数理解析研究所講究録 1370 巻 2004 年 62-68

Tightness of free (abelian) topological groups over metrizable spaces

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1. Introduction

All spaces are assume to be Tychonoff. Our notations and terminology follow [4].

Let X be a topological space. For $A \subseteq X$ we use $\overline{A}$ to denote the closure of A in X. Recall that the *tightness* $t(X)$ of a space X is the smallest infinite cardinal number κ such that for every point $x \in X$ and each $A \subset X$, if $x \in \overline{A}$, then $x \in \overline{B}$ for some $B \subset A$ with $|B| \leq \kappa$. The *weight* of a space X is denoted by $w(X)$. We note that $w(X)$ is always infinite. In what follows, $F(X)$ and $A(X)$ will denote the free topological group and the free abelian topological group over a space X, respectively.

The main goal of this paper is to completely compute the tightness of $F(X)$ and $A(X)$ when X is a metric space. In particular, we provide a consistent solution to the following

Problem 1. (Arhangel'skiĭ, Okunev and Pestov [1]) *Does $t(A(X)) = w(X')$ for a metrizable space X, where X' is the set of all non-isolated points of X?*

Let $\{c_n : n \in \omega\}$ be a faithfully indexed sequence converging to a point $c \notin \{c_n : n \in \omega\}$. and we define $C = \{c_n : n \in \omega\} \cup \{c\}$. Thus C is the infinite countable compact space with a single non-isolated point c.

For every cardinal κ, D_κ will denote the discrete space of size κ, and we define $C_\kappa = C \times D_\kappa$. Topologically, C_κ is (homeomorphic to) the disjoint sum of κ-many convergent sequences. Finally, S_κ will denote the *Fréchet-Urysohn fan of size* κ, i.e. the quotient space obtained from C_κ by identifying all non-isolated points (c, d) $(d \in D_\kappa)$ of C_κ to a single point which we will denote by $\{x^*\}$. Therefore, all points of S_κ except $\{x^*\}$ are isolated, and $\{W_f : f \in \omega^\kappa\}$ forms a basis of open neighborhoods of $\{x^*\}$, where

$$W_f = \{x^*\} \cup \{(c_n, d) : n \in \omega, d \in D_\kappa, n \geq f(d)\}$$

for $f \in \omega^\kappa$.

Recall that a *successor cardinal* is a cardinal of the form λ^+ for some cardinal λ, where λ^+ denotes the smallest cardinal strictly bigger that λ. A cardinal κ is *regular* provided that for every $\lambda < \kappa$ and each family $\{\kappa_\alpha : \alpha < \lambda\}$ of cardinals such that $\kappa_\alpha < \kappa$ whenever $\alpha < \lambda$, one has $\sup\{\kappa_\alpha : \alpha < \lambda\} < \kappa$. It is well-known that successor cardinals are regular, see [8].

2. Embeddings of finite powers of Fréchet-Urysohn fans into free (abelian) groups over metric spaces

We start with a well-known folklore fact.

Fact 2.1. *If $n \in \omega$ and $X = \oplus_{i=1}^n X_i$, then the map $f : \prod_{i=1}^n A(X_i) \to A(X)$ defined by*

$$f(g_1, g_2, \dots, g_n) = g_1 + g_2 + \dots + g_n$$

is a topological group isomorphism, i.e. both a homeomorphism and a group isomorphism.

Indeed, the above fact can be proved applying the following result.

Theorem 2.2 ([7, Theorem 6.11]). *Let G be a topological group and $N_1, N_2, \dots, N_m$ be normal subgroups of G with the following properties:*

1. *$N_1 N_2 \cdots N_m = G$;*
2. *$(N_1 N_2 \cdots N_k) \cap N_{k+1} = \{e\}$ for $k = 1, 2, \dots, m-1$;*
3. *if U_i is a neighborhood of e in N_i for for $i = 1, 2, \dots, m$, then $U_1 U_2 \cdots U_m$ contains a neighborhood of e in G.*

Then G is topologically isomorphic to $N_1 N_2 \cdots N_m$.

For a set D we define

$$Y_D = \{d c_n c^{-1} d^{-1} : n \in \omega, d \in D\} \subseteq F(C \oplus D)$$

and

$$Z_D = \{(c_n, d) - (c, d) : n \in \omega, d \in D\} \subseteq A(C \times D).$$

Then, applying Lemma 1.2 in [12], we obtain the following.

Lemma 2.3. *For each infinite cardinal κ the subspace Z_{D_κ} of $A(C_\kappa)$ is homeomorphic to S_κ.*

Fix $n \in \omega$. By Lemma 2.3, $A(C_\kappa)^n$ contains a subspace homeomorphic to S_κ^n. By Fact 2.1, $A(C_\kappa)^n$ is topologically isomorphic to $A(C_\kappa \oplus C_\kappa \oplus \cdots \oplus C_\kappa)$, where the direct sum has n summands. Since $C_\kappa \oplus C_\kappa \oplus \cdots \oplus C_\kappa$ is homeomorphic to C_κ, the latter group is topologically isomorphic to $A(C_\kappa)$. Consequently, we have the following theorem.

Theorem 2.4. *Let κ be an infinite cardinal. Then, for every $n \in \omega$, $A(C_\kappa)$ contains a subspace homeomorphic to S_κ^n.*

On the other hand, in the non-abelian case, we can show that S_κ^n can be embedded in $F(C \oplus D_\kappa)$, applying an idea in [11, Theorem 2.5].

Lemma 2.5. *For every infinite discrete space D, there exists a continuous group homomorphism $\psi_D : F(C \oplus D) \to A(C \times F(D))$ such that*

$$\psi_D(dc_n c^{-1} d^{-1}) = (c_n, d) - (c, d)$$

whenever $n \in \omega$ and $d \in D$.

Construction of the map. Let

$$\tau : F(D) \times (C \times F(D)) \to C \times F(D); (g, (x, h)) \mapsto (x, gh)$$

be the continuous action of $F(D)$ on $C \times F(D)$. Since $F(D)$ is a discrete space, we can get the continuous topological group automorphism

$$\overline{\tau} : F(D) \times A(C \times F(D)) \to A(C \times F(D)).$$

Let $G = F(D) \ltimes_{\overline{\tau}} A(C \times F(D))$ be the semidirect product formed with respect to $\overline{\tau}$, that is, for $\forall g, h \in F(D)$ and $\forall a, b \in A(C \times F(D))$

$$(g, a) \cdot (h, b) = (gh, a + \overline{\tau}(g, b)).$$

Then, for each $d \in D$ and $x \in C$, we have

$$\begin{aligned} (d^{-1}, 0) \cdot (d, 0) &= (e, 0) \\ (d, 0) \cdot (d^{-1}, 0) &= (e, 0) \\ (e, -(x, e)) \cdot (e, (x, e)) &= (e, 0) \\ (e, (x, e)) \cdot (e, -(x, e)) &= (e, 0) \end{aligned}$$

Define a mapping $\varphi : C \oplus D \to G$ by

$$\varphi(t) = \begin{cases} (e, (t, e)) \in F(D) \ltimes_{\overline{\tau}} A(C \times F(D)), & \text{if } t \in C \\ (t, 0) \in F(D) \ltimes_{\overline{\tau}} A(C \times F(D)), & \text{if } t \in D \end{cases}$$

and let $\overline{\varphi} : F(C \oplus D) \to G$ be the continuous homomorphism extension on φ. Finally, put

$$\psi_D = \pi \circ \overline{\varphi} : F(C \oplus D) \to A(C \times F(D)),$$

where $\pi : G \to A(C \times F(D))$ is the projection.

Fix $n \in \omega$ and divide the set κ int sets $A_1, A_2, \dots, A_n$ such that $|A_i| = \kappa$ for each $i = 1, 2, \dots, n$. Put

$$X_i = \{d_\alpha x_j x^{-1} d_\alpha^{-1} : j \in \omega, \alpha \in A_i\} \cup \{e\}$$

for each $i = 1, 2, \dots, n$. Then, by Lemma 2.5 and Theorem 2.4, we can show that $X = X_1 X_2 \cdots X_n$ is homeomorphic to S^n_κ. Therefore, we obtain the following.

Theorem 2.6. *Let κ be an infinite cardinal. Then, for every $n \in \omega$, $F(C \oplus D_\kappa)$ contains a subspace homeomorphic to S^n_κ.*

Since tightness is hereditary by subspaces, from Theorem 2.4 and Theorem 2.6 we immediately obtain

Corollary 2.7. *For every infinite cardinal κ, we have*

$$\sup\{t(S^n_\kappa) : n \in \mathbb{N}\} \leq t(A(C_\kappa))$$

and

$$\sup\{t(S^n_\kappa) : n \in \mathbb{N}\} \leq t(F(C \oplus D_\kappa)).$$

3. ZFC RESULTS

Let κ be a regular cardinal with $\omega \leq \kappa \leq w(X')$, where X' is the set of all non-isolated points of X. Then C_κ can be embedded in X as a closed subset. Hence we obtain

Theorem 3.1. *Let X be a metrizable space and X' the set of all non-isolated points of X. Then*

$$t(A(X)) \geq \sup\{t(A(C_\kappa)) : \omega \leq \kappa \leq w(X') \text{ and } \kappa \text{ is a regular cardinal}\}.$$

Our next corollary demonstrates that the general problem of Arhangel'skiĭ, Okunev and Pestov (see Problem 1) is completely reduced to the particular case of metric spaces of type C_κ.

Corollary 3.2. *The following conditions are equivalent:*

(i) $t(A(X)) = w(X')$ *for every metrizable space* X*, and*
(ii) $t(A(C_\kappa)) = \kappa$ *for each successor cardinal* $\kappa \geq \omega_1$.

In the non-abelian case, we obtain the following, similarly.

Theorem 3.3. *Let* X *be a non-discrete metrizable space. Then*

$$t(F(X)) \geq \sup\{t(F(C \oplus D_\kappa)) : \omega \leq \kappa \leq w(X) \text{ and } \kappa \text{ is a regular cardinal}\}.$$

Corollary 3.4. *The following conditions are equivalent:*

(i) $t(F(X)) = w(X)$ *for every non-discrete metrizable space* X*, and*
(ii) $t(F(C \oplus D_\kappa)) = \kappa$ *for each successor cardinal* $\kappa \geq \omega_1$.

Theorem 3.5. *Let* X *be a non-discrete metric space. Then* $F(X)$ *has countable tightness if and only if* X *is separable.*

Theorem 3.6. *If* $\sup\{t(S^n_\kappa) : n \in \mathbb{N}\} = \kappa$ *for every successor cardinal* $\kappa \geq \omega_1$*, then:*

(i) $t(A(X)) = w(X')$ *for every metrizable space* X*, and*
(ii) $t(F(X)) = w(X)$ *for every non-discrete metrizable space* X.

4. Consistency results

Let $\kappa \geq \omega_1$ be a regular cardinal. Let $\square(\kappa)$ denote the statement that there exists a sequence $\{C_\alpha : \alpha \in \kappa\}$ such that:

(i) for $\alpha < \kappa$, C_α is closed and unbounded in α, and $C_{\alpha+1} = \{\alpha\}$,
(ii) if α is a limit point of C_β, then $C_\alpha = C_\beta \cap \alpha$,
(iii) if C is a closed and unbounded subset of κ, then there is a limit point α of C so that $C_\alpha \neq C \cap \alpha$.

Fact 4.1. (Todorčević) *Let* $\kappa \geq \omega_1$ *be a regular cardinal. Then:*

(i) $\square(\kappa)$ *implies* $t(S^2_\kappa) = \kappa$*, and*
(ii) *if* $\square(\kappa)$ *fails, then* κ *is weakly compact in* L.

Proof. Indeed, (i) follows from [2, Proposition 4.18], and (ii) coincides with [2, Theorem 4.6]. $\square$

Fact 4.2. $\square_\kappa$ *implies* $\square^T_\kappa$ *for every successor cardinal* $\kappa \geq \omega_1$.

Fact 4.3. $\square_\kappa^T$ *implies* $\square(\kappa)$ *for every successor cardinal* $\kappa \geq \omega_1$.

Fact 4.4. *Under the Axiom of Constructibility* $V = L$, $\square(\kappa)$ *holds for every successor cardinal* $\kappa \geq \omega_1$.

Theorem 4.5. *Assume* $\square(\kappa)$ *for every successor cardinal* $\kappa \geq \omega_1$. *Then:*

(i) $t(A(X)) = w(X')$ *for every metrizable space* X, *and*

(ii) $t(F(X)) = w(X)$ *for every non-discrete metrizable space* X.

Proof. From the assumption of our theorem and Fact 4.1(i) we conclude that $\kappa = t(S_\kappa^2) \leq \sup\{t(S_\kappa^n) : n \in \mathbb{N}\} = \kappa$ for every successor cardinal $\kappa \geq \omega_1$. Now the result follows from Theorem 3.6. $\square$

Item (i) of our next corollary provides a positive consistent answer to Problem 1.

Corollary 4.6. *Under the Axiom of Constructibility* $V = L$, *one has:*

(i) $t(A(X)) = w(X')$ *for every metrizable space* X, *and*

(ii) $t(F(X)) = w(X)$ *for every non-discrete metrizable space* X.

Proof. Combine Theorem 4.5 with Fact 4.4. $\square$

Our next result shows that both statements (i) and (ii) from Theorem 4.5 have "large cardinal strength" in a sense that the failure of either of them implies the existence of large cardinal.

Corollary 4.7. *Suppose that either the statement (i) or the statement (ii) from Theorem 4.5 fails. Then there exists a weakly compact cardinal in* L.

Proof. The assumption of our corollary implies that the conclusion of Theorem 4.5 fails. Therefore, the assumption of Theorem 4.5 must also fail, i.e. $\square(\kappa)$ must fail for some regular cardinal $\kappa \geq \omega_1$. Now κ is weakly compact in L according to Fact 4.1(ii). $\square$

Corollary 4.8. *If there exists a counterexample to the Problem 1, then there exists a weakly compact cardinal in* L.

5. Open question

Question 1. *Does Theorem 4.5 hold in ZFC?*

Even a particular version of the above question seems interesting:

Question 2. *Are statements (i) and (ii) of Theorem 4.5 equivalent in ZFC?*

Question 3. *Does $t(F(C \oplus D_\kappa)) = t(A(C_\kappa))$ for every successor cardinal $\kappa \geq \omega_1$?*

Question 4. *Does $t(F(S_\kappa)) = t(A(S_\kappa))$ for every (successor) cardinal $\kappa \geq \omega_1$?*

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