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A DIRECT METHOD FOR FINDING DUCKS IN R^4

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ABSTRACT. The singular perturbation problem in $R^n(n>3)$ includes a possibility having a constrained surface with a 2-dimensional dfferentiable manifold. We will take up the system in R^4 having such a constrained surface. Although it is difficult to analyze these systems in general, we will show some sufficient conditions to make it possible. We will reduce the system to the problem in R^3 (indirect method) or R^2 (direct one) and show the existence of the duck solutions.

1. INTRODUCTION

S.A.Campbell, one of authors of [3], investigated first the coupled FitzHugh-Nagumo equations as a bifurcation problem. On this system, we have already proved the existence of the winding duck solutions in R^4 ([4]), reducing it to a system in R^3. This method uses an indirect way as using an approximated system. In this paper, we will reduce it to the system in R^2 directly and get a duck solution, which has a delayed jump along the vertical direction.

2. PRELIMINARIES

Let us consider a constrained system(2.1):

$$\begin{aligned} dx/dt &= f(x,y,z,u),\\ dy/dt &= g(x,y,z,u),\\ h(x,y,z,u) &= 0, \end{aligned} \tag{2.1}$$

where u is a parameter (any fixed) and f,g,h are defined in $R^3\times R^1$. Furthermore, let us consider the singular perturbation problem of the system (2.1):

$$\begin{aligned} dx/dt &= f(x,y,z,u),\\ dy/dt &= g(x,y,z,u),\\ \epsilon dz/dt &= h(x,y,z,u), \end{aligned} \tag{2.2}$$

where ϵ is infinitesimally small.

We assume that the system (2.1) satisfies the following conditions $(A1)-(A5)$:

$(A1)$ f and g are of class C^1 and h is of class C^2.

$(A2)$ The set $S=\{(x,y,z)\in R^3|h(x,y,z,u)=0\}$ is a 2-dimensional differentiable manifold and the set S intersects the set $T=\{(x,y,z)\in R^3|\partial h(x,y,z,u)/\partial z=0\}$ transversely so that the pli set $PL=\{(x,y,z)\in S\cap T\}$ is a 1-dimensional differentiable manifold.

$(A3)$ Either the value of f or that of g is nonzero at any point $p\in PL$.

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Let $(x(t,u), y(t,u), z(t,u))$ be a solution of (2.1). By differentiating $h(x,y,z,u)$ with respect to the time t, the following equation holds:

$$(2.3)\quad h_x(x,y,z,u)f(x,y,z,u) + h_y(x,y,z,u)g(x,y,z,u) + h_z(x,y,z,u)dz/dt = 0,$$

where $h_i(x,y,z,u) = \partial h(x,y,z,u)/\partial i$, $i = x,y,z$. The above system (2.1) becomes the following system:

$$(2.4)\quad \begin{aligned} &dx/dt = f(x,y,z,u),\\ &dy/dt = g(x,y,z,u),\\ &dz/dt = -\{h_x(x,y,z,u)f(x,y,z,u)+\\ &h_y(x,y,z,u)g(x,y,z,u)\}/h_z(x,y,z,u), \end{aligned}$$

where $(x,y,z) \in S \setminus PL$. The system (2.1) coincides with the system (2.4) at any point $p \in S \setminus PL$. In order to study the system (2.4), let consider the following system:

$$(2.5)\quad \begin{aligned} &dx/dt = -h_z(x,y,z,u)f(x,y,z,u),\\ &dy/dt = -h_z(x,y,z,u)g(x,y,z,u),\\ &dz/dt = h_x(x,y,z,u)f(x,y,z,u) + h_y(x,y,z,u)g(x,y,z,u). \end{aligned}$$

As the system(2.5) is well defined at any point of R^3, it is well defined indeed at any point of PL. The solutions of (2.4) coincide with those of (2.1) on $S \setminus PL$ except the velocity when they start from the same initial points.

($A4$) For any $(x,y,z) \in S$, either of the following holds;

$$(2.6)\quad h_y(x,y,z,u) \neq 0, h_x(x,y,z,u) \neq 0,$$

that is, the surface S can be expressed as $y = \varphi(x,z,u)$ or $x = \psi(y,z,u)$ in the neighborhood of PL. Let $y = \varphi(x,z,u)$ exist, then the projected system, which restricts the system (2.5) is obtained:

$$(2.7)\quad \begin{aligned} &dx/dt = -h_z(x,\varphi(x,z,u),z,u)f(x,\varphi(x,z,u),z,u),\\ &dz/dt = h_x(x,\varphi(x,z,u),z,u)f(x,\varphi(x,z,u),z,u)+\\ &h_y(x,\varphi(x,z,u),z,u)g(x,\varphi(x,\varphi(x,z,u),z,u). \end{aligned}$$

($A5$) All the singular points of (2.7) are nondegenerate, that is, the matrix induced from the linearized system of (2.7) at a singular point has two nonzero eigenvalues. Note that all the points contained in $PS = \{(x,y,z) \in PL | dz/dt = 0\}$, which is called *pseudo singular points* are the singular points of (2.5).

Definition2.1. Let $p \in PS$ and $\mu_1(u)$, $\mu_2(u)$ be two eigenvalues of the matrix associated with the linearized system of (2.7) at p. The point p is called *pseudo singular saddle* if $\mu_1(u) < 0 < \mu_2(u)$ and called *pseudo singular node* if $\mu_1(u) < \mu_2(u) < 0$ or $\mu_1(u) > \mu_2(u) > 0$.

Definition2.2. Solution $(x(t,u), y(t,u), z(t,u))$ of the systems(3.1), (3.2) are called *ducks*, if there exist standard $t_1 < t_0 < t_2$ such that

(1) $^*(x(t_0,u), y(t_0,u), z(t_0,u)) \in S$, where $^*(X)$ denotes the standard part of X,
(2) for $t \in (t_1, t_0)$ the segment of the trajectory $(x(t,u), y(t,u), z(t,u))$ is infinitesimally close to the attracting part of the slow curves,
(3) for $t \in (t_0, t_2)$, it is infinitesimally close to the repelling part of the slow curves, and
(4) the attracting and repelling parts of the trajectory are not infinitesimally small.

Theorem2.1(Benoit). If the system has a pseudo singular saddle or node point with no resonance, then it has duck solutions.

3. Slow-fast system in R^4

Let us consider the following slow-fast system:

$$\begin{aligned}
\epsilon dx_1/dt &= h1(x,y,u),\\
\epsilon dx_2/dt &= h2(x,y,u),\\
dy_1/dt &= f1(x,y,u),\\
dy_2/dt &= f2(x,y,u),
\end{aligned} \tag{3.1}$$

where $x^t = (x_1, x_2)$, $y^t = (y_1, y_2)$, are variables, $u \in R$ is a parameter and ϵ is infinitesimal in the sense of non-standard analysis of Nelson.

In the case we assume that $rank[Jh] = 2$ with respect to x, that is, there exists h_x^{-1} and therefore does a function ψ such that $x = \psi(y)$, where $\psi(y,u))^t = (\psi_1(y,u)), \psi_2(y,u)))$. In this state, using a relation $x_2 = \psi_2(y,u)$, the system (3.1) is reduced to the following slow-fast system in R^3:

$$\begin{aligned}
dy_1/dt &= f1(x_1, \psi_2(y,u), y, u),\\
dy_2/dt &= f2(x_1, \psi_2(y,u), y, u),\\
\epsilon dx_1/dt &= h1(x_1, \psi_2(y,u), y, u),
\end{aligned} \tag{3.2}$$

when $|dx_1/dt - dx_2/dt|$ is limited. Using the other relation $x_1 = \psi_1(y,u)$, we can get the following:

$$\begin{aligned}
dy_1/dt &= f1(\psi_1(y,u), x_2, y, u),\\
dy_2/dt &= f2(\psi_1(y,u), x_2, y, u),\\
\epsilon dx_2/dt &= h2(\psi_1(y,u), x_2, y, u).
\end{aligned} \tag{3.3}$$

Definition3.1. If the system is invariant for changing the coordinates with respect to (x_1, x_2), and (y_1, y_2), the system is called *symmetric*.

Definition3.2. If there exist ducks in the both systems (3.2) and (3.3) at the common pseudo singular point, they are called *ducks in* R^4. If there exists a duck in only one of the systems, it is called *a partial duck* in R^4.

Lemma3.1. Assume that the system (3.2), or the system (3.3) satisfies the generic conditions. If they have pseudo singular saddle or pseudo singular node points without resonance, they have partial ducks in R^4. If the system (3.1) is symmetric under the above conditions at the common pseudo singular point, they have a duck in R^4.

In order to introduce a direct method for analyzing ducks, we assume that $rank[Jh] = 2$, where $h^t = (h1, h2)$, with respect to y, that is, there exists h_y^{-1}. Then, the implicit function theorem ensures that y is uniquely described like as $y = \phi(x, u)$, using a smooth function ϕ. The constrained surface S; when ϵ equals to zero, is as follows:

$$S = \{(x, \phi(x, u)) | h(x, \phi(x, u), u) = 0\}. \tag{3.4}$$

In this state, let us define a generalized pli set, simply GPL.

$$GPL = \{(x, \phi(x, u)) \in S | det[h_x] = 0\}. \tag{3.5}$$

On the set S, differentiating both sides of $h = 0$ by x,

$$[h_x] + [h_y]D\phi = 0, \tag{3.6}$$

where $D\phi$ is a derivative with respect to x, therefore

$$D\phi(x) = -[h_y]^{-1}[h_x]. \tag{3.7}$$

On the other hand,

$$dy/dt = D\phi(x)dx/dt, \tag{3.8}$$

notice that $y = \phi(x)$. We can reduce the slow system to the following:

$$D\phi(x)dx/dt = f(x, \phi(x)), \tag{3.9}$$

where $f^t = (f1, f2)$.

Using (3.7), the system (3.9) is described by

$$[h_x]dx/dt = -[h_y]f(x, \phi(x)). \tag{3.10}$$

Put $[h_x] = A$ simply, then

$$dx/dt = -B[h_y]f(x, \phi(x)), \tag{3.11}$$

where $AB = BA = (detA)I$.

This is the time scaled reduced system projected into R^2. Again, we assume that a set $\{(x, y) | detA = 0\}$ is not empty and this system also satisfies the generic conditions $(A1) - (A5)$ in the section2.

Lemma3.2. If the system (3.11) has singular saddle or singular node points without resonance, the system (3.1) has a partial duck.

4. Coupled FitzHugh-Nagumo system

In the coupled FHN system,

$$
\begin{aligned}
h1 &= y_1 - x_1^3/3 + x_2,\\
h2 &= y_2 - x_2^3/3 + x_1,\\
f1 &= x_1^3 - x_2,\\
f2 &= x_2^3 - x_1.
\end{aligned} \tag{4.1}
$$

Because of satisfying $rank[Jh] = 2$ regarding especially x_1, x_2, the system (4.1) can be reduced to the slow-fast system projected in R^3:

$$
\begin{aligned}
dy_1/dt &= -(x_1 + by_1)/c,\\
dy_2/dt &= -(x_1^3/3 - y_1 + by_2)/c,\\
\epsilon dx_1/dt &= y_2 - (x_1^3/3 - y_1)^3/3 + x_1,
\end{aligned} \tag{4.2}
$$

under the condition, that $|dx_1/dt - dx_2/dt|$ is limited. On the constrained surface in the system (4.1), we can get the time scaled reduced system:

$$
\begin{aligned}
dy_1/dt &= -(x_1 + by_1)(1 - (x_1^3/3 - y_1)^2 x_1^2),\\
dy_2/dt &= -(x_1^3/3 - y_1 + by_2)(1 - (x_1^3/3 - y_1)^2 x_1^2),\\
dx_1/dt &= (x_1^3/3 - y_1)^2 (x_1 + by_1) + x_1^3 - y_1 + by_2.
\end{aligned} \tag{4.3}
$$

Then, the pseudo singular point, that is, the singular point of the system (4.3) is determined by

$$
\begin{aligned}
&(x_1^3/3 - y_1)^2 (x_1 + by_1) + x_1^3/3 - y_1 + by_2 = 0,\\
&1 - (x_1^3/3 - y_1)^2 x_1^2 = 0.
\end{aligned} \tag{4.5}
$$

Note that the second equation in (4.5) can be expressed as $x_1^3/3 - y_1 = +(-)1/x_1$. In the case $(-)$, there are 2 pseudo singular points $P_0 = (x_{1o}, y_{1o}, x_{2o}, y_{2o})$:
$(x_{1o}, y_{1o}, x_{2o}, y_{2o}) = (1, 4/3, -1, -4/3)$, and $(-1, -4/3, 1, 4/3)$.

These points do not depend on the parameter b, therefore they are structurally stable.

As the characteristic equation of the linearized system is

$$
\lambda(\lambda - (2 + 8b/3))(\lambda + 8b/3) = 0, \tag{4.6}
$$

we can conclude that these will be node if $-3/4 < b < 0$. Then there are duck solutions at the pseudo-singular node. This fact implies they are winding.

In the case $(+)$, there are 4 pseudo singular points which depend on the parameter b. The characteristic equation in this case is

$$
\lambda(A\lambda^2 + B\lambda + C)/(3 + D)^3 = 0, \tag{4.7}
$$

where

$$
\begin{aligned}
A &= -D^3 - 27D + 36b^2 - 108\\
B &= 2[(4b^2 - 9)D^3 + (16b^4 - 90b^2 + 243))D - 162b^2 + 486]/3b\\
C &- -4[405D^3 + (64b^6 - 720b^4 + 291b^2 - 3645)D + 576b^6 - 3024b^4 + 3888b^2]/9b^2\\
D &= \sqrt{(3-2b)(3+2b)}.
\end{aligned}
$$

If $0 < b < 3/2$, there exist the pseudo singular points, as

$$x_1 = \pm\sqrt{\left(3/b \pm \sqrt{9/b^2 - 4}\right)/2}. \tag{4.8}$$

The eigenvalues of all four singular points are the same due to the symmetry. They arise in some sort of pitchfork bifurcation from the singular points in the $(-)$equation at $b = 3/2$. If $0.388 < b < 1.4489$, there will be ducks at the pseudo-singular node and spirals if $0 < b < 0.388$ or $1.4489 < b < 3/2$.

Therefore, the following theorem is established by Lemma3.1.

Theorem4.1. There exists a partial duck in the system (4.1). Furthermore, it has a duck in R^4, as the system is symmetric.

Now, turn back to the system (4.1). According to the direct method, we will reduce the system further.

$$
\begin{aligned}
&[h_y] = I,\\
&h1_{x_1} = -x_1^2,\\
&h1_{x_2} = 1,\\
&h2_{x_1} = 1,\\
&h2_{x_2} = -x_2^2,
\end{aligned} \tag{4.9}
$$

so, we can easily get the matrices A and B. The time caled reduced system is as follows:

$$
\begin{aligned}
dx_1/dt &= (x_1 + by_1)x_2^2 + x_2 + by_2 = F1(x),\\
dx_2/dt &= x_1 + by_1 + (x_2 + by_2)x_1^2 = F2(x),
\end{aligned} \tag{4.10}
$$

where

$$
\begin{aligned}
y_1 &= x_1^3/3 - x_2,\\
y_2 &= x_2^3/3 - x_1.
\end{aligned} \tag{4.11}
$$

$$
\begin{aligned}
F1_{x_1} &= (1 + bx_1^2)x_2^2 - b,\\
F1_{x_2} &= -bx_2^2 + (x_1 + by_1)2x_2 + 1 + bx_2^2,\\
F2_{x_1} &= 1 + bx_1^2 - bx_1^2 + (x_2 + by_2)2x_1,\\
F2_{x_2} &= -b + (1 + bx_2^2)x_1^2,
\end{aligned} \tag{4.12}
$$

and two pseudo singular points are

$$P_1 = (x_1, x_2, y_1, y_2) = (1, -1, 4/3, -4/3), \quad P_2 = (-1, 1, -4/3, 4/3), \tag{4.13}$$

thus

$$JF(P_1) = JF(P_2) = \begin{pmatrix} 1 & -1 - 8b/3 \\ -1 - 8b/3 & 1 \end{pmatrix}. \tag{4.14}$$

The corresponding characteristic equation is

$$(\lambda - 1)^2 - (1 + 8b/3)^2 = 0. \tag{4.15}$$

Lemma3.2 and Lemma3.1 ensure the following theorem.

Theorem4.2. The system (4.1) has a partial duck in R^4. As the system is symmetric, it is a duck in R^4.

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