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個体群サイズがロジスティック成長する SIR 伝染病モデルのパーマネンス解析
Permanence of a Delayed SIR Epidemic Model with Logistic Process

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1 Introduction

The SIR epidemic model is an important population model and has been studied by many authors [1, 4, 6, 7, 8, 9, 10, 12]. In modeling the spread process of infectious disease, it is often convenient to divide the population into three components of susceptible, infective and recovered individuals, with numbers at time t denoted by $S(t)$, $I(t)$ and $R(t)$, respectively. The total number of the population is denoted by $N=S+I+R$.

It is the purpose of this paper to consider transmission dynamics for a delayed SIR epidemic model with density dependent birth and death rate. The model is assumed that the dynamics of the total population are governed by the following Logistic equation:

$$\dot{N}=\left(b-a\frac{rN}{K}\right)N-\left(d+(1-a)\frac{rN}{K}\right)N, \tag{1.1}$$

with $0\leq a\leq 1$, where $r=b-d>0$ is the intrinsic growth rate and $K>0$ is the carrying capacity of the population. The first term in (1.1) has a density dependent per capita birth rate coefficient $b-arN/K$ and the second term has a density dependent per capita death rate coefficient $d+(1-a)rN/K$, where b and d are positive constants. Our model is described as follows:

$$\begin{aligned}
\dot{S}(t)&=-\frac{\beta S(t)I(t-h)}{N(t-h)}-\left(d+(1-a)\frac{rN(t)}{K}\right)S(t)+\left(b-a\frac{rN(t)}{K}\right)N(t)\\
\dot{I}(t)&=\frac{\beta S(t)I(t-h)}{N(t-h)}-\left(d+(1-a)\frac{rN(t)}{K}\right)I(t)-\lambda I(t)\\
\dot{R}(t)&=\lambda I(t)-\left(d+(1-a)\frac{rN(t)}{K}\right)R(t),
\end{aligned} \tag{1.2}$$

where β and λ are positive constants. β is the average number of contacts per infective per day and λ expresses the recovery rate of the infectives. Further, the nonnegative constant h is a fixed time during which infectious agents develop in vectors and it is only after that

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time that the infected vector can infect a susceptible human. The model (1.2) assumes that the all newborns are susceptibles.

The initial condition of (1.2) is given by

$$S(\theta) = \phi_1(\theta),\ I(\theta) = \phi_2(\theta),\ R(\theta) = \phi_3(\theta), \quad (-h \le \theta \le 0) \tag{1.3}$$

where $\phi = (\phi_1, \phi_2, \phi_3)^T \in C$. C denotes the Banach space $C([-h, 0], \mathcal{R}^3)$ of continuous functions mapping the interval $[-h, 0]$ into $\mathcal{R}^3$.

(1.2) always has the infection-free equilibrium $E_0 = (K, 0, 0)$. Furthermore, if

$$R_0 \equiv \frac{\beta}{d + (1-a)r + \lambda} = \frac{\beta}{b - ar + \lambda} > 1, \tag{1.4}$$

then (1.2) also has the unique endemic equilibrium $E^* = (S^*, I^*, R^*)$, where

$$S^* = \frac{K}{R_0}, \quad I^* = \frac{(d + (1-a)r)K(R_0 - 1)}{\beta} \quad \text{and} \quad R^* = \frac{\lambda}{b - ar} I^*.$$

2 Main Result

Throughout of this paper, we focus on the dynamics of the solutions of (1.2) in the following restricted region:

$$\Omega = \{(S, I, R) | S \ge 0, I \ge 0, R \ge 0, 0 \le S + I + R = N \le K\}. \tag{2.1}$$

In this region the usual local existence, uniqueness and continuation results apply [3, 5]. Hence a unique solution $(S(t), I(t), R(t))$ starting the interior of Ω exists on $[0, \infty)$ if the solution remains bounded. The following lemma implies that solutions starting in Ω exist for all time $t \ge 0$. The proof is omitted.

Lemma 2.1. *Consider solutions of (1.2) with $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in Int\Omega$ for all $\theta \in [-h, 0]$. Then solutions stay in the interior of the region Ω for all time $t \ge 0$.*

We will assume throughout the rest of this paper that $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in \operatorname{Int}\Omega$ for all $\theta \in [-h, 0]$.

A linear analysis shows that $R_0 < 1$ guarantees that E_0 is locally asymptotically stable, and when $R_0 > 1$, E_0 is unstable and E_+ is locally asymptotically stable. Let us consider the global asymptotic behavior of the solution of (1.2) when $R_0 \le 1$. We have the following result.

Theorem 2.1. *If $R_0 < 1$, then the infection-free equilibrium E_0 of (1.2) is globally asymptotically stable. If $R_0 = 1$, then the infection-free equilibrium E_0 of (1.2) is globally attractive.*

Proof. For $t \geq 0$, we define a positive differentiable function $V(t)$ as follows,

$$V(t) = I(t) + \frac{\beta}{R_0}\int_{t-h}^{t} I(s)ds. \tag{2.2}$$

Then, using (1.4), the derivative of $V(t)$ satisfies

$$\begin{aligned}\dot{V}(t) &= \dot{I}(t) + \frac{\beta}{R_0}I(t) - \frac{\beta}{R_0}I(t-h)\\ &= \beta\left(\frac{S(t)}{N(t-h)} - \frac{1}{R_0}\right)I(t-h) + (1-a)r\left(1 - \frac{N(t)}{K}\right)I(t).\end{aligned} \tag{2.3}$$

From (1.1), it is obvious that $0 < N(t) < K$ for all $t \geq 0$ and $N(t)$ is strictly increasing in $t \geq 0$. Since $I(t) \leq N(t)$ for all $t \geq 0$, it follows from $R_0 \leq 1$ and (2.3) that for all $t \geq 0$

$$\begin{aligned}\dot{V}(t) &\leq \beta(S(t) - N(t))\frac{I(t-h)}{N(t-h)} + \beta(N(t) - N(t-h))\frac{I(t-h)}{N(t-h)} + \dot{N}(t)\\ &< -\frac{\beta}{K}(I(t) + R(t))I(t-h) + \beta(N(t) - N(t-h)) + \dot{N}(t).\end{aligned} \tag{2.4}$$

Integrating (2.4) from 0 to t for all $t \geq 0$,

$$\begin{aligned}V(t) + \frac{\beta}{K}\int_0^t (I(s) + R(s))\,I(s-h)ds &< \beta\int_{t-h}^{t} N(s)ds + N(t) - N(0) + V(0)\\ &< \beta Kh + K + V(0),\end{aligned} \tag{2.5}$$

which implies that

$$\int_0^{+\infty} (I(s) + R(s))\,I(s-h)ds < +\infty.$$

Note that the solution of (1.2) is bounded. We see from (1.2) that $S(t)$, $I(t)$ and $R(t)$ are also uniformly continuous for all $t \geq 0$. Thus, well known Barbălat lemma [2] shows that

$$\lim_{t\to\infty} (I(t) + R(t))\,I(t-h) = 0.$$

Hence we obtain $\lim_{t\to\infty} I(t) = 0$. Then by (1.2), $\lim_{t\to\infty} R(t) = 0$. It follows from (1.1) that $\lim_{t\to\infty} N(t) = \lim_{t\to\infty}(S(t)+I(t)+R(t)) = K$, which implies that $\lim_{t\to\infty} S(t) = K$. This proves that E_0 is globally attractive. Since E_0 is locally asymptotically stable when $R_0 < 1$, this completes the proof of the theorem. □

Finally, we investigate a permanence result [5]. The following is our main result of this paper. We will give the following result by using some techniques given in [8] and [11].

Theorem 2.2. *(1.4) is a necessary and sufficient condition for the permanence of (1.2).*

Proof. By Theorem 2.1, it is enough to consider the only sufficiency. Furthermore, from lemma 2.1, it suffices to show that $\liminf_{t\to\infty} S(t) \geq \frac{(b-ar)K}{\beta+d+(1-a)r} \equiv m_1$, $\liminf_{t\to\infty} I(t) \geq I^* e^{-(d+(1-a)r+\lambda)h} \equiv m_2$ and $\liminf_{t\to\infty} R(t) \geq \frac{\lambda m_3}{d+(1-a)r} \equiv m_3$.

Note that $0 < N(t) < K$ for all $t \geq 0$ and that $\lim_{t\to\infty} N(t) = K$. It is easy to see that $\liminf_{t\to\infty} S(t) \geq m_1$. In fact, let $\epsilon < K$ be arbitrary. Choose $T_1 > h$ so large that $N(t) > K - \epsilon$ for $t > T_1$. We have the following inequality:

$$\dot{S}(t) > -(\beta + d + (1-a)r)S(t) + (b - ar)(K - \epsilon),$$

for all $t \geq T_1$, which implies that

$$\liminf_{t\to\infty} S(t) \geq \frac{(b-ar)(K-\epsilon)}{\beta + d + (1-a)r}.$$

Note that ϵ may be arbitrarily small so that $\liminf_{t\to\infty} S(t) \geq m_1$.

Next, we will show $\liminf_{t\to\infty} I(t) \geq m_2$. For any $q : 0 < q < 1$, we see the inequality $S^* < (b-ar)K/(\frac{\beta q I^*}{K} + d + (1-a)r)$. There exists sufficiently large $\rho \geq 1$ and sufficiently small ϵ such that $S^* < \{(b-ar)(K-\epsilon)/(\frac{\beta q I^*}{K-\epsilon} + d + (1-a)r)\}(1 - e^{-(\frac{\beta q I^*}{K-\epsilon}+d+(1-a)r)\rho h}) \equiv S^\Delta$.

We show that $I(t_0) > qI^*$ for some $t_0 \geq \rho h$. In fact, if not, it follows from the first equation of (1.2) that, for all $t \geq \rho h + h \geq T_1 + h$,

$$\begin{aligned}\dot{S}(t) &\geq -\frac{\beta q I^* S(t)}{K-\epsilon} - (d + (1-a)r)S(t) + (b-ar)(K-\epsilon) \\ &= -\left(\frac{\beta q I^*}{K-\epsilon} + d + (1-a)r\right) S(t) + (b-ar)(K-\epsilon).\end{aligned}$$

Hence, for $t \geq \rho h + h$,

$$\begin{aligned}S(t) &\geq e^{-(\frac{\beta q I^*}{K-\epsilon}+d+(1-a)r)(t-\rho h-h)} \\ &\quad \times \left[S(\rho h + h) + (b-ar)(K-\epsilon)\int_{\rho h+h}^{t} e^{(\frac{\beta q I^*}{K-\epsilon}+d+(1-a)r)(\theta-\rho h-h)} d\theta\right] \\ &> \frac{(b-ar)(K-\epsilon)}{\frac{\beta q I^*}{K-\epsilon} + d + (1-a)r}(1 - e^{-(\frac{\beta q I^*}{K-\epsilon}+d+(1-a)r)(t-\rho h-h)}),\end{aligned}$$

which gives us, for $t \geq 2\rho h + h$,

$$\begin{aligned}S(t) &> \frac{(b-ar)(K-\epsilon)}{\frac{\beta q I^*}{K-\epsilon} + d + (1-a)r}(1 - e^{-(\frac{\beta q I^*}{K-\epsilon}+d+(1-a)r)\rho h}) \\ &= S^\Delta > S^*. \qquad (2.6)\end{aligned}$$

Using (2.2), we obtain the inequality, for $t \geq 2\rho h + h$,

$$\begin{aligned}\dot{V}(t) &= \beta\left(\frac{S(t)}{N(t-h)} - \frac{S^*}{K}\right)I(t-h) + (1-a)r\left(1 - \frac{N(t)}{K}\right)I(t)\\ &> \frac{\beta}{K}\left(S(t) - S^*\right)I(t-h)\\ &> \frac{\beta}{K}\left(S^\Delta - S^*\right)I(t-h).\end{aligned}$$

Let $\underline{i} = \min_{\theta\in[-h,0]} I(2\rho h + 2h + \theta)$. Now, let us show that $I(t) \geq \underline{i}$ for all $t \geq 2\rho h + h$. In fact, if there exists $T_2 \geq 0$ such that $I(t) \geq \underline{i}$ for $2\rho h + h \leq t \leq 2\rho h + 2h + T_2$, $I(2\rho h + 2h + T_2) = \underline{i}$ and $\dot{I}(2\rho h + 2h + T_2) \leq 0$. Direct calculation using the second equation of (1.2) and (2.6) gives

$$\begin{aligned}\dot{I}(2\rho h + 2h + T_2) &> [\frac{\beta}{K}S(2\rho h + 2h + T_2) - (d + (1-a)r + \lambda)]\underline{i}\\ &> (d + (1-a)r + \lambda)[\frac{S^\Delta}{S^*} - 1]\underline{i} > 0.\end{aligned}$$

This contradicts the definition of T_2. Thus, we have shown that $I(t) \geq \underline{i}$ for all $t \geq 2\rho h + h$. Hence, for all $t \geq 2\rho h + 2h$,

$$\dot{V}(t) > \frac{\beta}{K}(S^\Delta - S^*)\underline{i},$$

which implies that $V(t) \to +\infty$ as $t \to +\infty$. This contradicts the boundedness of $V(t)$. Consequently, $I(t_0) > qI^*$ for some $t_0 \geq \rho h$.

In the rest, we now need to consider two cases:

(I) $I(t) \geq qI^*$ for all large t.

(ii) $I(t)$ oscillates about qI^* for all large t.

We now need to show that $I(t) \geq qm_2$ for large t. Obviously, it suffices to show that it holds only for case (ii). We suppose that for any large T there exists $t_1, t_2 > T$ such that $I(t_1) = I(t_2) = qI^*$ and $I(t) < qI^*$ for $t_1 < t < t_2$. If $t_2 - t_1 \leq h$, the second equation (1.2) gives us $\dot{I}(t) > -(d + (1-a)r + \lambda)I(t)$, which implies that $I(t) > I(t_1)e^{-(d+(1-a)r+\lambda)(t-t_1)}$ on (t_1, t_2). Thus, $I(t) > qm_2$. On the other hand, if $t_2 - t_1 > h$, applying the same manner gives $I(t) \geq qm_2$ on $[t_1, t_1 + h]$, and hence the remaining work is to show $I(t) \geq qm_2$ on $[t_1 + h, t_2]$. In fact, assuming that there exists $T_3 > 0$ such that $I(t) \geq qm_2$ on $[t_1, t_1 + h + T_3]$, $I(t_1 + h + T_3) = qm_2$ and $\dot{I}(t_1 + h + T_3) \leq 0$, it follows from the equation (1.2) that

$$\begin{aligned}\dot{I}(t_1 + h + T_3) &\geq [\frac{\beta}{K}S(t_1 + h + T_3) - (d + (1-a)r + \lambda)]qm_2\\ &> (d + (1-a)r + \lambda)[\frac{S^\Delta}{S^*} - 1]qm_2 > 0.\end{aligned}$$

This contradicts the definition of T_3. Hence $I(t) \geq qm_2$ on $[t_1, t_2]$. Consequently, $I(t) \geq qm_2$ for large t in the case (ii). Therefore, $\liminf_{t\to\infty} I(t) \geq qm_2$. Note that q may be so close to 1 that $\liminf_{t\to\infty} I(t) \geq m_2$.

Finally, let us show that $\liminf_{t\to\infty} R(t) \geq \frac{\lambda m_2}{d+(1-a)r} \equiv m_3$. The third equation gives us

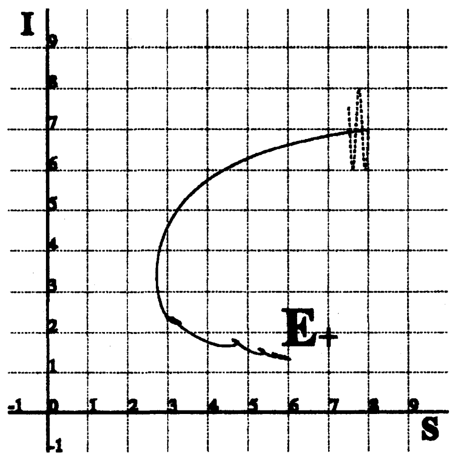
$$\begin{aligned}\dot{R}(t) &\geq \lambda I(t) - (d + (1-a)r)R(t)\\ &\geq \lambda q m_2 - (d + (1-a)r)R(t)\end{aligned}$$

for large t. Hence, $\liminf_{t\to\infty} R(t) \geq \frac{\lambda q m_2}{d+(1-a)r}$. In a similar manner, we could show $\liminf_{t\to\infty} R(t) \geq m_3$.

This proves the theorem. □

3 Conclusions

In this paper, we considered the asymptotic behavior of the SIR model with density dependent birth rate and death rate. We showed that $R_0 > 1$ implied that the disease would remain endemic for any time delay, which, together $R_0 = 1$ was the threshold between the extinction of the disease and the permanence of the disease. On the other hand, we have a conjecture that $R_0 > 1$ is also necessary and sufficient for the global asymptotic stability of the endemic equilibrium E_+ of (1.2) for any time delay. Unfortunately, we are unable to carry out the mathematical analysis, but for this conjecture there is computer evidence.



The computer evidence for this conjecture is illustrate by considering (1.2) with $\beta = 50$, $b = 50$, $d = 0.1$, $\lambda = 20$, $K = 10$, $a = 0.8$, $h = 1$ giving $\phi_1(\theta) = 0.5\theta + 8, \phi_2(\theta) = 7 + \sin(10\theta), \phi_3(\theta) = 1$. It is observed that the trajectory of solution of (1.2) converges to E_+.

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