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Algebraic invariants preserved by Bohr homeomorphisms*

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"In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics"

Hermann Weyl

1 Introduction

The encounter of Algebra and Topology in the field of Topological Groups is the best instance to observe how these disciplines can interact in a strong way. This is witnessed, in particular, by the remarkable (algebraic) properties of the homeomorphisms in the Bohr topology.

1.1 The Bohr topology

A Hausdorff abelian group G is *totally bounded* iff every non-empty open subset U of G admits a finite subset F of G such that $G = U + F$. In particular, the compact groups and their subgroups are totally bounded. It was proved by A. Weil that these are *all* totally bounded groups, i.e., the totally bounded groups are precisely the subgroups of the compact groups. On the other hand, the class of totally bounded groups is closed under arbitrary products. Hence every group topology of an abelian group G induced by a family H of homomorphisms $G \to \mathbb{T}$ is totally bounded. The proof of the much deeper fact that *every* totally bounded group topology of G has this form can be attributed to Følner (see [11] for a reasonably elementary exposition). For an abelian Hausdorff group $(G, \mathcal{T})$ let $\widehat{G}$ be the group of all continuous characters of $(G, \mathcal{T})$. The topology induced on G by the diagonal map $G \to \mathbb{T}^{\widehat{G}}$ is called the Bohr topology of $(G, \mathcal{T})$. The group G equipped with this topology is denoted by G^+. The group G is *maximally almost periodic* (brifely MAP) if G^+ is Haudorff. The completion bG of G^+ is widely known as the *Bohr compactification* of G ([28]). The continuous inclusion map $\rho_G : G \to bG$ is universal with respect to all continuous homomorphisms $f : G \to K$, where K is a compact group (i.e., there exists a unique continuos homomorphism $\tilde{f} : bG \to K$ such that $f = \tilde{f} \circ \rho_G$).

In this survey we shall be interested mainly in the Bohr topology of a *discrete* abelian group G. Clearly, this is the maximal totally bounded group topology of G. In this case the notation $G^\#$ is used instead of G^+. Hence this is the initial topology of *all* homomorphisms $G \to \mathbb{T}$. Since every discrete abelian group G is MAP, one has an embedding $G^\# \hookrightarrow \mathbb{T}^{Hom(G,\mathbb{T})}$. We keep the notation bG for the Bohr compactification of G. Clearly, this is the closure in $\mathbb{T}^{Hom(G,\mathbb{T})}$ of the image of G under this embedding.

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Now we list some properties of $G^{\#}$ in the next theorem. The first two are due to Comfort and Saks [4]:

Theorem 1.1 *Let G be an infinite abelian group. Then:*

1. *$G^{\#}$ is not pseudocompact.*

2. *every subgroup of $G^{\#}$ is closed.*

3. *If H is a subgroup of G, then $H^{\#}$ is a topological subgroup of $G^{\#}$ (i.e., $H \overset{alg.}{\hookrightarrow} G$ yields $H^{\#} \hookrightarrow G^{\#}$).*

For further properties of the Bohr topology the reader may see [18, 31, 32, 27, 24, 25, 29, 30, 6, 8, 16, 5].

1.2 The Bohr topology of the bounded abelian groups

The group G is *bounded*, if $mG = 0$ for some integer $m > 1$, where $mG = \{mx : x \in G\}$. A typical example to this effect is the group $\mathbb{V}_m^\kappa = \bigoplus_\kappa \mathbb{Z}_m$, where κ is cardinal and $\mathbb{Z}_m$ is the cyclic group of order m. Now the homomorphisms $G \to \mathbb{Z}_m$ suffice to describe the Bohr topology of G and a typical neighborhood of 0 in $G^{\#}$ is a *finite-index subgroup* of G (see [6, 8, 29, 16] for a more detailed descrition of the Bohr topology of $\mathbb{V}_m^\kappa$). It is not clear how much this specific fact has determined the best level of knowledge of the Bohr topology for the class of bounded abelian groups.

By Prüfer's theorem [20, Theorem 17.2] every abelian group G of finite exponent is a direct sum of cyclic groups, so has the form

$$G = \bigoplus_{p \in \mathbf{P}} \bigoplus_{k \in \omega} \mathbb{V}_{p^k}^{\kappa_{p,k}},$$

where only finitely many of the cardinals $\kappa_{p,k}$ are non-zero. The cardinals $\kappa_{p,k}$ are known as *Ulm-Kaplansky invariants* of G (for the definition of the Ulm-Kaplansky invariants of *arbitrary* abelian groups see [20, §37]).

For a bounded group G the *essential order* $eo(G)$ of G is *the smallest positive integer* m with mG finite (e.g., $eo(\mathbb{V}_{91}^2 \times \mathbb{V}_7^3 \times \mathbb{V}_2^\omega \times \mathbb{V}_3^{\omega_1}) = 6$). Then, $G = F \times H$, with $mH = 0$ and F finite.

1.3 van Douwen's homeomorphism problem

In the sequel we write $G \approx H$ ($G \approx_u H$) for topological groups G and H to denote that they are (uniformly) homeomorphic as topological (resp., unform) spaces. Since we are considering only abelian groups, all three uniformities appearing usually in the framework of topological groups coincide in this case.

E. van Douwen [19] posed the following challenging problem in 1987 [1, Question 515]:

Problem 1.2 (van Douwen) *Does $|G| = |H|$ for abelian groups G, H imply $G^{\#} \approx H^{\#}$?*

It is easy to see that most of the currently used topological cardinal invariants of a group of the form $G^{\#}$ depend only on the size $|G|$ (i.e., $w(G^{\#}) = \chi(G^{\#}) = 2^{|G|}$, $d(G^{\#}) = |G|$, $\psi(G^{\#}) = \log|G|$, $\dim G^{\#} = indG^{\#} = 0$, etc.). Hence topological cardinal invariants cannot help to answer this question. This suggests the idea to check whether some algebraic invariants of the group G are preserved by Bohr homeomorphisms. This turned out to be the right clue later on ([7]). Before touching this specific point we recall the relevant steps towards the solution of the problem.

The first instance of a pair of non-isomorphic groups that are Bohr homeomorphic was given by Trigos [32, Theorem 6.33] – if an abelian group G has a subgroup H of index n and $G \cong H$, then $G^{\#} \approx G^{\#} \times \mathbb{Z}_n$:

Theorem 1.3 (Trigos) *For $n < \omega$ if G admits a monomorphism $f : G \to G$ such that $[G : f(G)] = n$, then $G^{\#} \approx G^{\#} \times \mathbb{Z}_n$. In particular, $\mathbb{Z}^{\#} \approx \mathbb{Z}^{\#} \times \mathbb{Z}_n$.*

As a matter of fact, it is easy to see that for the subgroup $H = f(G)$ the obvious homeomorphism $G^{\#} \approx H^{\#} \times \mathbb{Z}_n$ is actually *uniform* (this holds true for every subgroup H of index n and makes no use of the monomorphism f). So $G^{\#} \approx_u H^{\#} \times \mathbb{Z}_n$, allong with the topological group isomorphism $G^{\#} \cong H^{\#}$ (due to the isomorphism $f : G \to H$) gives $G^{\#} \approx_u G^{\#} \times \mathbb{Z}_n$. Hence $\mathbb{Z}^{\#} \approx_u \mathbb{Z}^{\#} \times \mathbb{Z}_n$.

A negative solution to van Douwen's Problem was obtained in November 1996 by Kunen [29] and independently, almost at the same time, by Watson and the author [15] (even if the paper appeared in printed form somewhat later [16]).

Theorem 1.4 (Kunen [29]) *$\mathbb{V}_p^{\omega\#} \not\approx \mathbb{V}_q^{\omega\#}$ for primes $p \neq q$.*

Watson and the author [16] proved that $\mathbb{V}_2^{\kappa\#} \not\to \mathbb{V}_m^{\kappa\#}$ for $m \neq 2$ and $\kappa > 2^{2^{\mathfrak{c}}}$.

Following Hart and Kunen [24], call a pair G, H of abelian groups *almost isomorphic* if G and H have isomorphic finite index subgroups. The next theorem generalizes Theorem 1.3:

Theorem 1.5 (Hart and Kunen [24]) *If G, H are almost isomorphic abelian groups, then $G^{\#} \approx H^{\#}$.*

We give a detailed proof of this theorem in §2.1. Since the above theorem presents the only known positive *general* result on Bohr homeomorphisms, the next question, posed by Kunen [29], seems very natural:

Question 1.6 *Is the implication in Theorem 1.5 reversible?*

The answer to this question will be discussed in §2.2. In the same section we discuss also the following *uniform* version of van Douwen's Problem

Problem 1.7 *When $|G| = |H|$ for abelian groups G, H implies $G^{\#} \approx_u H^{\#}$?*

Clearly, the condition $G^{\#} \approx_u H^{\#}$ is more restrictive than just $G^{\#} \approx H^{\#}$. Hence Theorem 1.4 already gives the first answer "not always". On the other hand, $\mathbb{Z}^{\#} \approx_u \mathbb{Z}^{\#} \times \mathbb{Z}_n$ shows that non-isomorphic groups may be uniformly homeomorphic in the Bohr topology.

We are not discussing here another intersting van Douwen's problem concerning retracts in the Bohr topology (see [23, 21, 2, 9, 5]).

1.4 Group properties invariant under Bohr homeomorphisms

The negative solution of van Douwen's problem 1.2 makes the first three items in the following definition meaninglful. Call a pair G, H of infinite abelian groups:

1. *Bohr-equivalent* if $G^{\#} \approx H^{\#}$;
2. *strongly Bohr-equivalent* if G^{κ} and H^{κ} are Bohr-equivalent for every cardinal κ;
3. *uniformly Bohr-equivalent* if $G^{\#} \approx_u H^{\#}$;

4. *weakly Bohr-equivalent* if there exist embeddings $G^\# \hookrightarrow H^\#$ and $H^\# \hookrightarrow G^\#$;
5. *weakly isomorphic* if $|mG| \cdot |mH| \geq \omega$ implies $|mG| = |mH|$ for every $m \in \mathbf{N}$.
6. *c-equivalent* if G admits a compact group topology iff H does.
7. *cc-equivalent* if G admits a countably compact group topology iff H does.
8. *psc-equivalent* if G admits a pseudocompact group topology iff H does.

In these terms Kunen [29] proved that $\mathbb{V}_p^\omega$ and $\mathbb{V}_q^\omega$ are not even weakly Bohr-equivalent for distinct primes p, q, while Theorem 1.5 asserts that almost isomorphic groups are always Bohr-equivalent. Our purpose will be to clarify the relations between these properties.

2 Around almost isomorphism

2.1 Proof of Theorem 1.5

According to van Douwen [17], if X is a regular countable homogeneous space, then every pair U and V of non-empty clopen sets of X are homeomorphic. For the sake of completeness we give a proof of a slightly more precise version of this fact in the case when $X = G^\#$ for a countable abelian group G.

Claim 1. If G is a countably infinite abelian group and U, V are a non-empty clopen set of $G^\#$, then there exist clopen partitions $U = \bigcup_m A_m$ and $V = \bigcup_n B_m$, and a homeomorphism $h : U \to V$ such that for every m the restriction h_m of h to A_m is a translation t_m of the group G carrying A_m onto B_m.

Proof. Let $U = \{g_1, \dots, g_n, \dots\}$ and $V = \{x_1, \dots, x_n, \dots\}$. Let h_1 be the translation carrying g_1 to x_1. Since $G^\#$ is zero-dimensional and U, V are clopen, there exist proper clopen subsets $g_1 \in A_1 \subset U$ and $x_1 \in B_1 \subset V$ such that $h_1(A_1) = B_1$. Then $U_1 = U \setminus A_1$ and $V_1 = V \setminus B_1$ are non-empty clopens sets. Let n_1 and k_1 be minimal such that $g_{n_1} \in U_1$ and $x_{k_1} \in V_1$. Choose analogously clopen proper clopen subsets $g_{n_1} \in A_2 \subseteq U_1$ and $x_{k_1} \in B_2 \subseteq V_1$ so that the translation $x \mapsto x + x_{n_1} - g_{n_1}$ carries A_2 onto B_2. Build analogously $A_3, \dots, A_m, \dots$ and $B_3, \dots, B_m, \dots$ and note that $\bigcup_{i=1}^k A_i$ contains at least $g_1, \dots, g_k$ and $\bigcup_{i=1}^k B_i$ contains at least $x_1, \dots, x_k$, therefore, $U = \bigcup_m A_m$ and $V = \bigcup_m B_m$. QED

It follows from the above claim that if G, H are countably infinite abelian groups that are not weakly Bohr-equivalent, then one can find either a non-empty clopen set of $G^\#$ that cannot be embedded in $H^\#$, or a non-empty clopen set of $H^\#$ that cannot be embedded in $G^\#$.

The proof given below follows the lines of the proof [24].

Proof of Theorem 1.5. If G is a countably infinite abelian group and H is a finite index subgroup of G, then H is clopen (being a closed subgroup of finite index). So the above claim gives a homeomorphism $h : G^\# \to H^\#$ with the above mentioned properties.

If the group G is uncountable, then there exists a subgroup N of H such that the quotient G/N is countably infinite. Let $f : G \to G/N$ be the canonical homomorphism. Then $f(H)$ is a finite index subgroup of G/N. By Claim 1 there exists clopen partitions $G/N = \bigcup_m A_m$, $f(H) = \bigcup_n B_m$ and a family of elements a_m of G/N such that the translation $t_m : x \mapsto x + a_m$ of G/N carries A_m onto B_m. For every m let b_m be an element of G such that $f(b_m) = a_m$. Let $A'_m = f^{-1}(A_m)$ and $B'_m = f^{-1}(B_m)$. Then $G = \bigcup_m A'_m$ and $H = \bigcup_n B'_m$ are clopen partitions. Finally, let s_m be the traslation $y \mapsto y + b_m$ of the group G. Then $f \circ s_m = t_m \circ f$, and consequently $s_m(A'_m) = B'_m$. Therefore the family (s_m) defines a homeomorphism $h : G^\# \to H^\#$ in the usual way (for every m define h to coincide on A'_m with s_m). QED

Example 2.1 It is easy to see that Theorem 1.5 cannot be extended to uniform homeomorphisms. It suffices to see that there exists no uniform homeomorphism $h : \mathbb{Q}^\# \to (\mathbb{Q} \times \mathbb{Z}_2)^\#$. Indeed, if such an h exists, then it can be extended to the completions to give a homeomorphism between $b\mathbb{Q}$ and $b(\mathbb{Q} \times \mathbb{Z}_2)$. Since $b\mathbb{Q}$ is connected and $b(\mathbb{Q} \times \mathbb{Z}_2) = b\mathbb{Q} \times \mathbb{Z}_2$ is not, we arrive at a contradiction.

The same argument proves

Theorem 2.2 *If D is a divisible abelian group and $G^\# \approx_u D^\#$, then also G is divisible.*

Inspired by the above example and by Theorem 1.3 let us consider for infinite abelian groups G and H the following conditions:

(a) there exist finite groups F, F' such that $G \times F' \cong H \times F$;

(b) G and H are almost isomorphic, denoted by $G \sim H$ in the sequel;

(c) all infinite Ulm-Kaplanski invariants of G coincide with the respective Ulm-Kaplanski invariants of H.

In general these conditions need not be equivalent. It is easy to see that (a) is equivalent also to the following

(a′) there exist finite subgroups F, F' of G and H respectively, such that $G = G_1 \times F$, $H = H_1 \times F'$ and $G_1 \cong H_1$.

Lemma 2.3 *Let G and H be infinite abelian groups. Then always a) $\Rightarrow$ b) $\Rightarrow$ c) $\Rightarrow$ d). If the groups G, H are bounded, all they are equivalent.*

The easy proof of the lemma is based on the fact that all binary relations defined above are equivalence relations (in the larger sense) satisfying the following easy to check propeties:

(i) all three conditions (a)-(c) are preserved under taking finite products;

(ii) all three conditions are local (i.e., if G and H satisfy some of them, then also their p-primary components do).

(iii) if G and H satisfy (a), then $t(G)$ and $t(G)$ satisfy (a) and $G/t(G) \cong H/t(H)$ (where $t(G)$ denotes the torsion subgroup of the group G);

(iv) if $G \sim H$, then $t(G) \sim t(H)$ and $G/t(G) \sim H/t(H)$;

(v) $G \sim H$ implies $t_p(G) \cong t_p(H)$ for almost all p and $t_p(G) \sim t_p(H)$ for all p (where $t(G)$ denotes the p-torsion subgroup of G). If H and G are torsion, the conjunction of these two properties implies $G \sim H$.

(vi) $G \sim H$ iff their maximal divisible subgroups $d(G)$, $d(H)$ are isomorphic and the reduced groups $G/d(G)$ and $H/d(H)$ are almost isomorphic (it suffices to note that every finite index subgroup contains the maximal divisible subgroup).

By means of these properties one can complete Lemma 2.3 and determine the precise relations between the properties (a)–(c) in various classes of groups.

(A) For divisible abelian groups the relations (a) and (b) coincide with the usual $\cong$, while any pair of divisible abelian groups vacuously satisfies (c).

(B) For torsion-free groups (a) coincides with $\cong$ (by (iii)), while $G \sim H$ need not imply $G \cong H$. Indeed, there exist (finite rank) torsion-free abelian group G non-isomorphic to its subgroups of finite index. Hence, (b) is a weaker condition than (a) in the class of torsion-free abelian groups. Finally, any pair of torsion-free abelian groups vacuously satisfies (c).

(C) Combining the properties (i)-(vi), one can limit the torsion case to the reduced p-torsion one. More precisely, if for all pairs of reduced p-torsion group G, H (b) implies (a), then also for all pairs of torsion groups G, H (b) implies (a). In particular, this gives: For all pairs G, H of torsion abelian groups such that each primary component is bounded conditions (a) and (b) are equivalent, while the condition (c) is properly weaker (just take the groups $G = \bigoplus_p \mathbb{Z}_p$ and $H = G^2$).

2.2 Answer to Question 1.6

The following theorem of Comfort, Hernández and Trigos [2] opened new insights on Bohr homeomorphisms:

Theorem 2.4 [2] *Let G be an abelian group and let A be a subgroup of G that is either finitely generated or has finite index. Then $G^\# \approx (G/A)^\# \times A^\#$.*

As a corollary it provides an immediate negative answer to Question 1.6.

Example 2.5 ([Comfort-Hernández-Trigos [2]) $\mathbb{Q}^\# \approx (\mathbb{Q}/\mathbb{Z})^\# \times \mathbb{Z}^\#$, *but* $\mathbb{Q} \not\sim \mathbb{Q}/\mathbb{Z} \times \mathbb{Z}$, *according to (iv).*

As another application of 2.4 we show how this theorem can be used as a formidable tool for creating Bohr homeomorphisms "out of nothing".

Since every abelian group G having a subgroup H of index $n < \omega$ satisfies $G^\# \approx H^\# \times_u \mathbb{Z}_n$ (see the comment after Theorem 1.3), clearly Theorem 1.5 follows from the next:

Claim 2. If H is a abelian group then $H^\# \approx H^\# \times \mathbb{Z}_n$ for every $n < \omega$.

We do not know whether Claim 2 has a proof simpler than Hart-Kunen's proof of Theorem 1.5 given above. The next observation shows that this is the case for non-torsion H.

Observation 2.6 If $n < \omega$ and H is a non-torsion abelian group then $H^\# \approx G^\# \times \mathbb{Z}_n$. Indeed, let c be a generator of $\mathbb{Z}_n$ and let a be a non-torsion element of H. Then (a, c) is a non-torsion element of $G = H \times \mathbb{Z}_n$. The cyclic subgroup of G generated by (a, c) is infinite, so also $C_1 = C \cap H$ is an infinite cyclic group with $C \cong C_1$ and $C/C_1 \cong \mathbb{Z}_n$. By Theorem 2.4 $G^\# \approx (G/C)^\# \times C^\#$, while $G/C = (H + C)/C \cong H/C_1$. Hence, Theorem 2.4 applied to H gives

$$G^\# \approx (H/C_1)^\# \times C^\# \approx (H/C_1)^\# \times C_1^\# \approx H^\#.$$

2.3 Kunen's conjecture in the realm of bounded groups

Here we give evidence to support the hope for a positive answer to Question 1.6 in the realm of bounded groups.

Using the fact that a discrete abelian group G has a totally disconnected Bohr compactification iff G is bounded torsion, one obtains an immediate proof of the following fact:

Theorem 2.7 *If H is bounded and $G^\#$ admits a uniform embedding into $H^\#$, then also the group G is bounded. Uniform Bohr-equivalence preserves boundedness.*

Using appropriate "hypergraph spaces" (in the line of a similar approach exploiting the chromatic number of graphs from [30]) Givens and Kunen [22] obtained the following much stronger statement as well as a series of important results that we give below.

Theorem 2.8 (Givens and Kunen [22]) *If H is bounded and $G^\# \hookrightarrow H^\#$, then also G is bounded. Consequently, the weak Bohr-equivalence preserves boundedness.*

Theorem 2.9 (Givens and Kunen [22]) *If p is a prime and K is an infinite abelian group of exponent p, then the following are equivalent for an abelian group G:*

(a) $G^\#$ is homeomorphic to a subset of $K^\#$;

(b) G is almost isomorphic to a subgroup of K;

Clearly, if $|G| = |H|$ in the above theorem, then the equivalent conditions imply $G^\# \approx H^\#$.

Theorem 2.10 (Givens and Kunen [22]) *$eo(G) = eo(H)$ for weakly Bohr-equivalent bounded groups G, H such that one of them is either countable or has a prime exponent.*

3 The full power of the weak Bohr-equivalence

Here we see that Theorem 2.10 can be strenthened as follows (see also Corollary 3.5).

Theorem 3.1 *For bounded abelian groups G, H*

"weakly isomorphic" $\Rightarrow$ "weakly Bohr-equivalent" $\Rightarrow$ $eo(G) = eo(H)$.

All three properties coincide in the case of countable groups.

3.1 The Straightening Law and its corollaries

The proof of Theorem 3.1 is based on the following Straightening Law (a preliminary form was announced by the author in Prague 2001 [8]):

Straightening Law Theorem. *Let $m > 1$ and $\pi : \mathbb{V}_m^{\kappa\,\#} \hookrightarrow H^\#$ be an embedding with $\pi(0) = 0$ into an abelian group H. If either H is bounded or $\kappa > \beth_{2m-1}$, then there exists an infinite subset A of $\mathbb{V}_m^\kappa$ such that:*

(a) $\langle A \rangle \cong \mathbb{V}_m^\kappa$;

(b) $\pi \restriction_A = \ell \restriction_A$ for some injective homomorphism $\ell : \langle A \rangle \to H$.

Let us underline the importance of the fact that the continuous embeddings covered by the Straightening Law have domain $\mathbb{V}_m^{\kappa}\#$. In fact, the homeomorphism from Example 2.5 provides an embedding $\pi : (\mathbb{Q}/\mathbb{Z})^\# \hookrightarrow \mathbb{Q}^\#$ such that for no non-empty subset $A \neq \{0\}$ of $\mathbb{Q}/\mathbb{Z}$ the restriction $\pi\restriction_A$ may coincide with the restriction $\ell\restriction_A$ of some injective homomorphism $\ell : \langle A\rangle \to \mathbb{Q}$.

For a prime p and $s \in \omega$ let

$$\gamma_{p,s}(G) := \sup\{\kappa_{p,l}(G) : l \geq s\}.$$

This cardinal invariant captures perfectly weak isomorphisms. Indeed, it is easy to see that $\gamma_{p,s}(G) \geq \kappa$ if and only of $\mathbb{V}_{p^s}^{\kappa} \overset{alg.}{\hookrightarrow} G$

The next lemma ensures the first implication in Theorem 3.1:

Lemma 3.2 *For bounded abelian groups G and H the following are equivalent:*

(a) G, H are weakly isomorphic;

(b) $\gamma_{p,s}(G) = \gamma_{p,s}(H)$ for every prime p and every $s < \omega$;

(c) $G \overset{alg.}{\hookrightarrow} H$ and $H \overset{alg.}{\hookrightarrow} G$.

The next claim is proved in [10] by means of the Straightening Law:

Claim 3. If $\mathbb{V}_{p^s}^{\kappa}\# \hookrightarrow H^\#$ with $\kappa \geq \omega$ and $0 < s < \omega$, then $\mathbb{V}_p^{\kappa} \times \mathbb{V}_{p^s}^{\omega} \overset{alg.}{\hookrightarrow} H$.

Now, to prove the second implication in Theorem 3.1 note that $p^s|eo(G)$ if and only if $\mathbb{V}_{p^s}^{\omega} \overset{alg.}{\hookrightarrow} G$. Hence $G^\# \hookrightarrow H^\#$ implies $\mathbb{V}_{p^s}^{\omega}\# \hookrightarrow G^\# \hookrightarrow H^\#$, so by the Claim $\mathbb{V}_{p^s}^{\omega} \overset{alg.}{\hookrightarrow} H$, and consequenlty $p^s|eo(H)$, whenever $p^s|eo(G)$.

Lemma 3.3 $G^\# \hookrightarrow H^\# \;\Rightarrow\; r_p(G) \leq r_p(H)$ *if* $r_p(G) \geq \omega$.

Note that $r_p(G) \geq \kappa \;\Rightarrow\; \mathbb{V}_p^{\kappa} \overset{alg.}{\hookrightarrow} G$, so $\mathbb{V}_p^{\kappa\#} \hookrightarrow G^\# \hookrightarrow H^\#$ when $\kappa \geq \omega$, hence the Claim gives $\mathbb{V}_p^{\kappa} \overset{alg.}{\hookrightarrow} H$.

This next corollary answers (for $p = 2$ and $q = 3$) a question from [22].

Corollary 3.4 $\mathbb{V}_p^{\omega_1\#} \not\hookrightarrow (\mathbb{V}_p^{\omega} \times \mathbb{V}_q^{\omega_1})^\#$ *for distinct primes p, q.*

Indeed, $r_p(\mathbb{V}_p^{\omega_1}) = \omega_1 > \omega = r_p(\mathbb{V}_p^{\omega} \times \mathbb{V}_q^{\omega_1})$, so Lemma 3.3 applies.

Corollary 3.5 *If G and H are bounded weakly Bohr-equivalent groups, then $eo(G) = eo(H)$ and $r_p(G) = r_p(H)$ whenever at least one of these cardinals is infinite.*

Theorem 3.6 *If G and H are strongly Bohr-equivalent abelian groups, then they are simultaneously torsion-free (resp. p-torsion-free, for any prime p).*

Indeed, assume that $r_p(G) > 0$. Then $\mathbb{V}_p^{\omega} \overset{alg.}{\hookrightarrow} G^{\omega}$, so $\mathbb{V}_p^{\omega\#} \hookrightarrow G^{\omega\#} \approx H^{\omega\#}$, hence Corollary 3.5 applies to give $r_p(G) = r_p(H)$.

In case G and H are *not* bounded torsion, ω has to be replaced by $\beth_{2p-1}$ [6].

Example 3.7 Almost isomorphic abelian groups need not be strongly Bohr-equivalent. Indeed, take $G = \mathbb{V}_2^{\omega}$, $H = \mathbb{Z}_3 \times \mathbb{V}_2^{\omega}$ and apply Corollary 3.5 to the groups G^{ω} and H^{ω} to conclude that G^{ω} and H^{ω} cannot be Bohr homeomorphic since $r_3(G^{\omega}) = 0$ and $r_3(H^{\omega}) = \omega$.

3.2 Almost homogeneous bounded abelian groups

Definition 3.8 *A bounded abelian groups G is* almost homogeneous *if for every prime p at most one $\kappa_{p,s}(G) \geq \omega$ $(0 < s < \omega)$.*

Example 3.9 *(a) Bounded groups of square-free essential order are almost homogeneous (i.e., groups of the form $G = H \times F$, where F is finite and $p_1p_2 \dots p_nH = 0$ for distinct primes $p_1, p_2, \dots, p_n$).*

(b) An infinite p-group G is almost homogeneous iff $G = F \times \mathbb{V}_{p^s}^{\kappa}$ for some finite p-group F, $s \in \omega$ and $\kappa = |G|$.

(c) Every almost homogeneous bounded abelian group is almost isomorphic to a group of the form $\bigoplus_{i=1}^{n} \mathbb{V}_{p_i^{s_i}}^{\kappa_i}$, where $p_1, p_2, \dots, p_n$ are distinct primes.

Theorem 3.10 *For almost homogeneous abelian groups G, H TFAE:*

(a) G and H are Bohr-equivalent,

(b) G and H are weakly Bohr-equivalent;

(c) G and H are weakly isomorphic,

(d) G and H are almost isomorphic;

(e) $eo(G) = eo(H)$ and $r_p(G) = r_p(H)$ whenever $r_p(G) + r_p(G) \geq \omega$.

This gives:

Corollary 3.11 *If G and H are countably infinite abelian groups of finite square-free essential exponent, then there exists a homeomorphism $\pi : G^\# \to H^\#$ iff $G \sim H$.*

Example 3.7 shows that strong Bohr-equivalence cannot be added to this list.

4 c-equivalence, psc-equivalence and cc-equivalence

Conjecture 1 *If G and H are almost isomorphic abelian groups, then G and H are c-equivalent.*

By $d(G) \cong d(H)$, the conjecture is restricted to the case of reduced groups ($d(G) = d(H) = 0$).

Theorem 4.1 *Weakly isomorphic bounded abelian groups are psc-equivalent.*

This follows immediately from the description of the torsion abelian groups admitting pseudocompact group topologies obtained by Shakhmatov and the author in [12]. Indeed, this description depends only on the invariants $\gamma_{p,s}(G)$, so that Lemma 3.2 applies.

Corollary 4.2 *If G and H are almost homogeneous and weakly Bohr-equivalent, then they are psc-equivalent. In particular, if $G^\# \approx H^\#$ and G, H are almost homogeneous, then they are psc-equivalent.*

Question 4.3 *Does $G^\# \approx H^\#$ and G, H always imply that G and H are psc-equivalent?*

Recent results of Tkachenko and the author [14] imply

Theorem 4.4 *[MA] Weakly isomorphic bounded abelian groups of size $\leq \mathfrak{c}$ are cc-equivalent.*

Indeed, one can derive from the description given in [14], under the assumption of MA, that a group G of size $\leq \mathfrak{c}$ admits a countably compact group topology if and only if all $\gamma_{p,s}(G)$ are either finite or $\mathfrak{c}$.

Corollary 4.5 (MA) *For almost homogeneous bounded abelian groups of size $\leq \mathfrak{c}$ weak Bohr-equivalence yields cc-equivalence.*

Shakhmatov and the author [13] introduced for every cardinal $\kappa \geq \omega_2$, a set-theoretic axiom ∇_κ consistent with ZFC and implying $\mathfrak{c} = \omega_1, 2^{\mathfrak{c}} = \kappa$ (with $2^{\mathfrak{c}}$ "arbitrarily large"). From the the main results of [13] it follows that, under the assumption of ∇_κ, an abelian group G of size $\leq 2^{\mathfrak{c}}$ admits a countably compact group topology if and only if all $\gamma_{p,s}(G)$ are either finite or $\mathfrak{c}$. This description gives these two corollaries:

Theorem 4.6 *Under ∇_κ, weakly isomorphic bounded abelian groups of size $\leq 2^{\mathfrak{c}}$ are cc-equivalent.*

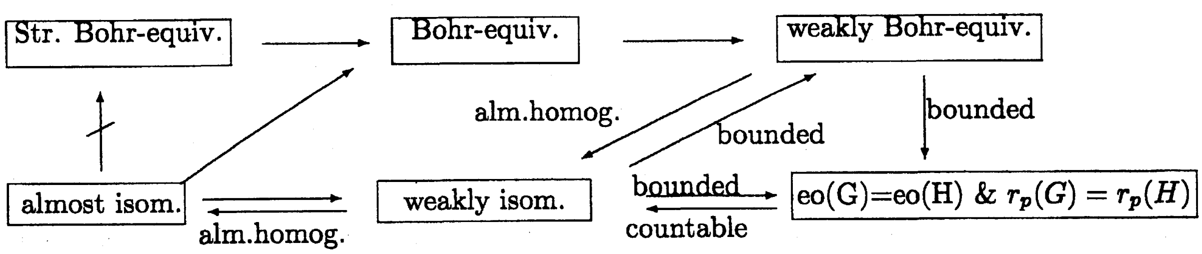
Corollary 4.7 *Under ∇_κ, weak Bohr-equivalence yields cc-equivalence for almost homogeneous bounded abelian groups of size $\leq 2^{\mathfrak{c}}$.*

Corollary 4.8 *Under ∇_κ, if G and H are weakly Bohr-equivalent almost homogeneous abelian groups and G admits a* separable pseudocompact *group topology then H admits a* countably compact and hereditarily separable group topology without infinite compact subsets.

It is not clear whether 4.4–4.8 remain true in ZFC.

5 Open questions

For reader's convenience we collect in the next diagram most of the relations between various levels of Bohr-equivalence and the various levels of "weak" isomorphisms discussed in the paper. Arrows accompanied by a property (e.g., "bounded", "countable", etc.) are implications valid for pairs of abelian groups with that specific property.



In spite of the results of §3, it still remais unclear where weak Bohr-equivalence should be placed. Our open qeustions aim to clarify its real position with respect to the remaining three adjacent conditions: Bohr-equivalence, weak isomorphism and

(*) $eo(G) = eo(H)$ and $r_p(G) = r_p(H)$ for all p with $r_p(G) + r_p(H) \geq \omega$.

According to Theorem 3.1 and Corollary 3.5 weak Bohr-equivalence is captured between weak isomorphism and the weaker condition $(*)$. Since countable abelian group G, H with $eo(G) = eo(H)$ are weakly isomorphic, this yields that the three properties coincide for countable groups.

Obviously, weak Bohr-equivalence follows from Bohr-equivalence, this is why we start the questions by discussing this (easiest) implication.

The groups $\mathbb{V}_4^\omega$ and $\mathbb{V}_2^\omega \times \mathbb{V}_4^\omega$ are weakly isomorphic, hence weakly Bohr-equivalent by Theorem 3.1.

Question 5.1 *(a) (Kunen [29]) Are $\mathbb{V}_4^\omega$ and $\mathbb{V}_2^\omega \times \mathbb{V}_4^\omega$ Bohr-equivalent?*

(b) Are weakly Bohr-equivalent groups always Bohr-equivalent?

A positive answer to (a) will answer negatively Question 1.6 for bounded abelian groups.

Let us discuss now the implication

$$\text{weakly Bohr-equivalent} \overset{?}{\Rightarrow} \text{weakly isomorphic}$$

The groups $\mathbb{V}_4^{\omega_1}$ and $\mathbb{V}_2^{\omega_1} \times \mathbb{V}_4^\omega$ are not weakly isomorphic, so it makes sense to ask

Question 5.2 *Are $\mathbb{V}_4^{\omega_1}$ and $\mathbb{V}_2^{\omega_1} \times \mathbb{V}_4^\omega$ weakly Bohr-equivalent (i.e., does $(\mathbb{V}_4^{\omega_1})^\# \hookrightarrow (\mathbb{V}_2^{\omega_1} \times \mathbb{V}_4^\omega)^\#$)?*

Or the strongest form:

Question 5.3 *Are $\mathbb{V}_{p^s}^\kappa$ and $\mathbb{V}_p^\kappa \times \mathbb{V}_{p^s}^\omega$ weakly Bohr-equivalent for all possible $s \in \omega, p \in P$, $\kappa \geq \omega$? Can this depend on p?*

If the answer to Question 5.3 is positive, then for any pair G, H of bouned abelian groups weak Bohr-equivalence is equivalent to $(*)$.

The next question is an equivalent form of the strongest negative answer to Question 5.3.

Question 5.4 *Is it true that for every prime p, for every $0 < k < \omega$ and every uncountable cardinal κ*

$$(\mathbb{V}_{p^k}^\kappa)^\# \hookrightarrow (\mathbb{V}_{p^{k-1}}^\kappa \times \mathbb{V}_{p^k}^\lambda)^\# \implies \lambda \geq \kappa?$$

Another equivalent form is the following

Question 5.5 *Assume there exists an embedding $\pi : G^\# \hookrightarrow H^\#$ for some bounded abelian group H. Is it true that $\gamma_{p,k}(G) \leq \omega \cdot \gamma_{p,k}(H)$ for every prime p and for every $0 < k < \omega$?*

In particular, if G and H are weakly Bohr-equivalent and H is bounded, are then G and H weakly isomorphic?

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