. Namely we have

$$|\omega^{(\nu_j)}_{\nu_j+p_j+1}| \geq |\omega^{(\nu_j)}_{\nu_j}|^{8\beta} \geq \frac{e^{-8\beta r_j}}{r_j^{16\beta}}.$$

As a result we obtain

$$\frac{|\omega^{(\nu_{j+1})}|}{|\omega^{(\nu_j)}|} \leq 256De^{-r_{j+1}+8\beta r_j} r_j^{16\beta} < e^{-r_{j+1}+9\beta r_j},$$

where it is possible to hold the last inequality by choosing sufficiently small δ, because D stays bounded as $\delta \to 0$.

The exceptional case needs different analysis. A similar argument is invalid since a binding period is not associated with substantial escapes. There are further two possibilities to be considered according to the position of $\omega^{(\nu_0)}_{\nu_1}$, namely, whether $\omega^{(\nu_0)}_{\nu_1} \subset \Delta^+$ or otherwise. The distinction arises when applying Lemma 9.2. In the first case, we can take $\overline{\omega}$ in the statement as the whole $\omega^{(\nu_0)}$. In the second case, $\overline{\omega}$ is taken as the maximal subinterval of $\omega^{(\nu_0)}$ whose image via c_{ν_1} is contained in Δ^+.

First, we treat the case $\omega^{(\nu_0)}_{\nu_1} \subset \Delta^+$. There is no chopping time of $\omega^{(\nu_0)}$ between ν_0 and ν_1, and hence $\omega^{(\nu_0)}$ survives as an element of $\mathcal{P}^{(\nu_1-1)}$ and satisfies $BR(\alpha,\delta)_{\nu_1-1}$. Then, Lemma 5.2 claims that c_{ν_1} is a diffeomorphism on $\omega^{(\nu_0)}$. By using the mean value theorem and Lemma 9.2, we have

$$\frac{|\omega^{(\nu_1)}|}{|\omega^{(\nu_0)}|} = \frac{|c'_{\nu_1}(a)|}{|c'_{\nu_1}(b)|} \cdot \frac{|\omega^{(\nu_1)}_{\nu_1}|}{|\omega^{(\nu_0)}_{\nu_1}|} < \overline{D}\frac{|\omega^{(\nu_1)}_{\nu_1}|}{|\omega^{(\nu_0)}_{\nu_1}|}$$

for some $a \in \omega^{(\nu_0)}$ and $b \in \omega^{(\nu_1)}$. Corollary 5.2.1 gives a lower estimate of the denominator in the form of $|\omega_{\nu_1}^{(\nu_0)}| \geq \frac{1}{4}|\omega_{\nu_0}^{(\nu_0)}| \geq \delta^{\iota}/4$, and therefore

$$\frac{|\omega^{(\nu_1)}|}{|\omega^{(\nu_0)}|} < 4\overline{D}\frac{|\omega_{\nu_1}^{(\nu_1)}|}{|\omega_{\nu_0}^{(\nu_0)}|} < \overline{D}e^{-r_1}\delta^{-\iota} < \overline{D}e^{-r_1}\delta^{-4\beta} < e^{-r_1}e^{9\beta r_\delta},$$

as long as $\iota < 4\beta$. The last inequality follows by taking δ sufficiently small so that $\overline{D} < e^{5\beta r_\delta}$, which is possible because $\overline{D}$ stays bounded as δ tends to 0.

Next we consider the case $\omega_{\nu_1}^{(\nu_0)} \not\subset \Delta^+$, which includes the case $\nu_0 = 0$. With a similar argument and applying Lemma 9.2 with $\varpi = \varpi^{(\nu_0)} := c_{\nu_1}^{-1}(\Delta^+ \cap \omega_{\nu_1}^{(\nu_0)})$, we obtain

$$\frac{|\omega^{(\nu_1)}|}{|\omega^{(\nu_0)}|} \leq \frac{|\omega^{(\nu_1)}|}{|\varpi^{(\nu_0)}|} = \frac{|c'_{\nu_1}(a)|}{|c'_{\nu_1}(b)|}\frac{|\omega_{\nu_1}^{(\nu_1)}|}{|\varpi_{\nu_1}^{(\nu_0)}|} < \overline{D}\frac{|\omega_{\nu_1}^{(\nu_1)}|}{|\varpi_{\nu_1}^{(\nu_0)}|}.$$

By definition, $\varpi_{\nu_1}^{(\nu_0)}$ intersects both $\partial\Delta^+$ and $\partial\Delta$. Thus $|\varpi_{\nu_1}^{(\nu_0)}| > \delta^{\iota} - \delta > \delta^{\iota}/2$ holds and we can proceed an estimate similar to the above. This completes the proof of Sublemma 10.1.1. □

10.2 Proof of Proposition 10.

By definition, we have

$$\int_{\Omega^{(n-1)}} e^{\mathcal{E}^{(n)}(a)/2}da = \sum_{\omega \in \mathcal{Q}_n^{(n)}} e^{\mathcal{E}^{(n)}(\omega)/2}|\omega|,$$

and hence

$$\sum_{\omega \in \mathcal{Q}_n^{(n)}} e^{\mathcal{E}^{(n)}(\omega)/2}|\omega| = \prod_{i=0}^{n-1} \sum_{\omega^{(i+1)} \in \mathcal{Q}_n^{(i+1)}(\omega^{(i)})} e^{\Delta\mathcal{E}^{(i)}(\omega^{(i)},\omega^{(i+1)})/2}|\omega^{(i+1)}|,$$

where $\omega^{(0)} := [2-\epsilon, 2]$. Applying Lemma 10.2 to each factor yields

$$\sum_{\omega^{(i+1)} \in \mathcal{Q}_n^{(i+1)}(\omega^{(i)})} e^{\Delta\mathcal{E}^{(i)}(\omega^{(i)},\omega^{(i+1)})/2}|\omega^{(i+1)}| = \sum_{\omega^{(i+1)} \in \mathcal{Q}_n^{(i+1)}(\omega^{(i)},0)} |\omega^{(i+1)}| + \sum_{R \geq r_\delta} e^{R/2} \sum_{\omega^{(i+1)} \in \mathcal{Q}_n^{(i+1)}(\omega^{(i)},R)} |\omega^{(i+1)}|$$

$$< \left(1 + \sum_{R \geq r_\delta} e^{(10\beta - 1/2)R}\right)|\omega^{(i)}| \leq (1 + e^{-r_\delta/3})|\omega^{(i)}| \leq e^{3/r_\delta}|\omega^{(i)}|$$

for $0 \leq i \leq n-1$ and $\omega^{(i)} \in \mathcal{Q}_n^{(i)}$. Applying this formula to the above nested expression we obtain

$$\int_{\Omega^{(n-1)}} e^{\mathcal{E}^{(n)}(a)/2}da = \sum_{\omega \in \mathcal{Q}_n^{(n)}} e^{\mathcal{E}^{(n)}(\omega)/2}|\omega| \leq e^{3n/r_\delta}|\Omega_\epsilon|.$$

□

References

[BC85] M. Benedicks and L. Carleson - On iterations of $1-ax^2$ on $(-1,1)$, *Ann. of Math.* **122** (1985), 1-25.

[BC91] M. Benedicks and L. Carleson - The dynamics of the Hénon map, *Ann. of Math.* **133** (1991), 73-169.

[BD] C. Bonatti and L. J. Díaz - Persistent nonhyperbolic transitive diffeomorphisms, *Ann of Math.* **143** (1996), 357-396.

[CE] P. Collet and J. P. Eckmann - Positive Lyapunov exponents and absolute continuity for maps of the interval, *Ergod. Th. and Dyn. Sys.* **3** (1983), 13-46.

[DV] W. de Melo and S. van Strien - *One-Dimensional Dynamics*, Springer, 1993.

[GS] J. Graczyk and G. Świątek - Generic hyperbolicity in the logistic family, *Ann. of Math.* **146** (1997), 1-52.

[Gu] J. Guckenheimer - Renormalization of one-dimensional mappings and strange attractors, *Contemp. Math.* **58**, III, Amer. math. Soc. Province, RI. (1987), 143-160.

[Ja] M. Jacobson - Absolutely continuous invariant measures for one-parameter families of one-dimensional maps, *Comm. Math. Phys.* **81(1)** (1981), 39-88.

[Lu] S. Luzzatto - Bounded recurrence of critical points and Jacobson theorem, *London Math, Soc, Lecture Note.* Ser **274** (1999). 173-210.

[Lyu] M. Lyubich - Dynamics of quadratic polynomials, I, II, *Acta Math.* **178** (1997), 185-247, 247-297.

[M] R. Mañé - contributions to the stability conjecture, Topology **17** (1978), 383-396.

[Ne70] S. Newhouse - Nondensity of axiom A on S^2. *Global Analysis pp. Amer. Math. Soc.* (1970), 191-20

[PV] J. Palis & M. Viana - High dimension diffeomorphisms displaying infinitely many periodic attractors, *Ann of Math.(2)* **140** (1994), no.1, 207-250.

[S] M. Shub - Topological transitive diffeomorphisms on T^4, Lecture notes in Math. vol. **206**, 39, Springer-Verlag, 1971.

[Tsu93a] M. Tsujii - Positive Lyapunov exponents in families of one-dimensional dynamical systems, *Invent. Math.* **111** (1993), 113-137.

[Tsu93b] M. Tsujii - A proof of Benedicks-Carleson-Jacobson theorem, *Tokyo J. Math.* **16** (1993), 295-310.

[WY01] Q. Wang and L-S. Young - Strange attractors with one direction of instability, *Comm. Math. Phys.* **218** (2001), 1-97.

[WY] Q. Wang and L-S. Young - "Strange attractors with one direction of instability in n-dimensional spaces," preprint.

[Yoc91] J-C. Yoccoz - Polynômes quadratiques et attracteur de Hénon, *Séminaire Bourbaki.* Vol. 1990/91, 143-165.

[Yoc99] J-C. Yoccoz - Dynamique des polynômes quadratiques, *Panor. Synthèses.* **8** (1999), 187-222.