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On various kinds of the Gray-type Theorem.

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0 Introduction

The goals of this short article are to introduce various kinds of tangent distributions on manifolds, for those Gray type theorems hold. Especially we introduce a recent result by the author ([A1]) on a generalization of the Gray theorem.

Global stabilities of various kinds of distributions on manifolds are important and interesting issues. For example, the well-known Gray theorem (see [Gr]) states that a deformation of a contact structure, through contact structures, on a compact manifold is represented by a family of global diffeomorphisms. We observe in this article some theorems of this type for various kinds of distributions. We consider sufficient conditions under which such distributions are globally stable, that is, the deformations can be represented by families of global diffeomorphisms of the underlying manifolds.

We observe the following results concerning the stability of tangent distributions. A *tangent distribution* (or *distribution* for short) D of rank k on an n-dimensional manifold M is a distribution of k-dimensional subspaces $D_x \subset T_xM$ of a tangent space at each point $x \in M$, in a strict sense. In other words, it is a subbundle of the tangent bundle.

In Section 2, we review the original Gray theorem ([Gr]). It is a global stability theorem where the deformed distributions are contact structures.

A *contact structure* is a distribution of corank one on an odd-dimensional manifold which is completely non-integrable. The Gray Theorem claims that deformations of a contact structure on a compact manifolds are represented by global isotopies (see [Gr]). There is no obstruction for representing a family of contact structures with a family of global diffeomorphisms. In the following, we introduce some studies of this type for some distributions on manifolds, which observe obstructions for the representation by global isotopies.

In Section 3, we introduce results due to R. Montgomery and M. Zhitomirskiĭ in [MZh]. They studied in [MZh] Goursat flags. A *Goursat flag* is a sequence of derived distributions rank of each of those is different by one from the next one (see Section 1 for precise definition). In subsection 3.1, we observe the case of distributions of corank one. A generalization of the Gray theorem is obtained here. A notion of *Cauchy characteristic distribution* (see Section 1 for definition) plays an important role. It is proved that a deformation of such a distribution preserving the Cauchy characteristic distribution is represented by a family of global diffeomorphisms. In subsection 3.2, we observe a Gray type theorem for Goursat flags. It is proved that a deformation of Goursat flag preserving the Cauchy characteristic distribution of the derived distribution of corank one is represented by a family of global diffeomorphisms. A Goursat flag of length 2 is, especially, called an *Engel structure.* In other words, an Engel structure is a distribution of rank 2 on a 4-dimensional manifold which is maximally non-integrable. Engel structures had been studied by F. Engel, E. Goursat, E. Cartan, and many other mathematicians for a long time. R. Montgomery and A. Golubev proved that Engel structures are globally stable under some condition about a certain line field (see [Mo], [Go]). A result due to R. Montgomery and M. Zhitomirskiĭ can be considered as an extension of this result.

In Section 4, we introduce a result in [A1] about global stability of distributions of higher coranks of derived length one. A subdistribution $K(D) \subset D$ of a distribution D is defined here. It is known that distributions with integrable $K(D)$ whose derived distribution is the tangent bundle has unique local normal form (see [KR]). It is proved that a deformation of such a distribution preserving the subdistribution K is represented by a family of global diffeomorphisms.

In Section 5, we introduce results due to B. Jakubczyk and M. Zhitomirskiĭ in [JZh3]. They studied global stability of Pfaffian equations. Pfaffian equations and tangent distributions of corank one have been studied for a long time. J. Martinet and M. Zhitomirskiĭ studied their local normal forms (see [Ma], [Zh2]). Recently, B. Jakubczyk and M. Zhitomirskiĭ obtained some results on the classification of Pfaffian equations (see [JZh1], [JZh2], [JZh3]). In the case of Pfaffian equations on odd-dimensional manifolds, non-contact loci and certain characteristic line fields on the loci played an important role for the classification. And, in the case of Pfaffian equations on even-dimensional manifolds, certain characteristic line fields played an important role for the classification. They obtained a sufficient condition for global stability in terms of above notions.

In Section 6, we introduce examples of distributions of rank 2 on 4-manifolds with non-Engel locus. M. Zhitomirskiĭobtained them in [Zh1] as normal forms. We mention about results, in the forthcoming paper [A2], about global stability of such distributions. Non-Engel loci and characteristic line fields play an important role.

1 Preliminaries

In this section, we define some basic notions needed in the following sections. Some notions needed in one of the sections is defined in each section. *Strong derived distributions* D^i, $i = 1, 2, \ldots, k$, of a distribution D are defined pointwise inductively as follows,

$$\begin{cases} D^1 = D, \\ D^{i+1} = D^i + [D^i, D^i]. \end{cases}$$

Note that it is defined pointwise in terms of sheaves of vector fields which are cross-section of the subbundle $D \subset TM$. Therefore ranks of D^i_p, $p \in M$, might be different for each point $p \in M$. We say a distribution D to be *Lie square regular* if all D^i are distributions of constant ranks. The *Cauchy characteristic distribution* $L(D)$ of a distribution D is defined pointwise as follows,

$$\begin{aligned} L(D)_p &= \{X \in D_p \mid [X,Y] \in D_p, \text{ for any } Y \in D_p\}, \\ &= \{X \in D_p \mid X \lrcorner d\omega|_{D_p} = 0, \text{ for any } \omega \in \mathcal{S}(D)\}. \end{aligned}$$

When $L(D)$ is a distribution of constant rank, the distribution $L(D)$ is integrable according to the Frobenius theorem. Note that $L(D)$ is a distribution of rank 0 when D is a contact structure.

2 The original Gray theorem

In this section, we introduce the well-known Gray theorem proved in [Gr]. It is one of the most important Theorem for contact topology.

Theorem 2.1 (Gray). *Let D_t be a family of contact structures on a compact orientable manifold M. Then there exists a family $\Phi_t\colon M \to M$ of global diffeomorphisms which satisfies $\varphi_0 = \mathrm{id}$ and $(\varphi_t)_* D_0 = D_t$.*

3 Lie square regular distributions and Goursat flags

In this section, we introduce some results due to R. Montgomery and M. Ya. Zhitomirskiĭ ([MZh]). They studied Goursat flags from the geometric view point. The first step of their inductive proof is a generalization of the Gray theorem.

3.1 Lie square regular distributions of corank one.

The following is proved in [MZh].

Theorem 3.1 (Montgomery-Zhitomirskiĭ). *Let D_t, $t \in [0,1]$, be a one-parameter family of distributions of corank $k = 1$ on a compact orientable manifold M. It is assumed that D_t has the Cauchy characteristic distribution $L(D_t) \equiv L$ of constant rank for any $t \in [0,1]$. Then, there exists a family of global diffeomorphisms $\varphi_t \colon M \to M$, $t \in [0,1]$, which satisfies $\varphi_0 = \mathrm{id}$ and $(\varphi_t)_* D_0 = D_t$ for any $t \in [0,1]$.*

Distributions appeared in the theorem above is Lie square regular. Theorem 3.1 is a generalization of the Gray theorem 2.1 since $\operatorname{rank} L(D_t) = 0$ if D_t are contact structures.

3.2 Goursat flags.

They also proved a Gray type theorem for Goursat flags. A *Goursat flag* of length s on a manifold M is a sequence

$$(F) : D_s \subset D_{s-1} \subset \cdots \subset D_1 \subset D_0 = TM, \ \ s \geq 2$$

of distributions on M which satisfies the following conditions:

$$\begin{cases} \mathrm{corank} D_i = i, & i = 1, 2, \ldots, s, \\ D_{i-1} = D_i^2 = D_i + [D_i, D_i], & i = 1, 2, \ldots, s. \end{cases}$$

The following is proved in [MZh].

Theorem 3.2 (Montgomery-Zhitomirskiĭ). *Let $(F_t) : D_{s,t} \subset \cdots \subset D_{1,t} \subset TM, t \in [0,1]$ be a family of Goursat flags of Length s on a compact orientable manifold M. Suppose that $L(D_{1,t}) \equiv L(D_{1,0}$ for any $t \in [0,1]$. Then there exists a family $\varphi \colon M \to M$ of diffeomorphisms which satisfies $\varphi_0 = \mathrm{id}$ and $(\varphi_t)_*(F_t) = (F_0)$.*

A distribution of rank 2 on a 4-manifold which construct a Goursat flag of length 2 is called an *Engel structure*. It is known that Engel structures have a unique local normal form. A Gray type theorem for Engel structure is studied independently (see [Go], [Mo]).

Theorem 3.3 (Golubev, Montgomery). *Let D_t be a family of Engel structures on a compact orientable 4-manifold M. Suppose that $L({D_t}^2) \equiv L({D_0}^2)$ for any $t \in [0,1]$. Then there exists a family $\varphi \colon M \to M$ of diffeomorphisms which satisfies $\varphi_0 = \mathrm{id}$ and $(\varphi_t)_*(D_t) = (D_0)$.*

Note that Theorem 3.2 can be considered as an extension of Theorem 3.3.

4 Lie square regular distributions of higher coranks

In this section, we introduce a result obtained in [A1]. It is a Gray type theorem for Lie square regular distributions those coranks are greater than one.

First of all, we define a certain subdistribution $K(D)$ of a distribution D. It is defined in terms of the Pfaffian system $\mathcal{S}(D)$. We define a covariant system associated to a Pfaffian system $\mathcal{S} \subset T^*M$ according to A. Kumpera and J. L. Rubin (see [KR]), as follows. The bundle map $\delta \colon \mathcal{S} \to \bigwedge^2(T^*M/\mathcal{S})$ defined on local sections of $\mathcal{S}$ as $\delta(\omega) = d\omega \pmod{\mathcal{S}}$ is called the *Martinet structure tensor* (see [Ma]). We define the *polar space* $\mathrm{Pol}(\mathcal{S})_p$ of $\mathcal{S}$ at $p \in M$ as

$$\mathrm{Pol}(\mathcal{S})_p := \left\{ w \in T_p^*M/\mathcal{S}_p \mid w \wedge \delta(\omega) = 0, \text{ for any } \omega \in \mathcal{S} \right\}.$$

When the polar space $\mathrm{Pol}(\mathcal{S})_p$ has a constant rank on M, we define the *covariant system* $\widehat{\mathcal{S}}$ associated to $\mathcal{S}$ as $\widehat{\mathcal{S}} := q^{-1}(\mathrm{Pol}(\mathcal{S}))$, where $q \colon T^*M \to T^*M/\mathcal{S}$ is the quotient map. For a distribution $D \subset TM$, let $K(D)$ denote the subdistribution of D which is annihilated by the covariant system $\widehat{\mathcal{S}}(D)$ associated to the Pfaffian system $\mathcal{S}(D)$.

Example 4.1. We give an example of the polar space and the covariant system for the standard distribution on $J^1(1,k) \cong \mathbb{R}^{2k+1}$. Let $D_0 = \{\omega_1 = 0, \ldots, \omega_k = 0\}$, where $\omega_i := dx_{2i-1} + x_{2i}dt$, be a distribution on $\mathbb{R}^{2k+1}$ with the standard coordinates $(x_1, \ldots, x_{2k}, t)$. When $k > 1$, the distribution of polar spaces of $\mathcal{S}(D_0)$ is obtained as follows:

$$\begin{aligned}\mathrm{Pol}(\mathcal{S}(D_0)) &= \{w \in T^*M/\mathcal{S}(D_0) \mid \\ &\qquad w \wedge dx_{2i} \wedge dt \equiv 0 \pmod{\mathcal{S}(D_0)},\ i = 1, 2, \ldots, k\} \\ &= \{dt\}.\end{aligned}$$

Then the covariant system is obtained as follows:

$$\widehat{\mathcal{S}}(D_0) = \{\omega_1, \ldots, \omega_k, dt\} = \{dx_1, dx_3, \ldots, dx_{2k-1}, dt\}.$$

Then we have $K(D_0) = \langle \partial/\partial x_2, \partial/\partial x_4, \ldots, \partial/\partial x_{2k}\rangle$. They are clearly integrable. When $k = 1$, $D_0 = \{dx_1 - x_2 dt = 0\}$ is the standard contact structure on $\mathbb{R}^3$. Then we have $\mathrm{Pol}(\mathcal{S}(D_0)) = \{dx_2, dt\}$, $\widehat{\mathcal{S}}(D_0) = \{dx_1, dx_2, dt\}$, and $K(D_0) = \langle 0 \rangle$.

The main theorem in [A1] is the following.

Theorem 4.2. *Let D_t, $t \in [0, 1]$, be a one-parameter family of distributions of corank $k \geq 1$ on a compact orientable manifold M. Suppose, for any $t \in [0, 1]$:*

(1) *the first derived distributions coincide with the tangent bundle of M: $D_t^2 = TM$,*

(2) *there exists a constant integrable subdistribution $K \subset D_t$ of corank one.*

Then, there exists a family $\varphi_t \colon M \to M$, $t \in [0, 1]$, of global diffeomorphisms which satisfies $\varphi_0 = \mathrm{id}$ and $(\varphi_t)_* D_0 = D_t$ for any $t \in [0, 1]$.

Note that Theorem 4.2 is obtained from Theorem 3.1 when $k = 1$ since $L(D_t) = K(D_t)$ then and they are integrable from the definition of the Cauchy characteristic distribution.

It is also proved in [KR] that such distributions as in Theorem 4.2 have have a unique normal form as in Example 4.1.

Proposition 4.3 (Kumpera-Rubin). *Let D be a distribution of corank $k > 1$ on a manifold M whose derived distribution coincides with the tangent bundle: $D^2 = TM$. If the distribution $K(D)$ is integrable, then at each point $p \in M$ the distribution D admits the following local normal form: $D = \{\omega_1 = 0, \dots, \omega_k = 0\}$,*

$$\omega_1 = dx_1 + x_2 dt,\ \omega_2 = dx_3 + x_4 dt,\ \dots,\ \omega_k = dx_{2k-1} + x_{2k} dt,$$

where the coordinates x_i, t vanish at $p \in M$.

5 Distributions of corank 1 with singularities

In this section, we introduce some results obtained by B. Jakubczyk and M. Zhitomirskiĭ in [JZh3]. They studied in [JZh3] global stability of distributions with degeneracy loci. The first typical example is the Martinet normal form.

Example 5.1 (Martinet). Set $D = \{\alpha = dz - y^2 dx = 0\}$ on $\mathbb{R}^3$ with coordinates (x, y, z). Then, we obtain the non-contact locus Σ as follows:

$$\Sigma = \{p \in \mathbb{R}^3 \mid (\alpha \wedge d\alpha)_p = (2y dx \wedge dy \wedge dz)_p = 0\} = \{y = 0\}.$$

The results in [JZh3] is mentioned in two cases: (1)the case $\dim M = 2k$, and (2) the case $\dim M = 2k+1$, where M is an underlying manifold. **Case(1)** Let $P = (\omega)$ be a Pfaffian equation on a manifold M of dimension $n = 2k$, and ω a generator of P. A *characteristic vector field* X of P is defined the relation $X \lrcorner \omega = \omega \wedge (d\omega)^{k-1}$, where Ω is a volume form. The line field $L(P)$ generated by a characteristic vector field X is called the *characteristic line field* of P. Let $\mathrm{Sing}(L)$ denote the set of singular points of $L(P)$. We introduce an important notion concerning singular points of $L(P)$. Let $I_p(X)$ be an ideal in the ring of function germs at $p \in \mathrm{Sing}(L)$, generated by the coefficients $a_1, \dots, a_n$ of a characteristic vector field X, with respect to some coordinate system around p. Set $d_p(P) := \mathrm{depth} I_p(X)$. Then the following condition is introduced in [JZh3]:

$$(A) \qquad d_p(P) \geq 3 \text{ for any point } p \in \mathrm{Sing}(L).$$

It is proved in [JZh3] that this condition is a genericity condition. The statement of the result is the following:

Theorem 5.2 (Jakubczyk-Zhitomirskiĭ). *Let P_t, $t \in [0,1]$, be a family of Pfaffian equations on M^{2k}, $k \geq 2$, which satisfies the following conditions:*

(1) *all P_t define the common characteristic line field $L = L(P_t)$,*

(2) *all P_t satisfy condition (A).*

Then, there exists a family $\Phi_t \colon M \to M$ of diffeomorphisms sending P_t to P_0.

Case(2) As we observe in Example 5.1, there is a non-contact locus called the *Martinet hypersurface*: $S = \{p \in M \mid (\omega \wedge (d\omega)^k)_p = 0\}$. In a similar way to Case(1) above, the characteristic line field $L(P)$ on the Martinet hypersurface S, and the depth $d_p(P)$ at singular point $p \in \mathrm{Sing}(L)$ are defined. In this case we need further condition concerning the Martinet hypersurface. Let H be a function defined as $H = \omega \wedge (d\omega)^k/\Omega$, where Ω is a volume form. H determines the Martinet hypersurface S as its zero level. The ideal (H) is called the *Martinet ideal.* The Martinet ideal (H) is said to have the *property of zeros* if the ideal generated by the germ H_p of H at $p \in S = \{H = 0\}$ in the ring of all function germs at p coincides with the ideal consisting of function germs vanishing on the germ at p of S in the same ring, for any $p \in S$. We need one more condition. Let $C^\infty(M)$ be the Freché space of smooth functions on M, and $C^\infty(M,S)$ its closed subspace consists of functions which vanish on S. Set $C^\infty(S) = C^\infty(M)/C^\infty(M,S)$. The Martinet hypersurface S is said to have the *extension property* if there exists a continuous linear operator $\lambda \colon C^\infty(S) \to C^\infty(M)$ which satisfy $\lambda(f)|_S = f$ for all $f \in C^\infty(S)$. The statement of the result is the following:

Theorem 5.3 (Jakubczyk-Zhitomirskiĭ). *Let P_t, $t \in [0,1]$, be a family of Pfaffian equations on a compact orientable manifold M^{2k+1}, $k \geq 1$, which satisfies the following conditions:*

(1) *all P_t have the common Martinet hypersurface S, which has the extension property, and the Martinet ideals have the property of zeros,*

(2) *all P_t define the common characteristic line field $L = L(P_t)$,*

(3) *all P_t satisfy condition (A).*

Then, there exists a family $\Phi_t \colon M \longrightarrow M$ of diffeomorphisms sending P_t to P_0.

6 Distributions of corank 2 on 4-manifolds with non-Engel locus

In this section, we consider distributions of rank 2 on 4-manifolds with non-Engel loci. First, we define the notion of non-Engel loci. Let D be an Engel structure. Recall that the derived distributions D^2, D^3 may not be distributions in a strict sense. In fact, they may have a point where the rank of distribution degenerates. We set

$$\Sigma_1(D) := \{p \in M \mid \operatorname{rank} D_p^2 < 3\}, \quad \Sigma_2(D) := \{p \in M \mid \operatorname{rank} D_p^3 < 4\},$$

and call them the first and the second *non-Engel loci* of D respectively. We call the union $\Sigma_1(D) \cup \Sigma_2(D) =: \Sigma(D)$ just the *non-Engel locus* of D.

Example 6.1. We observe normal forms obtained by M. Zhitomirskiĭ in [Zh1]. We regard them distributions on $\mathbb{R}^4$ with coordinates (x, y, z, w).

(1) $D = \{\omega_1 = dx + z^2 dwz - 0,\ \omega_2 = dy + zwdw = 0\}$
$\quad = \langle \partial/\partial w - z^2\partial/\partial x - zw\partial/\partial y, \partial/\partial z \rangle$

In this case,

$$D^2 = \left\langle \frac{\partial}{\partial w} - z^2\frac{\partial}{\partial x} - zw\frac{\partial}{\partial y}, \frac{\partial}{\partial z}, 2z\frac{\partial}{\partial x} + w\frac{\partial}{\partial y} \right\rangle,$$

$$D^3 = \left\langle \frac{\partial}{\partial w} - z^2\frac{\partial}{\partial x} - zw\frac{\partial}{\partial y}, \frac{\partial}{\partial z}, 2z\frac{\partial}{\partial x} + w\frac{\partial}{\partial y}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle.$$

Therefore,

$$\Sigma_1 = \{\dim D_p^2 < 3\} = \{z = 0, w = 0\}, \quad \Sigma_2 = \{\dim D_p^3 < 4\} = \emptyset.$$

(2) $D = \{\omega_1 = dx + zdw = 0,\ \omega_2 = dy + z^2 w dw = 0\}$
$= \langle \partial/\partial w - z\partial/\partial x - z^2 w\partial/\partial y, \partial/\partial z \rangle$

In this case,

$$D^2 = \left\langle \frac{\partial}{\partial w} - z\frac{\partial}{\partial x} - z^2 w\frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial x} + 2zw\frac{\partial}{\partial y} \right\rangle,$$

$$D^3 = \left\langle \frac{\partial}{\partial w} - z\frac{\partial}{\partial x} - z^2 w\frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial x} + 2zw\frac{\partial}{\partial y}, 2w\frac{\partial}{\partial y}, 2z\frac{\partial}{\partial y} \right\rangle.$$

Therefore,

$$\Sigma_1 = \{\dim D_p^2 < 3\} = \emptyset, \quad \Sigma_2 = \{\dim D_p^3 < 4\} = \{w = 0, z = 0\}.$$

In a similar way to Section 5, a characteristic line field for the derived distribution D^2 can be defined. Non-Engel loci and the characteristic line fields play an important role in the arguments on Gray type theorem for distributions with non-Engel loci, in the forthcoming paper [A2].

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