

# Robustness of limit cycles of a planar system under the delayed feedback control

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## 1 Introduction

In 1992 Pyragas [8] proposed a new scheme for controlling chaos which is known as delayed feedback control (DFC). The idea of this scheme is as follows. Consider an  $n$  dimensional chaotic system

$$(1) \quad \dot{x} = F(x),$$

and the system with an  $m$  dimensional control force  $u(t)$ ,

$$(2) \quad \dot{x}(t) = F(x(t)) + Bu(t),$$

where  $B$  is an  $n \times m$  constant matrix which represents the accessible elements. To stabilize an unstable periodic orbit (UPO) embedded within a strange attractor of (1), we take the difference  $u(t) = -K(x(t) - x(t - \tau))$  as a control force, where the control amplitude  $K$  is  $m \times n$  matrix. If the period  $T$  of the UPO which we intend to stabilize is known a priori, we could achieve the successful control by setting  $\tau = T$ . DFC has been applied to many systems numerically because of the simplicity to use. Though the analysis is very difficult, the analytical understanding also has been gained just over the past few years [1, 6, 7, 9].

The aim of this work is to prove the effectiveness (or ineffectiveness) of DFC scheme on planar systems analytically. In this paper we give some examples for which DFC scheme with a scalar control force  $u$  has no influence on Hopf bifurcating solutions. Note that "a scalar control force" means that the control amplitude  $K$  is given by an  $1 \times n$  matrix, that is, a column vector.

Consider planar ordinary differential equations with a scalar parameter  $\mu$ .

$$(3) \quad \frac{dx}{dt} = F(x; \mu),$$

Assume that the Hopf bifurcation occurs at  $\mu = 0$ . Especially, in this article we consider only the following normalized case when

$$F(x; \mu) = A(\mu)x + f(|x|x), \quad A(\mu) = \begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix},$$

where  $x = \text{col}(x_1, x_2)$ ,  $|x| = \sqrt{x_1^2 + x_2^2}$ , and  $f: [0, \infty) \rightarrow \mathbf{R}$  is a  $C^k$  ( $k \geq 3$ ) function satisfying  $f(0) = 0$ . Note that we refer to Theorem 11.15 in [3] as the Hopf bifurcation theorem reformulated to be more directly applicable to this system. To see the bifurcation diagram for periodic orbits of (3) precisely, let us introduce polar coordinates  $(r, \theta)$  defined by

$$x_1 = r \cos \theta, \quad x_2 = -r \sin \theta.$$

Then we obtain

$$\begin{cases} \frac{dr}{dt} = (\mu + f(r))r \\ \frac{d\theta}{dt} = 1. \end{cases}$$

**Example 1.1.** For  $f(r) = -r^2$ : there is a unique nontrivial periodic orbit whose amplitude is  $\rho = \sqrt{\mu}$  if  $\mu > 0$ . The periodic orbit is orbitally asymptotically stable. See Fig. 1 for the bifurcation diagram. Because the bifurcation curve emanates from the origin to the right, the bifurcation is called supercritical.

**Example 1.2.** For  $f(r) = -(r^2 - c)^2 + c^2$  with  $c > 0$  a fixed constant: there are two nontrivial periodic orbits, one orbitally unstable and the other orbitally asymptotically stable, for  $-c^2 < \mu < 0$  with amplitudes  $\rho = \sqrt{c \pm \sqrt{\mu + c^2}}$ . See Fig. 2 for the bifurcation diagram. Because the bifurcation curve emanates from the origin to the left, the bifurcation is called subcritical.

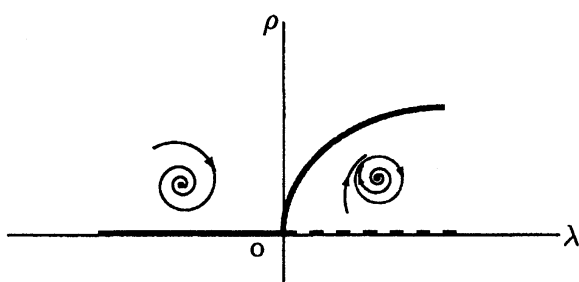


Fig. 1. Supercritical Hopf bifurcation (Ex. 1.1).

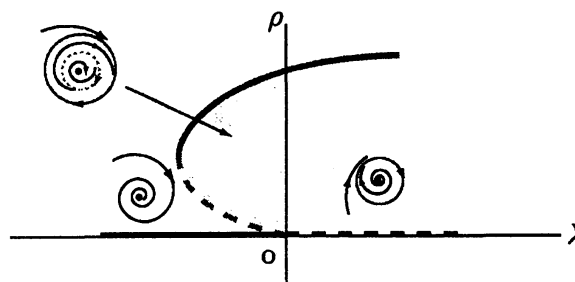


Fig. 2. Subcritical Hopf bifurcation (Ex. 1.2).

Adding the DFC with a scalar control force  $u$  to the system (3), we have

$$(4) \quad \frac{dx}{dt} = A(\mu)x + f(|x|x) + bu(t), \quad A(\mu) = \begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}, \quad u(t) = -k^T(x(t) - x(t - \tau))$$

where  $b$  and  $k$  are in  $\mathbf{R}^2$ , and  $k^T$  represents a transpose of  $k$ .

The linearized equation of (4) at  $x = 0$  becomes

$$(5) \quad \frac{dx(t)}{dt} = A(\mu)x(t) - bk^T(x(t) - x(t - \tau)).$$

The characteristic equation of (5) is

$$p(z) := \det [zI - A(\mu) + (1 - e^{-\tau z})bk^T] = 0.$$

Define two sets of the characteristic roots of (5) as

$$\Lambda := \{z: p(z) = 0 \text{ and } \Re z > 0\}, \quad \Lambda_0 := \{z: p(z) = 0 \text{ and } \Re z = 0\}.$$

Here  $\Re z$  represents the real part of a complex number  $z$ .

To avoid complication in calculating, throughout this paper we assume

$$(H1) \quad k^T b = 0.$$

Then it is convenient to write  $b$  and  $k$  as follows:

$$b = \hat{b} \begin{pmatrix} \cos \delta \\ -\sin \delta \end{pmatrix} \quad \text{and} \quad k = \hat{k} \begin{pmatrix} \sin \delta \\ \cos \delta \end{pmatrix},$$

where  $\hat{b} \leq 0$ ,  $\hat{k} \in \mathbf{R}$  and  $0 \leq \delta < 2\pi$ .

**Lemma 1.1 (RIMS Kokyuroku No. 1309, 2003).** Suppose that (H1) holds and  $|\hat{b}\hat{k}|$  is sufficiently small. If  $\tau = 2\pi$ , then for any small  $|\mu|$

(i)  $\mu < 0$  implies  $\Lambda \cup \Lambda_0 = \emptyset$ .

(ii)  $\mu = 0$  implies  $\Lambda = \emptyset$ ,  $\Lambda_0 = \{\pm i\}$ . Moreover the multiplicity of  $\pm i$  is 1 and  $\Re \frac{dz}{d\mu} \Big|_{\substack{\mu=0 \\ z=\pm i}} > 0$ .

(iii)  $\mu > 0$  implies  $\Lambda_0 = \emptyset$  and  $\Lambda$  has two elements.

This lemma shows that if we set  $\tau = 2\pi$  then the DFC has no influence on the Hopf bifurcation point of (3).

## 2 Hopf bifurcating solution & DFC

In this section we will calculate the Hopf bifurcating solution under the DFC when  $\tau = 2\pi$ . Note that for  $\mu = 0$  (5) has a pair of purely imaginary characteristic roots  $\pm i$ , while all the other eigenvalues have negative real parts by Lemma 1.1. Thus the asymptotic behavior of solutions of (4) near the equilibrium  $x = 0$  for small  $|\mu|$  governed by the dynamics on the center manifold.

Let  $\mu = 0$  in (5), that is, consider

$$(6) \quad \frac{dx}{dt} = A(0)x(t) - bk^T(x(t) - x(t - \tau)), \quad \tau = 2\pi.$$

Denote the generalized eigenspace of (6) associated with  $\Lambda_0 = \{\pm i\}$  by  $P$ . A basis of  $P$  is

$$\Phi(s) = (\phi_1(s), \phi_2(s)), \quad \phi_1(s) = \begin{pmatrix} \cos s \\ -\sin s \end{pmatrix}, \quad \phi_2(s) = \begin{pmatrix} \sin s \\ \cos s \end{pmatrix}$$

and the following relation holds.

$$\Phi(s) = \Phi(0)e^{A(0)s}.$$

If  $\phi \in P$ , then there exists  $a \in \mathbf{R}^2$  such that  $\phi = \Phi a$  and the solution  $x(\phi)$  of (6) through  $\phi$  at  $t = 0$  satisfies  $x_t(\phi) = e^{A(0)t}\Phi a$ . In other words, the solution of (6) on the generalized eigenspace of  $P$  behaves essentially an ordinary differential equation

$$\frac{dy}{dt} = A(0)y.$$

To compute the corresponding projection onto the generalized eigenspace  $\mu_0$ , consider the adjoint equation of (6)

$$(7) \quad \frac{dy(s)}{ds} = -y(s) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + (y(s) - y(s + \tau))bk^T$$

with respect to the bilinear form

$$(\psi, \phi) = \psi(0)\phi(0) + \int_{-\tau}^0 \psi(\xi + \tau)bk^T\phi(\xi)d\xi$$

for all  $\psi \in C^* = C([0, \tau], \mathbf{R}^{1 \times 2})$  and  $\phi \in C = C([-\tau, 0], \mathbf{R}^2)$ . The basis of the generalized eigenspace of (7) associated with  $\mu_0$  is

$$\Psi(s) = \begin{pmatrix} \psi_1(s) \\ \psi_2(s) \end{pmatrix}, \quad \psi_1(s) = (\cos s, -\sin s), \quad \psi_2(s) = (\sin s, \cos s).$$

It is convenient to introduce a notation of a matrix  $R(\theta)$ ,

$$R(\theta) := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

Then  $\Phi(s) = R(-s)$  and  $\Psi(s) = R(s)$ . Therefore we obtain

$$\begin{aligned} (\Psi, \Phi) &= \Psi(0)\Phi(0) + \int_{-\tau}^0 \Psi(\xi + \tau)bk^T\Phi(\xi)d\xi \\ &= I + \int_{-2\pi}^0 R(\xi + 2\pi)bk^T R(-\xi)d\xi \\ &= I + \int_{-2\pi}^0 R(\xi)b\{R(\xi)k\}^T d\xi \\ &= I + \int_{-2\pi}^0 \hat{b} \begin{pmatrix} \cos(\xi - \delta) \\ \sin(\xi - \delta) \end{pmatrix} \hat{k}(-\sin(\xi - \delta), \cos(\xi - \delta))d\xi \\ &= I + \hat{b}\hat{k} \int_{-2\pi}^0 \begin{pmatrix} -\cos(\xi - \delta)\sin(\xi - \delta) & \cos^2(\xi - \delta) \\ -\sin^2(\xi - \delta) & \cos(\xi - \delta)\sin(\xi - \delta) \end{pmatrix} d\xi \\ &= I + \frac{\hat{b}\hat{k}}{2} \int_{-2\pi}^0 \begin{pmatrix} -\sin 2(\xi - \delta) & \cos 2(\xi - \delta) + 1 \\ \cos 2(\xi - \delta) - 1 & \sin 2(\xi - \delta) \end{pmatrix} d\xi \\ &= I + \hat{b}\hat{k}\pi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = I + \hat{b}\hat{k}\pi R\left(-\frac{\pi}{2}\right), \end{aligned}$$

and

$$(\Psi, \Phi)^{-1} = \frac{1}{1 + (\hat{b}\hat{k}\pi)^2} \left\{ I + \hat{b}\hat{k}\pi R\left(\frac{\pi}{2}\right) \right\}.$$

In the following we denote  $(\Psi, \Phi)^{-1} = \Psi_0$  and  $q = 1 + (\hat{b}\hat{k}\pi)^2$ . For any  $\phi \in C$ , we can decompose it as follows:

$$\phi = \Phi c + \phi^Q, \quad \text{where } c = \Psi_0(\Psi, \phi) \quad \text{and} \quad (\Psi, \phi^Q) = 0.$$

Now let  $x$  be a solution of (4) through  $\phi \in C$  at  $t = 0$ , and let  $x_t^Q, y(t)$  be defined by

$$\begin{aligned} x_t &= \Phi y + x_t^Q, & y(t) &= \Psi_0(\Psi, x_t), \\ \phi &= \Phi c + \phi^Q, & c &= \Psi_0(\Psi, \phi). \end{aligned}$$

Using the similar arguments in [2]  $x_t^Q = O(\mu)$  as  $\mu \rightarrow 0$  whenever  $|x_t| < \mu$ , so that the basic problem lies in the investigation of the ordinary differential equation

$$(8) \quad \frac{dy}{dt} = A(0)y + \Psi_0 \{ \mu y + f(|y|)y \}.$$

Again we introduce polar coordinates  $(r, \theta)$  defined by

$$y_1 = r \cos \theta, \quad y_2 = -r \sin \theta.$$

Then we have

$$\begin{pmatrix} \frac{dr}{dt} \\ \frac{d\theta}{dt} \\ -r \frac{d\theta}{dt} \end{pmatrix} = R(\theta) \frac{dy}{dt}$$

$$\begin{aligned}
&= R(\theta) [A(0) + (\mu + f(r))\Psi_0] y \\
&= R(\theta) \left[ R\left(-\frac{\pi}{2}\right) + \frac{\mu + f(r)}{q} \left\{ I + \hat{b}\hat{k}\pi R\left(\frac{\pi}{2}\right) \right\} \right] y \\
&= \begin{pmatrix} 0 \\ -r \end{pmatrix} + \frac{(\mu + f(r))r}{q} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \hat{b}\hat{k}\pi \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.
\end{aligned}$$

Therefore we obtain

$$(9) \quad \begin{cases} \frac{dr}{dt} = \frac{1}{1+(\hat{b}\hat{k}\pi)^2}(\mu + f(r))r \\ \frac{d\theta}{dt} = 1 - \frac{\hat{b}\hat{k}\pi}{1+(\hat{b}\hat{k}\pi)^2}(\mu + f(r)). \end{cases}$$

**Remark 2.1.** Equations (9) shows that DFC has no effects on the Hopf bifurcating solutions of the planar system (3) when we take the time delay  $\tau = 2\pi$ .

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