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Bifurcation Control of Kapitza Pendulum (Kapitza Pendulum の分岐制御)

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1 Introduction

A simple pendulum has an unstable equilibrium state in the upright position. The horizontal movement of the supporting point according to state feedback control is often treated as a good example of the application of modern control theory (for example [1]). In the present study, we focus on another stabilization method without feedback control. Vertical high-frequency excitation of the supporting point (Fig. 1) realizes the stabilization of the unstable upright position without feedback control, which is the so-called "dynamic stabilization," and the stabilized pendulum is called "Kapitza pendulum" [2] [3]. In this study, nonlinear characteristics of a pendulum under high-frequency excitation are theoretically examined based on bifurcation theory. In the analytical approach, it is necessary to separate the motion into components depending on a slow time scale and a fast time scale[4]. The dynamics expressed by the slow time scale dominates the stability of the high-frequency excited pendulum. Therefore, it is essential in the theoretical analysis to separate the components consistently. Under the suitable scaling for the parameters of the system, we introduce three time scales and seek an approximate solution. We transform the equation of motion which is a non-autonomous system into an autonomous system (averaged equation). Then bifurcation analysis is performed by using this autonomous system.

2 Analytical model and equation of motion

We consider a pendulum whose supporting point is excited in the x' direction as shown in Fig. 2. When the excitation displacement is expressed as

$$x'_e = \beta \cos \omega t, \tag{1}$$

the equation of motion with respect to the angle θ is expressed as

$$\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + \frac{g}{l}\sin\theta + \frac{\beta}{l}\omega^2 \cos\omega t \sin(\theta - \gamma) = 0, \tag{2}$$

where c, g, l, β, and ω are viscous damping coefficient, gravity acceleration, length of the pendulum, excitation amplitude, and excitation frequency. For seeking an analytical solution under the high-frequency excitation, the inverse of the excitation frequency $1/\omega$ is introduced as the representative time and then the dimensionless equation of motion is obtained as follows:

$$\ddot{\theta} + \mu\dot{\theta} + \sigma \sin\theta + a\cos t^* \sin(\theta - \gamma) = 0, \tag{3}$$

where $\dot{()}$ denotes the derivative with respect to the dimensionless time, $t^* = \omega t$, and the dimensionless parameters, a, σ, and μ, are expressed as follows:

$$a = \beta/l, \quad \sigma = (g/l)/\omega^2, \quad \mu = c/\omega. \tag{4}$$

For the excitations in the neighborhood of horizontal and vertical directions, i.e., $\gamma = \pi/2 + \Delta\gamma$ and $\gamma = 0 + \Delta\gamma$, Eq. (3) are expressed respectively as follows:

$$\ddot{\theta} + \mu\dot{\theta} + \sigma \sin\theta - a\cos t^* \cos(\theta - \Delta\gamma) = 0 \tag{5}$$

$$\ddot{\phi} + \mu\dot{\phi} - \sigma \sin\phi - a\cos t^* \sin(\phi - \Delta\gamma) = 0, \tag{6}$$

where $\phi = \theta - \pi$. Letting $\Delta\gamma = 0$ for Eq. (6) leads to the equation of motion of Kapitza pendulum.

2.1 Theoretical analysis

We average Eqs. (5) and (6) by using the method of multiple scales. It is essential for obtaining the consistent result to do the suitable scaling based on the physical insight. In the case when bifurcations are produced under the high-frequency excitation, i.e. in the case when the pendulum does not point to the gravity direction, it is expected that the second order derivative term (inertia term) does not balance any terms in the equation of motion. The damping term $\mu\dot{\theta}$ ($\mu\dot{\phi}$) should be the same order as the effect of the gravity $\sigma\sin\theta$ ($\sigma\sin\phi$) to include the damping effect in the slow time scale

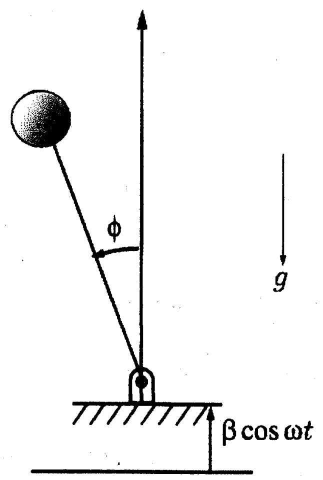

Figure 1: Vertically excited pendulum

dynamics. Taking into account the above physical insight, we perform the scaling of parameters as follows:

$$a \equiv \epsilon\hat{a} \ \ (0 < \epsilon << 1), \quad \sigma \equiv \epsilon^2\hat{\sigma}, \quad \mu \equiv \epsilon\hat{\mu}. \tag{7}$$

Furthermore, for Eqs. (5) and (6), we assume the approximate solutions as follows:

$$\theta(t;\epsilon) = \theta_0(t_0,t_1,t_2) + \epsilon\theta_1(t_0,t_1,t_2) + \epsilon^2\theta_2(t_0,t_1,t_2) + \cdots, \tag{8}$$

$$\phi(t;\epsilon) = \phi_0(t_0,t_1,t_2) + \epsilon\phi_1(t_0,t_1,t_2) + \epsilon^2\phi_2(t_0,t_1,t_2) + \cdots. \tag{9}$$

Introducing the multiple time scales:

$$t_0 = t \quad ,t_1 = \epsilon t \quad ,t_2 = \epsilon^2 t, \tag{10}$$

we seek an approximate solutions, Eqs. (8) and (9).

Hereafter, we show the analytical procedure for Kapitza pendulum (Eq. (6) with $\Delta\gamma = 0$), i.e.,

$$\ddot{\phi} + \mu\dot{\phi} - (\sigma + a\cos t^*)\sin\phi = 0. \tag{11}$$

Substituting Eq. (9) into Eq. (11) and equating the coefficients of like powers of ϵ yield the following equations for orders:

$$O(1) : D_0^2\phi_0 = 0 \tag{12}$$

$$O(\epsilon) : D_0^2\phi_1 = -2D_0D_1\phi_0 - \hat{\mu}D_0\phi_0 + \hat{a}\sin\phi_0\cos t_0 \tag{13}$$

$$\begin{aligned} O(\epsilon^2) : D_0^2\phi_2 = & -2D_0D_1\phi_1 - 2D_0D_2\phi_0 - D_1^2\phi_0 - \hat{\mu}(D_0\phi_1 + D_1\phi_0) \\ & + \hat{\sigma}\sin\phi_0 + \hat{a}\phi_1\cos\phi_0\cos t_0, \end{aligned} \tag{14}$$

where $D_i = \partial/\partial t_i$. The solution of Eq. (12) is expressed as follows:

$$\phi_0 = c_1(t_1,t_2)t_0 + c_0(t_1,t_2), \tag{15}$$

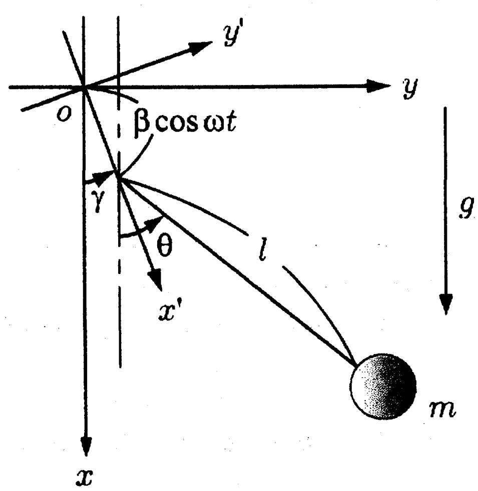

Figure 2: Periodically excited pendulum

where c_0 and c_1 are integral constants. We note that the first term is a secular term of Eq. (15). For a uniform expansion, this term must be eliminated by setting c_1 to zero. Then the general solution becomes

$$\phi_0 = c_0(t_1, t_2). \tag{16}$$

We substitute Eq. (16) into (13) and obtain

$$D_0^2\phi_1 = -2D_0D_1\phi_0 - \hat{\mu}D_0\phi_0 + \hat{a}\sin\phi_0\cos t_0 = \hat{a}\sin c_0\cos t_0. \tag{17}$$

The right-hand side does not contain any terms produce secular terms in ϕ_1. The particular solution of Eq. (17) is

$$\phi_1 = -\hat{a}\sin c_0\cos t_0. \tag{18}$$

Furthermore, substituting Eqs. (18) and (16) into Eq. (14) yields

$$\begin{aligned} D_0^2\phi_2 = &-D_1^2c_0 - \hat{\mu}D_1c_0 + \hat{\sigma}\sin c_0 - \frac{\hat{a}^2}{4}\sin 2c_0 \\ &- \hat{a}(2D_1\sin c_0 + \hat{\mu}\sin c_0)\sin t_0 - \frac{\hat{a}^2}{4}\sin 2c_0\cos 2t_0. \end{aligned} \tag{19}$$

Because the terms which do not explicitly contain t_0 produce a secular term in ϕ_2, the sum of these terms must be set to zero as follows:

$$D_1^2c_0 + \hat{\mu}D_1c_0 - \hat{\sigma}\sin c_0 + \frac{\hat{a}^2}{4}\sin 2c_0 = 0. \tag{20}$$

Then, multiplying both sides by ϵ^2 yields the following equation:

$$\ddot{c}_0 + \mu\dot{c}_0 - \sigma\sin c_0 + \frac{a^2}{4}\sin 2c_0 = 0. \tag{21}$$

Because from Eqs. (9) and (16), ϕ is equal to $c_0(=\phi_0)$ in neglecting the error of $O(\epsilon)$, we can approximately replace c_0 in Eq. (21) by ϕ. Therefore, the equation governing the motion of the pendulum can be approximately described as follows:

$$\ddot{\phi} + \mu\dot{\phi} - \sigma\sin\phi + \frac{a^2}{4}\sin 2\phi = O(\epsilon). \tag{22}$$

As a result, the governing equation (11), which is nonautonomous, is transformed into the autonomous differential equation, i.e., averaged equation (22) by using the method of multiple scales. It is very easy to perform bifurcation analysis for Eq. (22).

By the similar analytical procedure, the averaged equations for Eqs. (5) and (6) are expressed as follows:

$$\ddot{\theta} + \mu\dot{\theta} + \sigma\sin\theta - \frac{1}{4}a^2\sin 2(\theta - \Delta\gamma) = 0 \tag{23}$$

$$\ddot{\phi} + \mu\dot{\phi} - \sigma\sin\phi + \frac{1}{4}a^2\sin 2(\phi - \Delta\gamma) = 0. \tag{24}$$

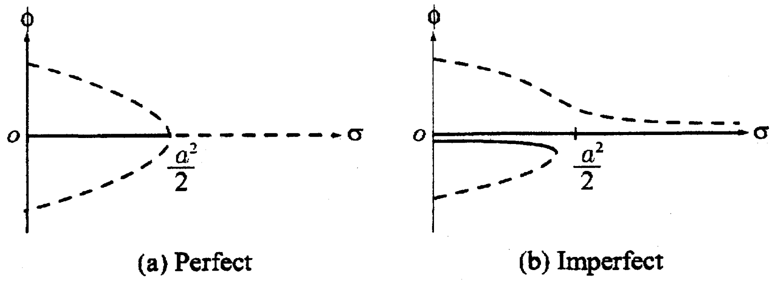

Figure 3: Bifurcation diagram under vertical high frequency excited pendulum (solid line: stable, dashed line: unstable)

2.1.1 Vertical excitation

Neglecting $O(\phi^5)$, $\Delta\gamma^2$, and $\Delta\gamma\phi$, in Eq. (24), we obtain

$$\ddot{\phi} + \mu\dot{\phi} - \left(\sigma - \frac{a^2}{2}\right)\phi - \left(\frac{a^2}{3} - \frac{\sigma}{6}\right)\phi^3 - \frac{a^2}{2}\Delta\gamma = 0. \tag{25}$$

The bifurcation equation is

$$-\left(\sigma - \frac{a^2}{2}\right)\phi - \left(\frac{a^2}{3} - \frac{\sigma}{6}\right)\phi^3 + \frac{a^2}{2}\Delta\gamma = 0. \tag{26}$$

In the case of $\Delta\gamma = 0$ which indicates the completely vertical excitation, the bifurcation diagram is a complete subcritical pitchfork bifurcation as Fig. 3(a)[1] . In the case of $\Delta\gamma \neq 0$ which indicates the excitation tilted from the vertical direction, the subcritical pitchfork bifurcation is perturbed as Fig. 3(b).

2.1.2 Horizontal excitation

Neglecting $O(\phi^5)$, $\Delta\gamma^2$, and $\Delta\gamma\phi$, in Eq. (23), we obtain

$$\ddot{\theta} + \mu\dot{\theta} + \left(\sigma - \frac{a^2}{2}\right)\theta + \left(\frac{a^2}{3} - \frac{\sigma}{6}\right)\theta^3 + \frac{a^2}{2}\Delta\gamma = 0. \tag{27}$$

The bifurcation equation is

$$\left(\sigma - \frac{a^2}{2}\right)\theta + \left(\frac{a^2}{3} - \frac{\sigma}{6}\right)\theta^3 + \frac{a^2}{2}\Delta\gamma = 0. \tag{28}$$

In the case of $\Delta\gamma = 0$ which indicates the completely horizontal excitation, the bifurcation diagram is a complete supercritical pitchfork bifurcation as Fig. 4(a) [2] . In the case of $\Delta\gamma \neq 0$ which indicates the excitation tilted from the horizontal direction, the supercritical pitchfork bifurcation is perturbed as Fig. 4(b).

[1] Near bifurcation point, the coefficient of cubic term is $-\left(\frac{a^2}{3} - \frac{\sigma}{6}\right) = -\left(\frac{a^2}{3} - \frac{a^2}{12}\right) = -a^2/4 < 0$

[2] Near the bifurcation point, the coefficient of cubic term is $\left(\frac{a^2}{3} - \frac{\sigma}{6}\right) = \left(\frac{a^2}{3} - \frac{a^2}{12}\right) = a^2/4 > 0$

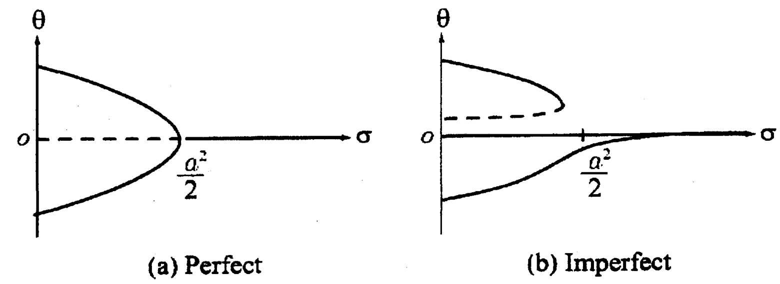

Figure 4: Bifurcation diagram under horizontal high-frequency excited pendulum (solid line: stable, dashed line: unstable)

3 Conclusions

This paper addresses the bifurcation of a high-frequency excited pendulum. Utilizing the method of multiple scales with the suitable scaling of the parameters based on the physical insight, the averaged equation governing the slow time scale dynamics is obtained. Vertical and horizontal high-frequency excitations induce subcritical and supercritical pitchfork bifurcations, respectively. Furthermore, the title of the excitation direction perturbs the above complete subcritical and supercritical pitchfork bifurcations. The validity of analytical results in this paper is experimentally confirmed in [8] and an application of the high-frequency excitation to the motion control of an underactuated manipulator without state feedback is shown in [9].

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