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Extensions by means of expansions and selections

– A summary –

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1. Introduction

The purpose of this report is to announce the principal results of authors' recent paper [15] on extensions of continuous mappings. We give only theorems and their corollaries omitting all proofs and most auxiliary lemmas. For the details, the reader is referred to [15], which will be published elsewhere.

Let λ be an infinite cardinal number. A subset A of a space X is *P^λ-embedded* in X if for every locally finite cozero-set cover $\mathcal{U}$ of A of cardinality $|\mathcal{U}| \leq \lambda$, there is a locally finite cozero-set cover $\mathcal{V}$ of X such that $\mathcal{U}$ is refined by $\mathcal{V} \cap A = \{V \cap A : V \in \mathcal{V}\}$. The notion "$P^\lambda$-embedded" in this sense is the same as "P^λ-embedded" in the sense of Shapiro [33] which was introduced by Arens [3] under the name "λ-normally embedded", see [33].

Our interest in P^λ-embedding was motivated by the following result in [24, Corollary 10] (see, also, [1, Corollary 2.4] and [30, Proposition 3.1]).

Theorem 1.1. *If λ is an infinite cardinal, then a subset A of a space X is P^λ-embedded in X if and only if for every Banach space Y of weight $w(Y) \leq \lambda$, every continuous map $g : A \to Y$ can be extended to a continuous map $f : X \to Y$.*

In the present report, we are concerned with some other embedding-like properties and their possible impact to the extension theory in the light of the above result. To become more specific, let us recall that a subset A of a space X is *C^*-embedded* in X if every bounded real-valued continuous function on A is continuously extendable to the whole of X. If this holds for all real-valued continuous functions on A, then A is called *C-embedded* in X.

Another special embedding we are interested in is given by uniformly locally finite families of sets. A family $\mathcal{U}$ of subsets of a space X is *uniformly locally finite* in X [17, 25, 29] if there exists a locally finite cozero-set cover $\mathcal{V}$ of X such that every

$V \in \mathcal{V}$ meets at most finitely many members of $\mathcal{U}$. Now, a subset A is *U^λ-embedded* in X [16] if every uniformly locally finite collection $\mathcal{U}$ of subsets of A, with $|\mathcal{U}| \leq \lambda$, is uniformly locally finite in X.

It should be mentioned that every C-embedded set is C^*-embedded but the converse fails [8]. In fact, a subset $A \subset X$ is C-embedded in X if and only if it is both U^ω- and C^*-embedded in X [26] (see [1, Proposition 1.6]), which can be expressed in an abstract setting as "$C = U^\omega + C^*$". Here, ω denotes the first infinite ordinal. On the other hand, a subset $A \subset X$ is C-embedded in X if and only if it is P^ω-embedded in X [7], hence we always have that $P^\omega = U^\omega + C^*$. As the reader may expect, the relation $P^\lambda = U^\lambda + C^*$ holds for any infinite cardinal λ, it was actually stated in [16] and shown in [26].

Going back to Theorem 1.1, we become especially interested to subdivide the property of a subset $A \subset X$ that "every continuous map $g : A \to Y$ in a Banach space Y, with $w(Y) \leq \lambda$, can be continuously extended to the whole of X" into two components corresponding to U^λ-embedding and, respectively, C^*-embedding.

Turning to this problem, we need a bit more terminology related to set-valued mappings. For a space Y, we use 2^Y to denote the set of all subsets of Y (not necessarily non-empty), and $\mathcal{C}(Y)$ that of all non-empty compact subsets of Y. A set-valued mapping $\varphi : X \to 2^Y$ is *lower* (*upper*) *semi-continuous*, or l.s.c. (respectively, u.s.c.), if the set $\varphi^{-1}(U) = \{x \in X : \varphi(x) \cap U \neq \emptyset\}$ is open (respectively, closed) in X for every open (respectively, closed) $U \subset Y$. Note that $\varphi : X \to 2^Y$ is u.s.c. if and only if $\varphi^\#(U) = \{x \in X : \varphi(x) \subset U\}$ is open in X for every open $U \subset Y$. A mapping $\varphi : X \to 2^Y$ is *continuous* if it is both l.s.c. and u.s.c. Finally, let us recall that a map $f : X \to Y$ (respectively, $\psi : X \to 2^Y$) is a *selection* for $\varphi : X \to 2^Y$ if $f(x) \in \varphi(x)$ (respectively, $\psi(x) \subset \varphi(x)$) for every $x \in X$. In this case, we also say that φ is an *expansion* of f (respectively, ψ).

The following two theorems will be obtained in this report.

Theorem 1.2. *Let λ be an infinite cardinal. Then, a subset A of a space X is U^λ-embedded in X if and only if for every Banach space Y, with $w(Y) \leq \lambda$, and every continuous map $g : A \to Y$, there exists a continuous mapping $\varphi : X \to \mathcal{C}(Y)$ such that $\varphi|A$ is an expansion of g.*

Theorem 1.3. *A subset A of a space X is C^*-embedded in X if and only if whenever Y is a Banach space and $\varphi : X \to \mathcal{C}(Y)$ is a continuous mapping, every continuous selection $g : A \to Y$ for $\varphi|A$ can be extended to a continuous map $f : X \to Y$.*

Let us stress the reader attention that, in Theorem 1.3, the extension f is not necessarily a selection for φ, but an extension of g which is a selection for φ does exist provided φ is convex-valued, see Theorem 4.1. It should be mentioned that the report provides also mapping-characterizations of some other embedding-like properties (such as C-embedding, z-embedding, etc.) which are in a good accordance with Theorem 1.1, see Sections 3 and 4. Some possible applications are demonstrated in Sections 5 and 6.

2. Covering properties of set-valued mappings

Throughout this section, we will work with indexed families. In their terms, a family $\{A_\gamma : \gamma \in \Gamma\}$ of subsets of a space X is *uniformly locally finite* in X [17, 25, 29] if there exists a locally finite cozero-set cover $\mathcal{V}$ of X such that $\{\gamma \in \Gamma : A_\gamma \cap V \neq \emptyset\}$ is finite for every $V \in \mathcal{V}$. Also, we shall say that $\{A_\gamma : \gamma \in \Gamma\}$ is *uniformly τ-locally finite* in X (for some cardinal $\tau \geq 1$) if for every $\alpha < \tau$ there exists a uniformly locally finite family $\{A_{(\gamma,\alpha)} : \gamma \in \Gamma\}$ of subsets of X such that $A_\gamma \subset \bigcup\{A_{(\gamma,\alpha)} : \alpha < \tau\}$ for every $\gamma \in \Gamma$.

Let X and Y be spaces, A be a subset of X, and $\tau \geq 1$ be a cardinal number. We shall say that $\varphi : A \to 2^Y$ is a *uniformly τ-locally finite lift* if $\{\varphi^{-1}(A_\gamma) : \gamma \in \Gamma\}$ is uniformly τ-locally finite in X for every locally finite family $\{A_\gamma : \gamma \in \Gamma\} \subset 2^Y$. Actually, we will use the same term for single-valued maps as we may consider every $f : A \to Y$ as a set-valued mapping that carries every $x \in A$ to the corresponding singleton $\{f(x)\}$.

We are now ready to state the main result of this section which provides the following characterization of uniformly τ-locally finite lifts in terms of "continuous expansions".

Theorem 2.1. *Let X be a space, A be a subset of X, Y be a connected and locally connected completely metrizable space, $\varphi : A \to 2^Y$, and let $\tau \geq 1$ be a cardinal number. Then φ is a uniformly τ-locally finite lift if and only if for every $\alpha < \tau$ there exists a continuous mapping $\varphi_\alpha : X \to \mathcal{C}(Y)$ such that*

$$\varphi(x) \subset \overline{\bigcup\{\varphi_\alpha(x) : \alpha < \tau\}}, \quad \textit{for every } x \in A.$$

To prove Theorem 2.1 we need the following theorem, which was proved by Nepomnyashchii [28] when $A = \emptyset$. In fact, we prove more than we need but our arguments are simpler and demonstrate that it follows from another result of Nepomnyashchii's in [27].

Theorem 2.2. *Let X be a paracompact space, Y be a completely metrizable space, and let $\Phi : X \to \mathcal{F}(Y)$ be an l.s.c. mapping such that the family $\{\Phi(x) : x \in X\}$ is equi-LCC in Y and each $\Phi(x)$, $x \in X$, is connected. Also, let $\theta : X \to \mathcal{C}(Y)$ be a u.s.c. selection for Φ, $A \subset X$ be closed, and let $\psi : A \to \mathcal{C}(Y)$ be a continuous selection for $\Phi|A$ such that $\theta(x) \subset \psi(x)$ for every $x \in A$. Then, ψ can be extended to a continuous selection $\varphi : X \to \mathcal{C}(Y)$ for Φ such that $\theta(x) \subset \varphi(x)$ for every $x \in X$.*

Since every connected and locally connected completely metrizable space is locally pathwise connected [5, 6.3.11], we have the following corollary which is a special case of Theorem 2.2 when $\Phi(x) = Y$, $x \in X$, and $A = \emptyset$.

Corollary 2.3 ([28]). *Let X be a paracompact space, Y be a connected and locally connected, completely metrizable space, and let $\theta : X \to \mathcal{C}(Y)$ be a u.s.c. mapping. Then, there exists a continuous mapping $\varphi : X \to \mathcal{C}(Y)$ such that $\theta(x) \subset \varphi(x)$ for every $x \in X$.*

We conclude this section demonstrating that, in Theorem 2.1 (and hence, in Corollary 2.3), the requirements on Y to be connected and locally connected are essential. To this end, let us observe that every u.s.c. and compact-valued (briefly, *usco*) mapping, with a metrizable domain, is a uniformly locally finite lift.

Proposition 2.4. *Let X be a metrizable space, Y be a space, and let $\theta : X \to \mathcal{C}(Y)$ be an usco mapping. Then, θ is a uniformly locally finite lift.*

In view of Proposition 2.4, our first example demonstrates that Theorem 2.1 fails if Y is supposed to be only locally connected.

Example 2.5. Let X be a connected space which has an infinite closed discrete set Y. Then, there exists an usco mapping $\theta : X \to \mathcal{C}(Y)$ which is not a selection of any continuous mapping $\varphi : X \to \mathcal{C}(Y)$. In particular, there exists an usco mapping $\theta : \mathbb{R} \to \mathcal{C}(\mathbb{N})$ which is not a selection of any continuous mapping $\varphi : \mathbb{R} \to \mathcal{C}(\mathbb{N})$.

In the same way, Theorem 2.1 fails if Y is supposed to be only connected which is the purpose of our next example.

Example 2.6. Let X be a connected and locally connected space having an infinite discrete closed subset (for instance, the real line $\mathbb{R}$), and let L be the long topologist's sine curve. Then, there exists an usco mapping $\theta : X \to \mathcal{C}(L)$ which is not a selection of any continuous mapping $\varphi : X \to \mathcal{C}(L)$.

Let us recall that the *long topologist's sine curve* L is the subspace

$$L = \{p_0\} \cup \bigcup\{K_n : n \in \mathbb{N}\}$$

of the Euclidean plane $\mathbb{R}^2$, where $p_0 = (0, 0)$ and

$$K_n = \{(x + n - 1, \sin(\pi/x)) : 0 < x \leq 1\}$$

for each $n \in \mathbb{N}$. Then, the space L is connected and completely metrizable.

3. Embedding properties and expansions

In this section, in fact, we provide some further examples of uniformly τ-locally finite lifts. To this end, let us recall that a subset A of a space X is *weakly z_λ-embedded* in X [34] if every uniformly locally finite collection $\mathcal{U}$ of subsets of A, with $|\mathcal{U}| \leq \lambda$, is uniformly ω-locally finite in X. Note that $A \subset X$ is weakly z_λ-embedded in X iff for every uniformly locally finite collection $\{U_\beta : \beta < \lambda\}$ of subsets of A there are uniformly locally finite collections $\{H_{(\beta,n)} : \beta < \lambda\}$, $n < \omega$, of subsets of X such that $U_\beta \subset \bigcup\{H_{(\beta,n)} : n < \omega\}$ for every $\beta < \lambda$. For some other characterizations of weakly z_λ-embedded sets we refer the interested reader to [34].

Now, we consider the following common point of view of both weak z_λ-embedding and U^λ-embedding which will play more technical role simplifying our arguments. Namely, we shall say that a subset A of a space X is *$U^\lambda L^\tau$-embedded* in X (suggesting "λ-Uniformly τ-Locally") if every uniformly locally finite collection $\{U_\beta : \beta < \lambda\}$ of subsets of A is uniformly τ-locally finite in X. Then, A is U^λ-embedded in X iff it is

$U^\lambda L^1$-embedded in X, while A is weakly z_λ-embedded in X iff it is $U^\lambda L^\omega$-embedded in X.

For a cardinal number λ, let $c_0(\lambda)$ be the Banach space of all real-valued functions s on λ such that, for each $\varepsilon > 0$, the set $\{\alpha < \lambda : |s(\alpha)| \geq \varepsilon\}$ is finite, where the linear operations on $c_0(\lambda)$ are defined pointwise, and $\|s\| = \sup\{|s(\alpha)| : \alpha < \lambda\}$ for every $s \in c_0(\lambda)$. It is well-known that $w(c_0(\lambda)) \leq \omega \cdot \lambda$. Note that we may consider a natural partial order in $c_0(\lambda)$ defined for points $s, t \in c_0(\lambda)$ by $s \leq t$ if $s(\alpha) \leq t(\alpha)$ for every $\alpha < \lambda$. Finally, for a subset $T \subset c_0(\lambda)$ and a point $s \in c_0(\lambda)$, let us agree to write that $s \leq \limsup T$ (respectively, $\liminf T \leq s$) if for every $\varepsilon > 0$ there exists $t \in T$, with $s(\alpha) < t(\alpha) + \varepsilon$ (respectively, $t(\alpha) - \varepsilon < s(\alpha)$) for every $\alpha < \lambda$.

Our first result unifies both U^λ-embedding and weak z_λ-embedding via expansion of mappings, and provides one of our basic examples of uniformly τ-locally finite lifts.

Theorem 3.1. *Let λ be an infinite cardinal, and $\tau \geq 1$ be a cardinal. For a subset A of a space X, the following conditions are equivalent*:

(a) *A is $U^\lambda L^\tau$-embedded in X.*
(b) *Whenever Y is a Banach space, with $w(Y) \leq \lambda$, every continuous mapping $\psi : A \to \mathcal{C}(Y)$ is a uniformly τ-locally finite lift.*
(c) *Every continuous map $g : A \to c_0(\lambda)$ is a uniformly τ-locally finite lift.*
(d) *Whenever $g : A \to c_0(\lambda)$ is a continuous map, there are continuous maps $\ell_\alpha, u_\alpha : X \to c_0(\lambda)$, $\alpha < \tau$, with $\liminf_{\alpha<\tau} \ell_\alpha(x) \leq g(x) \leq \limsup_{\alpha<\tau} u_\alpha(x)$ for every $x \in A$.*
(e) *Whenever $g : A \to c_0(\lambda)$ is a continuous map, there are continuous maps $f_\alpha : X \to c_0(\lambda)$, $\alpha < \tau$, with $g(x) \leq \limsup_{\alpha<\tau} f_\alpha(x)$ for every $x \in A$.*

Note that if $T = \{t\} \subset c_0(\lambda)$ is a singleton and $y \in c_0(\lambda)$, then $y \leq \limsup T$ (respectively, $\liminf T \leq y$) implies $y \leq t$ (respectively, $t \leq y$). Hence, by Theorem 2.1 and the case $\tau = 1$ of Theorem 3.1, we have the following immediate result. In particular, it provides the proof of Theorem 1.2 stated in the Introduction.

Corollary 3.2. *Let λ be an infinite cardinal. For a subset A of a space X, the following conditions are equivalent*:

(a) *A is U^λ-embedded in X.*
(b) *Whenever Y is a Banach space, with $w(Y) \leq \lambda$, and $\psi : A \to \mathcal{C}(Y)$ is a continuous mapping, there exists a continuous mapping $\varphi : X \to \mathcal{C}(Y)$ such that $\psi(x) \subset \varphi(x)$ for every $x \in A$.*
(c) *Whenever Y is a Banach space, with $w(Y) \leq \lambda$, and $g : A \to Y$ is a continuous map, there exists a continuous mapping $\varphi : X \to \mathcal{C}(Y)$ such that $g(x) \in \varphi(x)$ for every $x \in A$.*
(d) *Whenever $g : A \to c_0(\lambda)$ is a continuous map, there exist continuous maps $\ell, u : X \to c_0(\lambda)$ such that $\ell(x) \leq g(x) \leq u(x)$ for every $x \in A$.*
(e) *Whenever $g : A \to c_0(\lambda)$ is a continuous map, there exists a continuous map $f : X \to c_0(\lambda)$ such that $g(x) \leq f(x)$ for every $x \in A$.*

As usual, we write c_0 for $c_0(\omega)$. The equivalence of (a) and (c) of the following partial case of Corollary 3.2 was proven in [13].

Corollary 3.3. *For a subset A of a space X, the following conditions are equivalent*:

(a) *A is U^ω-embedded in X.*

(b) *Whenever $g : A \to c_0$ is a continuous map, there exists a continuous map $f : X \to c_0$ such that $g(x) \leq f(x)$ for every $x \in A$.*

(c) *Whenever $g : A \to \mathbb{R}$ is a continuous function, there exists a continuous function $f : X \to \mathbb{R}$ such that $g(x) \leq f(x)$ for every $x \in A$.*

In what follows, let us agree to say that a set-valued mapping $\psi : X \to \mathcal{F}(Y)$ is *lower σ-continuous* if there exists a sequence $\{\varphi_n : n < \omega\}$ of continuous mappings $\varphi_n : X \to \mathcal{C}(Y)$ such that

$$\psi(x) = \overline{\bigcup\{\varphi_n(x) : n < \omega\}}, \quad \text{for every } x \in X.$$

Note that every lower σ-continuous mapping is l.s.c. as a union of l.s.c. mappings, see [5, 1.7.17]. Concerning the inverse relation, we refer the reader to the next section where we provide a characterization of lower σ-continuous mappings in terms of "l.s.c factorizations" through metrizable spaces.

By Theorem 2.1 and the case $\tau = \omega$ of Theorem 3.1, we also have the following mapping-characterization of weak z_λ-embedding.

Corollary 3.4. *Let λ be an infinite cardinal. For a subset A of a space X, the following conditions are equivalent*:

(a) *A is weakly z_λ-embedded in X.*

(b) *Whenever Y is a Banach space, with $w(Y) \leq \lambda$, and $\psi : A \to \mathcal{C}(Y)$ is a continuous mapping, there exists a lower σ-continuous mapping $\varphi : X \to \mathcal{F}(Y)$ such that $\psi(x) \subset \varphi(x)$ for every $x \in A$.*

(c) *Whenever Y is a Banach space, with $w(Y) \leq \lambda$, and $g : A \to Y$ is a continuous map, there exists a lower σ-continuous mapping $\varphi : X \to \mathcal{F}(Y)$ such that $g(x) \in \varphi(x)$ for every $x \in A$.*

(d) *Whenever $g : A \to c_0(\lambda)$ is a continuous map, there are continuous maps $\ell_n, u_n : X \to c_0(\lambda)$, $n < \omega$, such that $\liminf_n \ell_n(x) \leq g(x) \leq \limsup_n u_n(x)$ for every $x \in A$.*

(e) *Whenever $g : A \to c_0(\lambda)$ is a continuous map, there are continuous maps $f_n : X \to c_0(\lambda)$, $n < \omega$, such that $g(x) \leq \limsup_n f_n(x)$ for every $x \in A$.*

Remark. The reader might be wonder if, in Corollary 3.4, for every continuous map $g : A \to c_0(\lambda)$ there exists a sequence $\{f_n : n < \omega\}$ of continuous maps $f_n : X \to c_0(\lambda)$ such that for every $x \in A$ one can find an $n(x) < \omega$, with $g(x) \leq f_{n(x)}(x)$. In general, this is not true which is demonstrated by the following example: Let $D(c_0)$ be the set $c_0 = c_0(\omega)$ endowed with the discrete topology, and let X be the one-point compactification of $D(c_0)$. Also, consider the identity map $g : D(c_0) \to c_0$ from the discrete space $D(c_0)$ to the Banach space c_0.

For an infinite cardinal λ, a space X is said to have the *property* (U^λ) if every locally finite collection $\mathcal{F}$ of subsets of X, with $|\mathcal{F}| \leq \lambda$, is uniformly locally finite, see [16]. Also, let us recall that a map $g : X \to c_0(\lambda)$ is *upper semi-continuous* if for every $x \in X$ and every $\varepsilon > 0$, there exists a neighbourhood G of x in X such that if $x' \in G$, then $g(x')(\alpha) < g(x)(\alpha) + \varepsilon$ for every $\alpha \in c_0(\lambda)$, see [14].

As another application of Theorem 2.1, we have the following expansion characterization of the property (U^λ).

Theorem 3.5. *For an infinite cardinal λ, the following conditions on a space X are equivalent:*

(a) *X has the property (U^λ).*
(b) *Whenever Y is a Banach space, with $w(Y) \leq \lambda$, and $\psi : X \to \mathcal{C}(Y)$ is a u.s.c. mapping, there exists a continuous mapping $\varphi : X \to \mathcal{C}(Y)$ such that $\psi(x) \subset \varphi(x)$ for each $x \in X$.*
(c) *Whenever $g : X \to c_0(\lambda)$ is an upper semi-continuous map, there exists a continuous map $f : X \to c_0(\lambda)$ such that $g(x) \leq f(x)$ for each $x \in X$.*

The next corollary follows from Theorem 3.5 and [14, Corollary 5.6].

Corollary 3.6. *For an infinite cardinal λ, a normal space X has the property (U^λ) if and only if X is λ-collectionwise normal and countably paracompact.*

As it was shown in [16], a space X has the property (U^ω) if and only if X is a *cb*-space in the sense of Mack [19]. Thus, the following corollary is a special case of Theorem 3.5, where the equivalence of (a) and (c) was proven by Mack in [18, Theorem 1].

Corollary 3.7. *The following conditions on a space X are equivalent:*

(a) *X is a cb-space.*
(b) *For every upper semi-continuous map $g : X \to c_0$, there exists a continuous map $f : X \to c_0$ such that $g(x) \leq f(x)$ for every $x \in X$.*
(c) *For every upper semi-continuous map $g : X \to \mathbb{R}$, there exists a continuous map $f : X \to \mathbb{R}$ such that $g(x) \leq f(x)$ for every $x \in X$.*

4. Embedding properties and selections

Here, we deal with another component of P^λ-embedding providing characterizations of weakly embedding properties in terms of controlled extensions of maps with values in arbitrary Banach spaces.

In what follows, a subset A of a space X is *z-embedded* in X if each zero-set of A is the restriction to A of a zero-set of X. Also, for a Banach space Y, we use $\mathcal{C}_c(Y)$ (respectively, $\mathcal{F}_c(Y)$) to denote all convex members of $\mathcal{C}(Y)$ (respectively, $\mathcal{F}(Y)$).

The following provides, in particular, Theorem 1.3 stated in the Introduction.

Theorem 4.1. *For a subset A of a space X, the following are equivalent:*

(a) *A is C^*-embedded in X.*

(b) *Whenever Y is a Banach space and $\varphi : X \to \mathcal{C}_c(Y)$ is continuous, every continuous selection $g : A \to Y$ for $\varphi|A$ can be extended to a continuous selection $f : X \to Y$ for φ.*
(c) *Whenever Y is a Banach space and $\varphi : X \to \mathcal{C}(Y)$ is continuous, every continuous selection $g : A \to Y$ for $\varphi|A$ can be extended to a continuous map $f : X \to Y$.*
(d) *Whenever λ is a cardinal and $\ell, u : X \to c_0(\lambda)$ are continuous maps such that $\ell(x) \le u(x)$ for every $x \in A$, every continuous map $g : A \to c_0(\lambda)$, with $\ell(x) \le g(x) \le u(x)$ for every $x \in A$, can be extended to a continuous map $f : X \to c_0(\lambda)$.*

Our next purpose is to characterize C-embedding in a similar way. To prepare for this, we first establish a result that sheds some light about the proper place of lower σ-continuous mappings.

Let Y be a metrizable space, $\mathcal{P}$ be a property of set-valued mappings, and let $\psi : X \to \mathcal{F}(Y)$ have the property $\mathcal{P}$, briefly $\psi \in \mathcal{P}$. A triple (Z, h, Ψ) is a $\mathcal{P}$-*factorization* for ψ (see [10]) if

(i) Z is a metrizable space, with $w(Z) \le w(Y)$,
(ii) $h : X \to Z$ is a continuous map,
(iii) $\Psi : Z \to \mathcal{F}(Y)$ is a mapping, with $\Psi \in \mathcal{P}$ and $\psi = \Psi \circ h$.

Finally, for a Banach space Y, we let $\mathcal{S}_c(Y) = \{S \in \mathcal{F}_c(Y) : S \text{ is separable}\}$.

Lemma 4.2. *Let Y be a Banach space. For a set-valued mapping $\psi : X \to \mathcal{S}_c(Y)$ the following conditions are equivalent:*

(a) *ψ is lower σ-continuous.*
(b) *ψ has a lower σ-continuous factorization (Z, h, Ψ).*
(c) *ψ has an l.s.c. factorization (Z, h, Ψ).*
(d) *There exists a countable set $\mathcal{T} \subset C(X, Y)$ such that $\{f(x) : f \in \mathcal{T}\}$ is dense in $\psi(x)$ for every $x \in X$.*

It is probably the place to remark that Lemma 4.2 may have some independent interest being a typical selection-factorization result. In fact, natural applications of that lemma could be related to the existence of continuous selections with some special properties which is demonstrated in this report as well. Towards this end, let us observe that lower σ-continuity is preserved by the usual operation of convex-closure.

Proposition 4.3. *Let X be a space, Y be a Banach space, and let $\varphi : X \to \mathcal{F}(Y)$ be lower σ-continuous. Define $\psi(x) = \overline{\operatorname{conv}(\varphi(x))}$ for every $x \in X$. Then, ψ is lower σ-continuous too.*

We are now ready for the promised characterization of C-embedding.

Theorem 4.4. *For a subset A of a space X, the following conditions are equivalent:*

(a) *A is C-embedded in X.*

(b) *Whenever Y is a Banach space and $\varphi : X \to \mathcal{F}_c(Y)$ is lower σ-continuous, every continuous selection $g : A \to Y$ for $\varphi|A$ can be extended to a continuous selection $f : X \to Y$ for φ.*
(c) *Whenever Y is a Banach space and $\varphi : X \to \mathcal{F}(Y)$ is lower σ-continuous, every continuous selection $g : A \to Y$ for $\varphi|A$ can be extended to a continuous map $f : X \to Y$.*
(d) *If λ is a cardinal and $g : A \to c_0(\lambda)$ is a continuous map such that there are continuous maps $\ell_n, u_n : X \to c_0(\lambda)$, $n < \omega$, with the property that $\liminf_{n<\omega} \ell_n(x) \leq g(x) \leq \limsup_{n<\omega} u_n(x)$ for every $x \in A$, then g can be extended to a continuous map $f : X \to c_0(\lambda)$.*
(e) *If λ is a cardinal and and $g : A \to c_0(\lambda)$ is a continuous map such that there are continuous maps $\ell_n, u_n : X \to c_0(\lambda)$, $n < \omega$, with the property that for every $x \in A$ there is an $n(x) < \omega$, with $\ell_{n(x)}(x) \leq g(x) \leq u_{n(x)}(x)$, then g can be extended to a continuous map $f : X \to c_0(\lambda)$.*

A few words about the proper place of Theorems 4.1 and 4.4 should be mentioned. First of all, let us stress the reader's attention that in the speacial case of a dense subset $A \subset X$, the equivalence (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a) of Theorem 4.1 was established by Sanchis in [32, Theorem 3.1], similarly for Theorem 4.4 (see [32, Theorem 4.1]). Also, let us stress the attention that both Theorems 4.1 and 4.4 remain valid if in (b) and (c) of these theorems the partial selection g is merely supposed to be non-empty compact-valued and continuous, i.e. $g : A \to \mathcal{C}(Y)$. In this case, the resulting extension will be a continuous mapping $f : X \to \mathcal{C}(Y)$ such that $f|A = g$. In fact, taking in mind that $g : A \to \mathcal{C}(Y)$ is a continuous mapping if and only if g is a continuous map of A into the space $(\mathcal{C}(Y), \tau_V)$, we can obtain this as a consequence of the corresponding statements for single-valued maps.

We complete this section with a similar selection-extension characterization of z-embedding. To this end, we shall say that a set-valued mapping $\theta : X \to \mathcal{C}(Y)$ is *upper δ-continuous* if there exists a sequence $\{\varphi_n : n < \omega\}$ of continuous mappings $\varphi_n : X \to \mathcal{C}(Y)$ such that $\theta(x) = \bigcap\{\varphi_n(x) : n < \omega\}$, for every $x \in X$. Let us stress the reader's attention that every upper δ-continuous mapping is u.s.c. as an intersection of usco mappings (see [5, 3.12.28]). In fact, modulo factorizations through metrizable spaces, the converse holds as well.

Lemma 4.5. *Let Y be a Banach space. For a set-valued mapping $\theta : X \to \mathcal{C}_c(Y)$, the following conditions are equivalent:*

(a) *θ is upper δ-continuous.*
(b) *θ has an upper δ-continuous factorization (Z, h, Θ).*
(c) *θ has a u.s.c. factorization (Z, h, Θ).*

Here is an important example of upper δ-continuous mappings.

Proposition 4.6. *Let Y be a Banach space, $\varphi : X \to \mathcal{C}(Y)$ be continuous, and let $\theta : X \to \mathcal{C}_c(Y)$ be a selection for φ such that $\theta^{-1}(F)$ is a zero-set of X for every closed $F \subset Y$. Then, θ is upper δ-continuous.*

We are now ready for our characterization of z-embedding.

Theorem 4.7. *For a subset A of a space X, the following conditions are equivalent:*

(a) *A is z-embedded in X.*

(b) *Whenever Y is a Banach space and $\varphi : X \to \mathcal{C}_c(Y)$ is continuous, every continuous selection $g : A \to Y$ for $\varphi|A$ can be extended to an upper δ-continuous selection $\theta : X \to \mathcal{C}_c(Y)$ for φ in sense that $\theta(x) = \{g(x)\}$ for every $x \in A$.*

(c) *Whenever Y is a Banach space and $\varphi : X \to \mathcal{C}(Y)$ is continuous, every continuous selection $g : A \to Y$ for $\varphi|A$ can be extended to an upper δ-continuous mapping $\theta : X \to \mathcal{C}_c(Y)$.*

(d) *Whenever λ is a cardinal and $\ell, u : X \to c_0(\lambda)$ are continuous maps such that $\ell(x) \leq u(x)$ for every $x \in A$, every continuous map $g : A \to c_0(\lambda)$, with $\ell(x) \leq g(x) \leq u(x)$ for every $x \in A$, can be extended to an upper δ-continuous mapping $\theta : X \to \mathcal{C}_c(c_0(\lambda))$.*

(e) *Every bounded continuous function $g : A \to \mathbb{R}$ can be extended to an upper δ-continuous mapping $\theta : X \to \mathcal{C}_c(\mathbb{R})$.*

Theorem 4.7 provides also a factorization property of z-embedding. Namely, it implies the following simple consequence which demonstrates that, with respect to continuous maps controlled by continuous compact-valued expansions, the z-embedded subsets are, in fact, subsets of metrizable spaces.

Corollary 4.8. *For a subset A of a space X, the following conditions are equivalent:*

(a) *A is z-embedded in X.*

(b) *Whenever Y is an infinte metrizable space, $\varphi : X \to \mathcal{C}(Y)$ is continuous, and $g : A \to Y$ is a continuous selection for $\varphi|A$, there exists a metrizable space Z, with $w(Z) \leq w(Y)$, a continuous map $h : X \to Z$, and a continuous map $f : h(A) \to Y$ such that $g = f \circ (h|A)$.*

(c) *Whenever $g : A \to \mathbb{R}$ is a continuous bounded function, there exists a separable metrizable space Z, a continuous map $h : X \to Z$, and a continuous function $f : h(A) \to \mathbb{R}$ such that $g = f \circ (h|A)$.*

5. Subdividing and generating extensions by means of expansions and selections

In this section we provide some possible applications of our extension results for weakly-embedding properties. In fact, we have the following three results suggesting the genesis of the extension property given by P^λ-embedding. The first one is an immediate consequence of Theorem 1.1, Corollary 3.2 and Theorem 4.1.

Corollary 5.1. *Let λ be an infinite cardinal, and A be a subset of a space X. Then, A is P^λ-embedded in X if and only if it is both U^λ-embedded and C^*-embedded in X, i.e.*

$$P^\lambda = U^\lambda + C^*.$$

In the same way, by Theorem 1.1, Corollary 3.4 and Theorem 4.4, we get the following consequence.

Corollary 5.2. *Let λ be an infinite cardinal, and A be a subset of a space X. Then, A is P^λ-embedded in X if and only if it is both weakly z_λ-embedded and C-embedded in X, i.e.*

$$P^\lambda = wz_\lambda + C.$$

To prepare for our third consequence, we first provide the following further extension property of P^λ-embedding.

Theorem 5.3. *Let λ be an infinite cardinal. For a subset A of a space X, the following conditions are equivalent:*

(a) *A is P^λ-embedded in X.*
(b) *Whenever Y is a Banach space, with $w(Y) \leq \lambda$, every continuous map $g : A \to Y$ can be extended to an upper δ-continuous mapping $\theta : X \to \mathcal{C}_c(Y)$.*

Combining Theorem 5.3 with Corollary 3.2 and Theorem 4.7, we finally get also the following result.

Corollary 5.4. *Let λ be an infinite cardinal, and A be a subset of a space X. Then, A is P^λ-embedded in X if and only if it is both U^λ-embedded and z-embedded in X, i.e.*

$$P^\lambda = U^\lambda + z.$$

6. Boundary avoiding selections and C-embedding

In this section, we provide some further applications of our mapping-characterizations of weakly-embedding properties. Towards this end, we first establish the following improvement in Theorem 4.4.

Theorem 6.1. *For a subset A of a space X, the following conditions are equivalent:*

(a) *A is C-embedded in X.*
(b) *If Y is an open convex subset of a Banach space E, $\varphi : X \to \mathcal{F}_c\left(\overline{Y}\right)$ is lower σ-continuous, and $g : A \to E$ is a continuous selection for $\varphi|A$, with $g^{-1}(Y) = \varphi^{-1}(Y) \cap A$, then g can be extended to a continuous selection $f : X \to E$ for φ such that $f^{-1}(Y) = \varphi^{-1}(Y)$.*
(c) *If Y is an open convex subset of a Banach space E, $\varphi : X \to \mathcal{C}_c\left(\overline{Y}\right)$ is continuous, and $g : A \to E$ is a continuous selection for $\varphi|A$, with $g^{-1}(Y) = \varphi^{-1}(Y) \cap A$, then g can be extended to a continuous selection $f : X \to E$ for φ such that $f^{-1}(Y) = \varphi^{-1}(Y)$.*

To prepare for the proof of Theorem 6.1, we need the following lemma which was actually proven in [4]. We can give a simple proof and demonstrate that it is, in fact, a consequence of the Michael's technique stated in [23].

Lemma 6.2. *Let X be a paracompact space, Y be an open convex subset of a Banach space E, $\varphi : X \to \mathcal{F}_c\left(\overline{Y}\right)$ be l.s.c., and let B be an F_σ-subset of X, with $B \subset \varphi^{-1}(Y)$. Then, φ has a continuous selection $\ell : X \to Y$ such that $B \subset \ell^{-1}(Y)$.*

In what follows, let us recall that a subset A of a space X is *well-embedded* if it is completely separated from any zero-set of X disjoint from A. The next result completes the preparation for the proof of Theorem 6.1, and, in particular, provides a mapping-like characterization of well-embedding.

Theorem 6.3. *For a subset A of a space X, the following conditions are equivalent:*

(a) *A is well-embedded in X.*

(b) *If Y is an open convex subset of a Banach space E, $\varphi : X \to \mathcal{F}_c(\overline{Y})$ is lower σ-continuous, and $g : X \to E$ is a continuous selection for φ, with $g^{-1}(Y) \cap A = \varphi^{-1}(Y) \cap A$, then there exists a continuous selection $f : X \to E$ for φ such that $f|A = g|A$ and $f^{-1}(Y) = \varphi^{-1}(Y)$.*

(c) *If Y is an open convex subset of a Banach space E, $\varphi : X \to \mathcal{C}_c(\overline{Y})$ is continuous, and $g : X \to E$ is a continuous selection for φ, with $g^{-1}(Y) \cap A = \varphi^{-1}(Y) \cap A$, then there exists a continuous selection $f : X \to E$ for φ such that $f|A = g|A$ and $f^{-1}(Y) = \varphi^{-1}(Y)$.*

We complete this report with two consequences. The first one demonstrates a generalization of a result in [6] which was established in [35].

Corollary 6.4 ([35]). *Let X be a space, A be a C-embedded subset of X, Z_0 and Z_1 be disjoint zero-sets in X, and let $g : A \to [0,1]$ be a continuous function, with $Z_i \cap A = g^{-1}(i)$, $i = 0, 1$. Then, g can be extended to a continuous function $f : X \to [0,1]$ such that $Z_i = f^{-1}(i)$, $i = 0, 1$.*

Our second consequence follows immediately from Theorems 4.1, 6.1 and 6.3. It demonstrates as the principle difference between the C^*- and C-embedding as an alternative proof of the formula $C = C^* +$ "well-embedded" (e.g. [2, Theorem 6.7] or [8, pp. 19]).

Corollary 6.5. *A subset A of a space X is C-embedded in X if and only if it is both C^*- and well-embedded in X, i.e.*

$$C = C^* + \text{"well-embedded"}.$$

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