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Fast Algorithms for Computing Jones Polynomials of Certain Links

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Abstract

We give fast algorithms for computing Jones polynomials of 2–bridge links, closed 3–braid links and Montesinos links from a progressive expression. The algorithms run with $\mathcal{O}(n)$ operations of polynomials of degree $\mathcal{O}(n)$, where n is the number of the crossings of the link diagram. We also give linear time algorithms for computing a progressive expression from the Tait graph of a link diagram of 2–bridge links and closed 3–braid links.

1 Introduction

In knot theory, various invariants are defined and studied for classifying and characterizing links. The Jones polynomial [4] is one of the well-studied invariants. It is powerful for distinguishing link types. The simplest way to define it is by using a slightly different polynomial: the bracket polynomial discovered by L.H. Kauffman [5]. But, it takes $\mathcal{O}\left(2^{\mathcal{O}(\sqrt{c(\widetilde{L})})}\right)$ operations of polynomials of degree $\mathcal{O}(c(\widetilde{L}))$ to compute a Jones polynomial in the way shown by Kauffman. Actually, computing the Jones polynomial is generally #**P**–hard [3, 12] and is expected to require exponential time in the worst case. K. Sekine, H. Imai and K. Imai [10] showed an algorithm that computes Jones polynomials in $\mathcal{O}\left(2^{\mathcal{O}(\sqrt{c(\widetilde{L})})}\right)$ time.

Recently, it has been recognized that it is important to compute Jones polynomials for links with reasonable restrictions. For any link diagram $\widetilde{L}$, we denote the number of the crossings of $\widetilde{L}$ by $c(\widetilde{L})$. J. A. Makowsky [6] showed that Jones polynomials are computed from the Tait graph G of $\widetilde{L}$ in polynomial time if the treewidth of G is a constant. J. Mighton [7] showed that Jones polynomials are computed from the Tait graph G of $\widetilde{L}$ with $\mathcal{O}(c(\widetilde{L})^4)$ operations of polynomials of degree $\mathcal{O}(c(\widetilde{L}))$ if the treewidth of G is at most 2. M. Hara, S. Tani and M. Yamamoto [2] showed that Jones polynomials of 2–bridge links are computed from the Tait graph of $\widetilde{L}$ with $\mathcal{O}(c(\widetilde{L})^2)$ operations of polynomials of degree $\mathcal{O}(c(\widetilde{L}))$, and Jones polynomials of closed 3–braid links and arborescent links are computed from the Tait graph of $\widetilde{L}$ with $\mathcal{O}(c(\widetilde{L})^3)$ operations of polynomials of degree $\mathcal{O}(c(\widetilde{L}))$. T. Utsumi and K. Imai [11] showed that Jones polynomials of pretzel links are computed from the Tait graph of $\widetilde{L}$ in $\mathcal{O}(c(\widetilde{L})^2)$ time.

In this paper, we give algorithms that compute Jones polynomials of 2–bridge links and closed 3–braid links from the Tait graph of $\widetilde{L}$ with $\mathcal{O}(c(\widetilde{L}))$ operations of polynomials of degree $\mathcal{O}(c(\widetilde{L}))$. We also show that Jones polynomials of Montesinos links are computed from a progressive expression of $\widetilde{L}$ with $\mathcal{O}(c(\widetilde{L}))$ operations of polynomials of degree $\mathcal{O}(c(\widetilde{L}))$.

2 Preliminaries

A *link* of n components is a subset of $\mathbf{R}^3$ that consists of n disjoint, simple closed curves. A link of one component is a *knot*. An image of a link by a natural projection from $\mathbf{R}^3$ to a plane is *regular* if it contains only finitely many multiple points, all multiple points are double points and these are traverse points. A regular image of a link is called a *link diagram* if the overcrossing line is marked at every double point in the image. Furthermore, the double points are called *crossings*. For any link diagram $\widetilde{L}$, we denote the number of the crossings of $\widetilde{L}$ by $c(\widetilde{L})$. A link is *oriented* if each of its components is given an orientation.

Definition 2.1 The *Kauffman bracket polynomial* is a function from link diagrams to the Laurent polynomial ring $\mathbf{Z}[A^{\pm 1}]$ with integer coefficients in an indeterminate A. It maps a link diagram $\widetilde{L}$ to $\langle \widetilde{L} \rangle \in \mathbf{Z}[A^{\pm 1}]$ and is characterized by

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$$\text{(i) } \langle \bigcirc \rangle = 1, \qquad \text{(ii) } \langle \widetilde{L} \sqcup \bigcirc \rangle = (-A^{-2} - A^{2})\langle \widetilde{L} \rangle, \qquad \text{(iii) } \langle \times \rangle = A\langle\,)(\, \rangle + A^{-1}\langle \asymp \rangle.$$

Here, $\bigcirc$ is the link diagram of the unknot without a crossing and $\widetilde{L} \sqcup \bigcirc$ is a link diagram consisting of the link diagram $\widetilde{L}$ together with an extra closed curve $\bigcirc$ that contains no crossing at all, neither with itself nor $\widetilde{L}$. In (iii) the formula refers to three link diagrams that are exactly the same except near a point where they differ in the way indicated.

The *writhe* $w(\widetilde{L})$ of an oriented link diagram $\widetilde{L}$ is the sum of the signs of the crossings of $\widetilde{L}$, where each crossing has sign $+1$ or -1 as defined (by convention) in Figure 1.

Definition 2.2 The *Jones polynomial* $V(L)$ of an oriented link L is the Laurent polynomial in $t^{1/2}$ with integer coefficients, defined by

$$V(L) = (-A)^{-3w(\widetilde{L})}\langle \widetilde{L} \rangle\Big|_{t^{1/2}=A^{-2}},$$

where $\widetilde{L}$ is any oriented link diagram for L.

Given any link diagram $\widetilde{L}$, we can color the faces black and white in such a way that no two faces with a common edge are the same color. We color the unique unbounded face white. We call this the *Tait coloring* of $\widetilde{L}$. As in Figure 2, we can get a signed planar graph G of $\widetilde{L}$; its vertices are the black faces of the Tait coloring and two vertices are joined by a signed edge if they share a crossing. The sign of the edge is $+1$ or -1 according to the (conventional) rule shown in Figure 3. G is the *Tait graph* of $\widetilde{L}$.

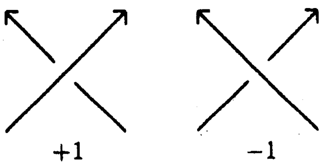


Figure 1: Signs of crossings

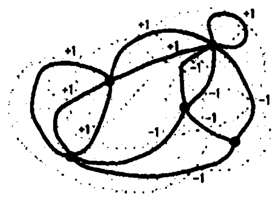


Figure 2: A Tait graph

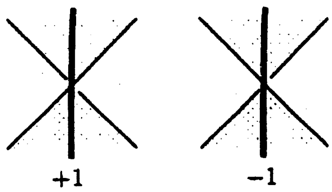


Figure 3: Signs of edges

A *tangle* is a portion of a link diagram from which there emerge just 4 arcs. The tangle consisting of two vertical strings without a crossing is called 0*-tangle*. The tangle twisted 0–tangle k times is called k*-tangle* and is denoted by I_k. They are called *integer tangles* (Figure 4). The tangle consisting of two horizontal strings without a crossing is called ∞*-tangle*. For a set S, we denote the number of the elements of S by $|S|$. For an integer z, sign(z) is defined by the following:

$$\text{sign}(z) = \begin{cases} 1 & \text{if } z \geq 0, \\ -1 & \text{if } z < 0. \end{cases}$$

Let $G = (V, E)$ be a graph, where V is the vertex set of G and E is the edge set of G. For any vertex $v \in V$, $\deg_G(v)$ denotes the degree of v in G and $N_G(v)$ denotes the set of the neighbors of v in G. For any subset V' of V, $G[V']$ denotes the induced subgraph of G by V'. For any vertices $u, v \in V$, edge_sign$_G$ is a function from $V \times V$ to $\mathbf{Z}$ and edge_sign$_G(u, v)$ is the sum of the signs of the edges of G that join u and v.

3 Algorithms for 2–bridge links

Schubert [9] defined a numerical link invariant called bridge number and many knot theorists have been studying it (see [8, 1]). In particular, the 2–bridge link is one of the most important link types. It is well known that any 2–bridge link has a diagram consisting of integer tangles I_{a_k} as in Figure 5 where a_k is an integer for $k = 1, \ldots, m$.

Definition 3.1 A link diagram as in Figure 5 is called a *normal diagram* of a 2–bridge link and is denoted by $\widetilde{R}(a_1, \ldots, a_m)$. The progression $(a_1, \ldots, a_m)$ is called a *progressive expression* of the normal diagram.

Lemma 3.2 *For any Tait graph $G = (V, E)$ of a normal diagram of a 2-bridge link, there exists a vertex $v \in V$ such that $G[V - \{v\}]$ is a path.*

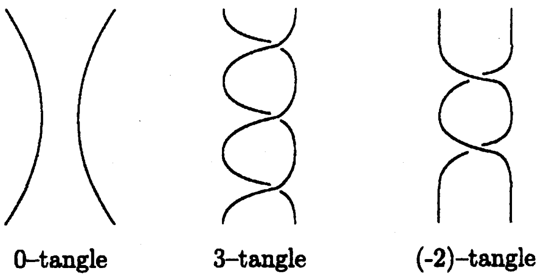


Figure 4: Integer tangles

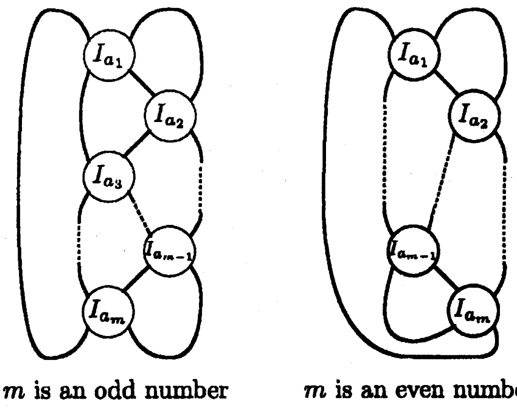


Figure 5: Normal diagrams of 2–bridge links

Lemma 3.3 *For any Tait graph $G = (V, E)$ of a normal diagram of a 2-bridge link, for a vertex $v \in V$, if $\deg_G(v)$ is the maximum degree of G, then $G[V - \{v\}]$ is a path.*

Given the Tait graph of a normal diagram $\widetilde{L}$ of a 2-bridge link, **Procedure progression_2–bridge** computes a progressive expression of $\widetilde{L}$.

Procedure progression_2–bridge
INPUT: The Tait graph $G = (V, E)$ of a normal diagram $\widetilde{L}$ of a 2–bridge link.
OUTPUT: A progressive expression $(a_1, \ldots, a_m)$ of $\widetilde{L}$.
$p \longleftarrow |V| - 1$.
Label a vertex $v \in V$ as "v_p" such that $\deg_G(v_p)$ is the maximum degree of G.
Label all vertices $v \in V - \{v_p\}$ as "$v_0, \ldots, v_{p-1}$" such that v_i and v_{i+1} are adjacent for $i = 0, \ldots, p-2$.
Compute edge_sign$_G(v_i, v_p)$ for $i = 0, \ldots, p-1$ and edge_sign$_G(v_j, v_{j+1})$ for $j = 0, \ldots, p-2$.
Initialize i as "0" and k as "1".
while $i < p - 1$ **do**
{ k is an odd number }
$a_k \longleftarrow$ −edge_sign$_G(v_i, v_p)$.
Increment k.
{ k is an even number }
Initialize a_k as "0".
repeat
$a_k \longleftarrow a_k +$ edge_sign$_G(v_i, v_{i+1})$.
Increment i.
until $\deg_G(v_i) \neq 2$ or $i = p - 1$
Increment k.
od
$a_k \longleftarrow$ −edge_sign$_G(v_{p-1}, v_p)$.

Lemma 3.4 *Given the Tait graph of a normal diagram $\widetilde{L}$ of a 2-bridge link,* **Procedure progression_2–bridge** *computes a progressive expression of $\widetilde{L}$ in $O(c(\widetilde{L}))$ time.*

Lemma 3.5 *For any normal diagram $\widetilde{R}(a_1, \ldots, a_m)$ of a 2-bridge link, the following recurrence formula holds.*

$$\langle \widetilde{R}(a_1, \ldots, a_m) \rangle = \begin{cases} A^{a_1}(-A^{-2} - A^2) + (-A)^{-3a_1 - 2\mathrm{sign}(a_1)} \sum_{k=1}^{|a_1|} \left(-A^{4\mathrm{sign}(a_1)}\right)^k & \text{if } m = 1, \\ A^{a_2}(-A^{-3})^{a_1} + (-A)^{-3a_2 - 2\mathrm{sign}(a_2)} \langle \widetilde{R}(a_1) \rangle \sum_{k=1}^{|a_2|} \left(-A^{4\mathrm{sign}(a_2)}\right)^k & \text{if } m = 2, \\ A^{a_m}(-A^{-3})^{a_{m-1}} \langle \widetilde{R}(a_1, \ldots, a_{m-2}) \rangle & \\ \quad + (-A)^{-3a_m - 2\mathrm{sign}(a_m)} \langle \widetilde{R}(a_1, \ldots, a_{m-1}) \rangle \sum_{k=1}^{|a_m|} \left(-A^{4\mathrm{sign}(a_m)}\right)^k & \text{if } m \geq 3. \end{cases}$$

Given a progressive expression of a normal diagram $\widetilde{L}$ of a 2–bridge link, **Procedure bracket_2–bridge** computes the Kauffman bracket polynomial of $\widetilde{L}$ by using the recurrence formula in **Lemma 3.5**. While the algorithm is running, every Kauffman bracket polynomial is computed once at most.

Procedure bracket_2–bridge

INPUT: A progressive expression $(a_1, \ldots, a_m)$ of a normal diagram $\widetilde{L}$ of a 2–bridge link.
OUTPUT: The Kauffman bracket polynomial $\langle \widetilde{R}(a_1, \ldots, a_m) \rangle$ of $\widetilde{L}$.
Compute $\langle \widetilde{R}(a_1) \rangle$ and $\langle \widetilde{R}(a_1, a_2) \rangle$.
Initialize i as "3".
while $i \leq m$ **do**

$$T \longleftarrow \sum_{k=1}^{|a_i|} \left(-A^{4\mathrm{sign}(a_i)}\right)^k.$$

Compute $\langle \widetilde{R}(a_1, \ldots, a_i) \rangle$ from $\langle \widetilde{R}(a_1, \ldots, a_{i-2}) \rangle$, $\langle \widetilde{R}(a_1, \ldots, a_{i-1}) \rangle$ and T.
Increment i.

od

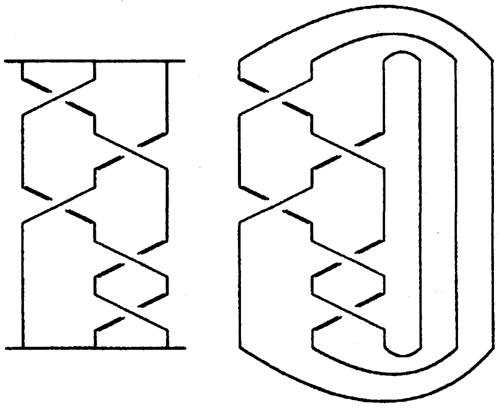
Lemma 3.6 *Given a progressive expression of a normal diagram $\widetilde{L}$ of a 2–bridge link,* **Procedure bracket_2–bridge** *computes the Kauffman bracket polynomial of $\widetilde{L}$ with $\mathcal{O}(c(\widetilde{L}))$ operations of polynomials of degree $\mathcal{O}(c(\widetilde{L}))$.*

Theorem 3.7 *The Jones polynomial of a 2–bridge link is computed from the Tait graph of a normal diagram $\widetilde{L}$ of the 2–bridge link with $\mathcal{O}(c(\widetilde{L}))$ operations of polynomials of degree $\mathcal{O}(c(\widetilde{L}))$.*

4 Algorithms for closed 3–braid links

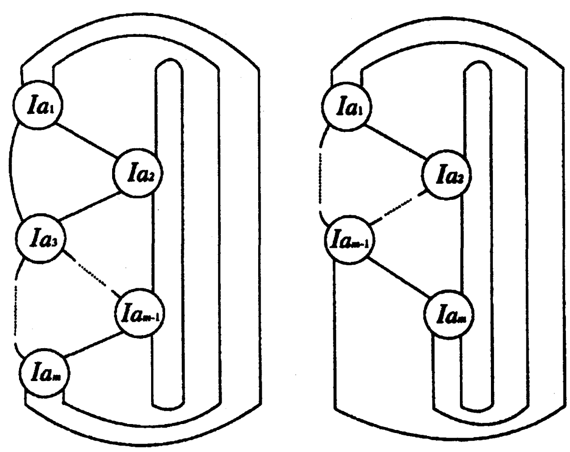
A *3–braid* is a set of 3 strings, all of which are attached to a horizontal bar at the top and at the bottom and each string intersects any horizontal plane between the two bars exactly once (see Figure 6 (a)). Given any 3–braid, its ends on the bottom edge may be joined to those on the top edge to produce the *closed 3–braid link* (see Figure 6 (b)). It is clear that any closed 3–braid link has a diagram consisting of integer tangles I_{a_k} as in Figure 7 where a_k is an integer for $k = 1, \ldots, m$.

Definition 4.1 A link diagram as in Figure 7 is called a *normal diagram* of a closed 3–braid link and is denoted by $\widetilde{B}(a_1, \ldots, a_m)$. The progression $(a_1, \ldots, a_m)$ is called a *progressive expression* of the normal diagram.



(a) (b)

Figure 6: (a) A 3–braid, (b) A closed 3–braid link



m is an odd number m is an even number

Figure 7: Normal diagrams of closed 3–braid links

Lemma 4.2 *For any Tait graph $G = (V, E)$ of a normal diagram of a closed 3–braid link, there exists a vertex $v \in V$ such that $G[V - \{v\}]$ is a cycle.*

Given the Tait graph of a normal diagram $\widetilde{L}$ of a closed 3–braid link, **Procedure progression_3–braid** computes a progressive expression of $\widetilde{L}$.

Procedure progression_3–braid

INPUT: The Tait graph $G = (V, E)$ of a normal diagram $\widetilde{L}$ of a closed 3–braid link.
OUTPUT: A progressive expression $(a_1, \ldots, a_m)$ of $\widetilde{L}$.

```
p ⟵ |V| − 1.
if there exists a vertex v ∈ V such that |N_G(v)| ≥ 4
  Label v as "v_p",
else if there exists a vertex v ∈ V such that |N_G(v)| = 0
  Label v as "v_p",
else if there exists a vertex v ∈ V such that |N_G(v)| = 1
  Label v as "v_p",
else if there exists no vertex v ∈ V such that |N_G(v)| ≠ 2   { for any vertex v ∈ V, |N_G(v)| = 2 }
  Label a vertex v ∈ V as "v_p" such that G[V − {v_p}] is a cycle,
else
  V' ⟵ {v | v ∈ V, |N_G(v)| = 3},
  Label a vertex v ∈ V as "v_p" such that ⋂_{v'∈V'−{v_p}} N_G(v') = {v_p}.
Label the all vertices v ∈ V − {v_p} as "v_0, ..., v_{p−1}"
                                such that v_i and v_{i+1 mod p} are adjacent for i = 0, ..., p − 1.
Compute edge_sign_G(v_i, v_p) and edge_sign_G(v_i, v_{i+1 mod p}) for i = 0, ..., p − 1.
Initialize i as "0" and k as "1".
repeat
  { k is an odd number }
  a_k ⟵ −edge_sign_G(v_i, v_p).
  Increment k.
  { k is an even number }
  Initialize a_k as "0".
  repeat
    a_k ⟵ a_k + edge_sign_G(v_i, v_{i+1 mod p}).
    Increment i.
  until deg_G(v_i) ≠ 2 or i = p
  Increment k.
until i = p
```

Lemma 4.3 *Given the Tait graph of a normal diagram $\widetilde{L}$ of a closed 3-braid link,* **Procedure progression_3–braid** *computes a progressive expression of $\widetilde{L}$ in $O(c(\widetilde{L}))$ time.*

Lemma 4.4 *For any normal diagram $\widetilde{B}(a_1, \ldots, a_m)$ of a closed 3-braid link, the following recurrence formula holds.*

$$\langle\widetilde{B}(a_1,\ldots,a_m)\rangle = \begin{cases} (-A^{-2}-A^{2})\langle\widetilde{R}(a_1)\rangle & \text{if } m=1,\\[2mm] A^{a_m}\langle\widetilde{B}(a_1,\ldots,a_{m-1})\rangle + (-A)^{-3a_m-2\mathrm{sign}(a_m)} & \text{if } m\ge 2 \text{ and}\\ \quad\times\langle\widetilde{R}(a_1,\ldots,a_{m-1})\rangle \displaystyle\sum_{k=1}^{|a_m|}\left(-A^{4\mathrm{sign}(a_m)}\right)^{k} & \quad m \text{ is an even number},\\[2mm] A^{a_m}\langle\widetilde{B}(a_1,\ldots,a_{m-1})\rangle + (-A)^{-3(a_m+a_1)-2\mathrm{sign}(a_m)} & \text{if } m\ge 3 \text{ and}\\ \quad\times\langle\widetilde{R}(a_2,\ldots,a_{m-1})\rangle \displaystyle\sum_{k=1}^{|a_m|}\left(-A^{4\mathrm{sign}(a_m)}\right)^{k} & \quad m \text{ is an odd number}. \end{cases}$$

Given a progressive expression of a normal diagram $\widetilde{L}$ of a closed 3-braid link, **Procedure bracket_3–braid** computes the Kauffman bracket polynomial of $\widetilde{L}$ by using the recurrence formulas in **Lemma 3.5** and **Lemma 4.4**. While the algorithm is running, every Kauffman bracket polynomial is computed once at most.

Procedure bracket_3–braid

INPUT: A progressive expression $(a_1, \ldots, a_m)$ of a normal diagram $\widetilde{L}$ of a closed 3–braid link.
OUTPUT: The Kauffman bracket polynomial $\langle \widetilde{B}(a_1, \ldots, a_m)\rangle$ of $\widetilde{L}$.
Compute $\langle \widetilde{R}(a_1)\rangle$, $\langle \widetilde{B}(a_1)\rangle$, $\langle \widetilde{B}(a_1, a_2)\rangle$, $\langle \widetilde{R}(a_1, a_2)\rangle$, $\langle \widetilde{R}(a_2)\rangle$, $\langle \widetilde{B}(a_1, a_2, a_3)\rangle$, $\langle \widetilde{R}(a_1, a_2, a_3)\rangle$ and $\langle \widetilde{R}(a_2, a_3)\rangle$.
Initialize i as "4".
while $i \leq m$ **do**
$$T \longleftarrow \sum_{k=1}^{|a_i|} \left(-A^{4\mathrm{sign}(a_i)}\right)^k.$$
if i is an even number
Compute $\langle \widetilde{B}(a_1, \ldots, a_i)\rangle$ from $\langle \widetilde{B}(a_1, \ldots, a_{i-1})\rangle$, $\langle \widetilde{R}(a_1, \ldots, a_{i-1})\rangle$ and T,
else
Compute $\langle \widetilde{B}(a_1, \ldots, a_i)\rangle$ from $\langle \widetilde{B}(a_1, \ldots, a_{i-1})\rangle$, $\langle \widetilde{R}(a_2, \ldots, a_{i-1})\rangle$ and T.
Compute $\langle \widetilde{R}(a_1, \ldots, a_i)\rangle$ from $\langle \widetilde{R}(a_1, \ldots, a_{i-2})\rangle$, $\langle \widetilde{R}(a_1, \ldots, a_{i-1})\rangle$ and T.
Compute $\langle \widetilde{R}(a_2, \ldots, a_i)\rangle$ form $\langle \widetilde{R}(a_2, \ldots, a_{i-2})\rangle$, $\langle \widetilde{R}(a_2, \ldots, a_{i-1})\rangle$ and T.
Increment i.
od

Lemma 4.5 *Given a progressive expression of a normal diagram $\widetilde{L}$ of a closed 3-braid link,* **Procedure bracket_3–braid** *computes the Kauffman bracket polynomial of $\widetilde{L}$ with $\mathcal{O}(c(\widetilde{L}))$ operations of polynomials of degree $\mathcal{O}(c(\widetilde{L}))$.*

Theorem 4.6 *The Jones polynomial of a closed 3-bridge link is computed from the Tait graph of a normal diagram $\widetilde{L}$ of the closed 3-braid link with $\mathcal{O}(c(\widetilde{L}))$ operations of polynomials of degree $\mathcal{O}(c(\widetilde{L}))$.*

5 Algorithms for Montesinos links

Definition 5.1 A link that has a link diagram as in Figure 8 is a *Montesinos link*. The link diagram is called a *normal diagram* of the Montesinos link and is denoted by $\widetilde{M}(a_{11}, \ldots, a_{1m_1}|\cdots|a_{l1}, \ldots, a_{lm_l})$. The progression $(a_{11}, \ldots, a_{1m_1}|\cdots|a_{l1}, \ldots, a_{lm_l})$ is called a *progressive expression* of the normal diagram. A link diagram as in Figure 9 is denoted by $\widetilde{N}(a_{11}, \ldots, a_{1m_1}|\cdots|a_{l1}, \ldots, a_{lm_l})$.

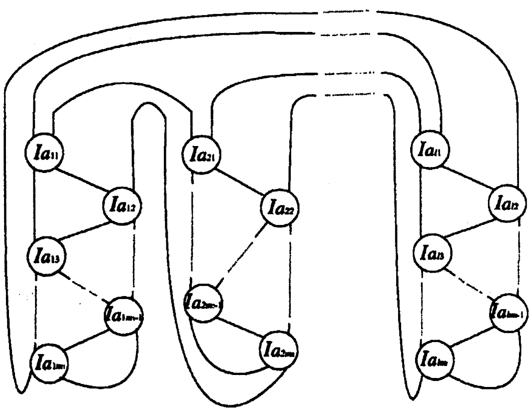


Figure 8: A normal diagram of a Montesinos link

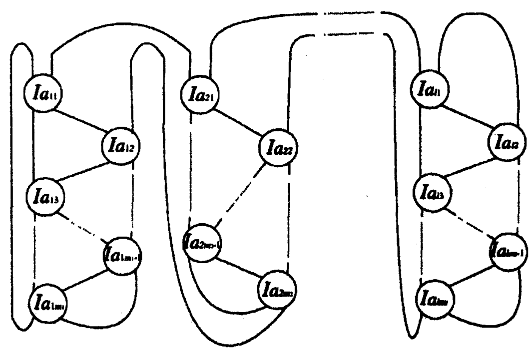


Figure 9: $\widetilde{N}(a_{11}, \ldots, a_{1m_1}|\cdots|a_{l1}, \ldots, a_{lm_l})$

Lemma 5.2 *For any normal diagram $\widetilde{M}(a_{11}, \ldots, a_{1m_1}|\cdots|a_{l1}, \ldots, a_{lm_l})$ of a Montesinos link, the following recurrence formula holds.*

$$
\langle \widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l})\rangle
= \begin{cases}
(-A^{-3})^{a_{11}} & \text{if } l=1 \text{ and } m_1=1,\\[2ex]
(-A^{-3})^{a_{11}}\langle \widetilde{R}(a_{12},\ldots,a_{1m_1})\rangle & \text{if } l=1 \text{ and } m_1\geq 2,\\[2ex]
A^{a_{l1}}\langle \widetilde{N}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l-11},\ldots,a_{l-1m_{l-1}})\rangle \\
\quad +(-A)^{-3a_{l1}-2\mathrm{sign}(a_{l1})}\langle \widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l-11},\ldots,a_{l-1m_{l-1}})\rangle \\
\quad\quad \times \sum_{k=1}^{|a_{l1}|}\left(-A^{4\mathrm{sign}(a_{l1})}\right)^k & \text{if } l\geq 2 \text{ and } m_l=1,\\[2ex]
A^{a_{l2}}(-A^{-3})^{a_{l1}}\langle \widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l-11},\ldots,a_{l-1m_{l-1}})\rangle \\
\quad +(-A)^{-3a_{l2}-2\mathrm{sign}(a_{l2})}\langle \widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l-11},\ldots,a_{l-1m_{l-1}}|a_{l1})\rangle \\
\quad\quad \times \sum_{k=1}^{|a_{l2}|}\left(-A^{4\mathrm{sign}(a_{l2})}\right)^k & \text{if } l\geq 2 \text{ and } m_l=2,\\[2ex]
A^{a_{lm_l}}(-A^{-3})^{a_{lm_l-1}}\langle \widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l-2})\rangle \\
\quad +(-A)^{-3a_{lm_l}-2\mathrm{sign}(a_{lm_l})}\langle \widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l-1})\rangle \\
\quad\quad \times \sum_{k=1}^{|a_{lm_l}|}\left(-A^{4\mathrm{sign}(a_{lm_l})}\right)^k & \text{if } l\geq 2 \text{ and } m_l\geq 3.
\end{cases}
$$

For any $\widetilde{N}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l})$, we also get a recurrence formula.

Theorem 5.3 *The Jones polynomial of a Montesinos link is computed from a progressive expression of a normal diagram $\widetilde{L}$ of the Montesinos link with $O(c(\widetilde{L}))$ operations of polynomials of degree $O(c(\widetilde{L}))$.*

References

[1] G. Burde, H. Zieschang. *Knots*, Walter de Gruyter, 1985.

[2] M. Hara, S. Tani, M. Yamamoto. A polynomial-time algorithm for computing the Jones polynomials of arborescent links (in Japanese). Information Technology Letters Vol. 1, pp.16-17 (2002).

[3] F. Jaeger, D. L. Vertigan, and D. J. A. Welsh. On the computational complexity of the Jones and Tutte polynomials. *Math. Proc. Camb. Phil. Soc.*, 108:35–53, 1990.

[4] V. F. R. Jones. A polynomial invariant for knots via Von Neumann algebras. *Bull. Amer. Math. Soc.*, 12:103–111, 1985.

[5] L. H. Kauffman. State models and the Jones polynomial. *Topology*, 26:395–407, 1987.

[6] J. A. Makowsky. Colored Tutte polynomials and Kauffman brackets for graphs of bounded tree width. Preprint.

[7] J. Mighton. *Knot Theory on Bipartite Graphs.* PhD thesis, Dept. of Math., University of Toronto, Canada, 1999.

[8] D. Rolfsen. *Knots and Links*, Publish or Perish, Inc., 1976.

[9] H. Schubert. Über eine Numerische Knoteninvariante. Math. Z., 61:245–288, 1954.

[10] K. Sekine, H. Imai, K. Imai. Computation of Jones Polynomial (in Japanese). Transaction of the Japan Society for Industrial and Applied Mathematics, 8 No. 3 (1998) 341–354.

[11] T. Utsumi, K. Imai. Computation of the Jones Polynomials for Pretzel Links (in Japanese). IPSJ SIG Technical Reports No.085 (2002).

[12] D. J. A. Welsh. *Complexity: Knots, Colorings and Counting*, Cambridge Univ. Press, 1993.