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On the field of definition for modularity of CM elliptic curves

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1 Introduction

Let E be a CM elliptic curve defined over an algebraic number field $F \subseteq \mathbb{C}$ whose $\mathbb{Q}$-algebra of endomorphisms defined over $\overline{\mathbb{Q}}$, denoted by $\mathrm{End}^0_{\overline{\mathbb{Q}}}(E)$, is isomorphic to an imaginary quadratic field $K \subseteq \mathbb{C}$. We take an integral ideal $\mathfrak{m}$ in K and denote by $I_K(\mathfrak{m})$ the group of fractional ideals in K prime to $\mathfrak{m}$. We consider a homomorphism $\lambda : I_K(\mathfrak{m}) \to \mathbb{C}^\times$ such that (i) $\lambda((\alpha)) = \alpha$ for any $\alpha \in K^\times$ s.t. $\alpha \equiv 1 \bmod^\times \mathfrak{m}$; (ii) λ is primitive, i.e. there is no proper divisor $\mathfrak{n}$ of $\mathfrak{m}$ such that λ has a extension $\widetilde{\lambda}$ to $I_K(\mathfrak{n})$ with the property: $\widetilde{\lambda}((\alpha)) = \alpha$ for any $\alpha \in K^\times$ s.t. $\alpha \equiv 1 \bmod^\times \mathfrak{n}$. Then we put

$$f_\lambda(z) := \sum_{\substack{\mathfrak{a} \in I_K(\mathfrak{m}) \\ \mathfrak{a}:\ \text{integral}}} \lambda(\mathfrak{a}) e^{2\pi i N(\mathfrak{a}) z} \quad (z \in \mathfrak{H},\ \text{the complex upper plane}),$$

where $N(\mathfrak{a})$ denotes the absolute norm of an ideal $\mathfrak{a}$. Let $-D$ be the discriminant of K and put $N := DN(\mathfrak{m})$. We define a Dirichlet character $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ by

$$\overline{a} \longmapsto \left(\frac{-D}{a}\right) \frac{\lambda((a))}{a} \quad (a \in \mathbb{Z},\ \ (a,\ N) = 1),$$

where if $a = p_1^{e_1} \cdots p_r^{e_r}$ is the factorization of a into prime factors,

$$\left(\frac{-D}{a}\right) = \prod_{i=1}^{r} \left(\frac{-D}{p_i}\right)^{e_i}, \quad \left(\frac{-D}{p_i}\right) = \begin{cases} 1 & \text{if } p_i \text{ splits in } K/\mathbb{Q} \\ -1 & \text{if } p_i \text{ is inert in } K/\mathbb{Q}. \end{cases}$$

By Hecke-Shimura, we have the following:

Fact 1. *f_λ is a normalized newform of weight two on $\Gamma_1(N)$ and ε is the Nebentypus of f_λ.*

By the Eichler-Shimura theory, for any normalized newform f of weight two on $\Gamma_1(M)$, we can associate the abelian variety J_f defined over $\mathbb{Q}$ which is a $\mathbb{Q}$-simple factor of $J_1(M)$, the jacobian variety of the modular curve $X_1(M)$. Shimura proved the following (see Proposition 1.6 and Remark 1.7 in [5]):

Fact 2. $\mathrm{Hom}_{\overline{\mathbb{Q}}}(E, J_f) \neq \{0\}$ *if and only if there exists an above* λ *such that* $f = f_\lambda$, *where* $\mathrm{Hom}_{\overline{\mathbb{Q}}}(E, J_f)$ *denotes the additive group of homomorphisms from* E *to* J_f *definrd over* $\overline{\mathbb{Q}}$.

For any imaginary quadratic field K, if we take an integral ideal $\mathfrak{m}_0$ in K such that

$$\zeta \in K, \ \ \zeta \text{ is a root of unity}, \ \ \zeta \equiv 1 \bmod^{\times}\mathfrak{m}_0 \Longrightarrow \zeta = 1$$

holds (we can always do so), there exists a homomorphism $\lambda : I_K(\mathfrak{m}_0) \to \mathbb{C}^\times$ satisfying the condition (i). Replacing $\mathfrak{m}_0$ by the minimal divisor $\mathfrak{m}$ of $\mathfrak{m}_0$ such that λ has an extension $\widetilde{\lambda}$ to $I_K(\mathfrak{m})$ and $\widetilde{\lambda}$ has also the proprety (i), we may assume that λ is primitive. Therefore we have

Fact 3. *For any CM elliptic curve* E *defined over an algebraic number field* F, *there exists a newform* f *such that a non-zero homomorphism* $\varphi : E \to J_f$ *defined over* $\overline{\mathbb{Q}}$ *exists, that is,* E *is modular over* $\overline{\mathbb{Q}}$.

In this paper we will consider the following questions.

Question 1. *Let* E/F *be as above. Under what condition does there exist a newform* f *such that a non-zero homomorphism* $\varphi : E \to J_f$ *defined over* F *exists, that is, when is* E *modular over* F*?*

Question 2. *Assume that* E/F *is modular over* F. *Therefore there exists a newform* f *with* $\mathrm{Hom}_F(E, J_f) \neq \{0\}$. *Then, how large is* $\mathrm{Hom}_F(E, J_f)$*? In other words, decide the multiplicity of* E *as* F*-simple factor of* J_f.

2 Preliminaries

Let E/F, K, $\lambda : I_K(\mathfrak{m}) \to \mathbb{C}^\times$, and $f = f_\lambda$ be as in the introduction. Let $f = \sum_{m\geq 1} a_m q^m$ ($q = e^{2\pi i z}$) be the Fourier expansion at $i\infty$ and put $H := \mathbb{Q}(a_m | m \geq 1)$ ($\subseteq \mathbb{C}$). Let n be the dimension of J_f, then H is an algebraic number field with $[H : \mathbb{Q}] = n$. A $\mathbb{Q}$-algebra isomorphism $\theta : H \xrightarrow{\sim} \mathrm{End}^0_{\mathbb{Q}}(J_f) = \mathrm{End}_{\mathbb{Q}}(J_f) \otimes_{\mathbb{Z}} \mathbb{Q}$ is defined by

$$a_m \longmapsto \text{the endomorphism of } J_f \text{ induced by the } m\text{-th Hecke operator w.r.t. } \Gamma_1(N)$$

($m = 1, 2, \ldots$). In [3] Shimura proved that J_f is isogenous to $E^n = E \times \cdots \times E$ (n terms) over $\overline{\mathbb{Q}}$, expressed by $J_f \sim_{\overline{\mathbb{Q}}} E^n$. So we have $\mathrm{End}^0_{\overline{\mathbb{Q}}}(J_f) \cong M_n(K)$, the algebra of $n \times n$-matrices with entries in K. Let Z be the center of $\mathrm{End}^0_{\overline{\mathbb{Q}}}(J_f)$. Then we have $Z \cong K$. We denote by T the sub $\mathbb{Q}$-algebra of $\mathrm{End}^0_{\overline{\mathbb{Q}}}(J_f)$ generated by Z and $\theta(H)$. Shimura uesd the following facts in the proof of Proposition 1.6 in [5] and we state them as a lemma without proof.

Lemma 2.1. (1) $Z \cap \theta(H) = \mathbb{Q}$. *Especially this implies that* $\dim_H T = 2$.
(2) $\mathrm{End}^0_K(J_f) = T$.

Therefore, as for the structure of T, we have the possibility of the following two cases:

Case 1 : T is isomorphic to an algebraic number field with degree $2n$ (over $\mathbb{Q}$) $\iff K \not\subseteq H$;
Case 2 : $T \cong H \oplus H \iff K \subseteq H$.

Let $F' = \langle F, K \rangle$ be the subfield of $\mathbb{C}$ generated by F and K. It is well known that $\mathrm{End}^0_{\overline{\mathbb{Q}}}(E) = \mathrm{End}^0_{F'}(E)\,(\cong K)$. We put $\mathcal{M} := \mathrm{Hom}_{\overline{\mathbb{Q}}}(E, J_f) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ over F acts on $\mathcal{M}$ by the action on coefficients of homomorphisms. If we know the structure of $\mathcal{M}$ as Galois module, we will be able to answer Questions 1 and 2. Therefore our purpose in this paper is to determine the structure of $\mathcal{M}$ as $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-module. On the other hand we have the following.

Lemma 2.2. $\mathrm{Hom}_{F'}(E, J_f) \neq \{0\} \iff \mathrm{Hom}_F(E, J_f) \neq \{0\}$.

By this lemma, for answer to Question 1, it is enough to study the structure of $\mathcal{M}$ as $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$-module. But, for answer to Question 2, this does not seem to be enough. Nevertheless, as we will see later, under assumption $\mathrm{Hom}_{F'}(E, J_f) \neq \{0\}$ the structure of $\mathcal{M}$ as $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-module can be easily recovered from that of $\mathcal{M}$ as $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$-module. Therefore, in the following we will study the $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$-module structure.

By composition of homomorphisms, $\mathcal{M}$ has the structure of left T- and right K-module:

$$T = \mathrm{End}^0_K(J_f) \curvearrowright \mathcal{M} \curvearrowleft \mathrm{End}^0_{F'}(E) \cong K.$$

As $J_f \sim_{\overline{\mathbb{Q}}} E^n$, we have

$$\mathcal{M} \cong \mathrm{Hom}_{\overline{\mathbb{Q}}}(E, E^n) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K^{\oplus n}$$

as $\mathbb{Q}$-vector space. In particular we have $\dim_{\mathbb{Q}} \mathcal{M} = n \times \dim_{\mathbb{Q}} K = 2n$. On the other hand $H \xrightarrow{\sim} \theta(H) \subseteq T$, we can view $\mathcal{M}$ as H-vector space. Since $[H : \mathbb{Q}] \times \dim_H \mathcal{M} = \dim_{\mathbb{Q}} \mathcal{M} = 2n$, we have $\dim_H \mathcal{M} = 2$.

Proposition 2.3. *$\mathcal{M}$ is a free left T-module of rank* 1.

Let ℓ be a prime number and put

$$V_\ell(E) := T_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, \quad V_\ell(J_f) := T_\ell(J_f) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, \quad \mathcal{M}_\ell := \mathcal{M} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell,$$

where $T_\ell(E)$ and $T_\ell(J_f)$ are Tate modules. We can consider the following actions:

- $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \curvearrowright \mathcal{M}_\ell \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell} V_\ell(E)$ by diagonal;
- $H \hookrightarrow T \curvearrowright \mathcal{M}_\ell \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell} V_\ell(E)$ by the action on $\mathcal{M}$.

We define a homomorphism $\nu : \mathcal{M}_\ell \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell} V_\ell(E) \longrightarrow V_\ell(J_f)$ by

$$(\varphi \otimes a) \otimes x \longmapsto a\varphi(x).$$

Proposition 2.4. *ν is an isomorphism of (left) $H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell[\mathrm{Gal}(\overline{\mathbb{Q}}/F)]$-modules and is also an isomorphism of (left) $T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell[\mathrm{Gal}(\overline{\mathbb{Q}}/F')]$-modules, where $H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell[\mathrm{Gal}(\overline{\mathbb{Q}}/F)]$ (resp. $T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell[\mathrm{Gal}(\overline{\mathbb{Q}}/F')]$) denotes the group algebra of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ (resp. $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$) over $H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ (resp. $T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$).*

3 The action of $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$ on $\mathcal{M}_\ell \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell} V_\ell(E)$

We review the known results about the structure of $V_\ell(E)$ as $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$-module. By changing $\iota : K \xrightarrow{\sim} \mathrm{End}^0_{F'}(E)$ if necessary, we may assume that the CM-type of (E, ι) is $(K; \{id\})$. Then there exists a lattice $\mathfrak{a}$ of K such that the following commutative diagram holds:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{a} & \longrightarrow & K_{\mathbb{R}} & \longrightarrow & K_{\mathbb{R}}/\mathfrak{a} & \longrightarrow & 0 \quad \text{(exact)} \\
 & & \downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} & & \\
0 & \longrightarrow & q(\mathfrak{a}) & \longrightarrow & \mathbb{C} & \underset{\xi}{\longrightarrow} & E(\mathbb{C}) & \longrightarrow & 0 \quad \text{(exact),}
\end{array}$$

where $K_{\mathbb{R}} := K \otimes_{\mathbb{Q}} \mathbb{R}$ and $q(a \otimes x) = ax$. By the theory of complex multiplication, the following is well known (see Theorem 19.8, p. 134 in [6]).

Theorem 3.1. (1) *Every point of $E(\mathbb{C})$ with finite order is F'_{ab}-rational, where F'_{ab} denotes the maximal abelian extension of F'.*
(2) *There exists a unique homomorphism $\alpha_{E/F'} : F'^{\times}_{\mathbb{A}} \longrightarrow K^\times$ (where $F'^{\times}_{\mathbb{A}}$ denotes the idele group of F') such that*

- $\mathrm{Ker}(\alpha_{E/F'})$ *is open in $F'^{\times}_{\mathbb{A}}$;*
- *For any $x \in F'^{\times}_{\mathbb{A}}$, $\alpha_{E/F'}(x) N_{F'/K}(x)^{-1}\mathfrak{a} = \mathfrak{a}$, where $N_{F'/K}$ is the norm map from $F'^{\times}_{\mathbb{A}}$ to $K^\times_{\mathbb{A}}$;*

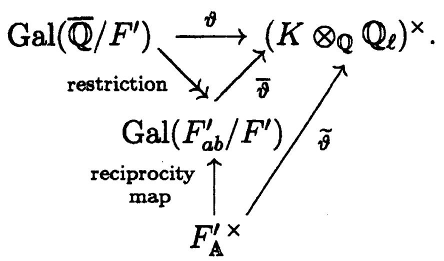
- *For any $x \in F_{\mathbb{A}}'^{\times}$, $\alpha_{E/F'}(x)\rho(\alpha_{E/F'}(x)) = N(il(x))$, where $\rho(v)$ is the complex conjugate of a complex number v and $il(x)$ is the fractional ideal of F' associated to an idele element x;*

- *For any $x \in F_{\mathbb{A}}'^{\times}$ and $w \in K/\mathfrak{a}$, ${}^{[x,\,F']}r(w) = r(\alpha_{E/F'}(x)N_{F'/K}(x)^{-1}w)$, where $[x,\,F']$ is the element of $\mathrm{Gal}(F'_{ab}/F')$ corresponding to x by the reciprocity law of class field theory.*

Since $V_\ell(E)$ is viewed as free $K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-module of rank 1 by ι, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$ on $V_\ell(E)$ determines the homomorphism

$$\vartheta : \mathrm{Gal}(\overline{\mathbb{Q}}/F') \longrightarrow (K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^{\times}.$$

Then ϑ factors through the restriction map to F'_{ab}. So we denote by $\overline{\vartheta}$ the induced map from $\mathrm{Gal}(F'_{ab}/F')$ to $(K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^{\times}$ and by $\widetilde{\vartheta}$ the composition of the reciprocity map for F' and $\overline{\vartheta}$. Thus we have the following commutative diagram:



Then Theorem 3.1 implies the following:

Corollary 3.2. *For any $x \in F_{\mathbb{A}}'^{\times}$, $\widetilde{\vartheta}(x) = (\alpha_{E/F'}(x)N_{F'/K}(x)^{-1})_\ell$, where $(\ )_\ell$ denotes the ℓ-component.*

By Proposition 2.3, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$ on $\mathcal{M}$ determines the homomorphism

$$\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/F') \longrightarrow T^{\times}.$$

Let χ_ℓ be the composition of χ and the canonical map $T^{\times} \longrightarrow (T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^{\times}$, then χ_ℓ corresponds to the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$ on $\mathcal{M}_\ell$. In other words, taking a basis η of $\mathcal{M}$ over T, we have ${}^{\sigma}\eta = \chi(\sigma) \circ \eta$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F')$.

Firstly we consider Case 1. Since K acts T-linearly on $\mathcal{M}$, we can take a $\mathbb{Q}$-algebra isomorphism $\kappa : K \xrightarrow{\sim} Z \subseteq T$ such that $\eta \circ \iota(a) = \kappa(a) \circ \eta$ for any $a \in K$, denoted by $\eta a = a\eta$ for short. We take a basis v of $V_\ell(E)$ over $K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Then $\omega := \eta \otimes v$ becomes a free basis of $\mathcal{M}_\ell \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell} V_\ell(E)$ over $T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ and it holds that

$$\begin{aligned} {}^{\sigma}\omega &= {}^{\sigma}\eta \otimes {}^{\sigma}v = (\chi_\ell(\sigma) \circ \eta) \otimes (\vartheta(\sigma)v) = (\chi_\ell(\sigma) \circ \eta \circ (\iota \otimes 1)(\vartheta(\sigma))) \otimes v \\ &= (\chi_\ell(\sigma)\vartheta(\sigma)\,\eta) \otimes v = \chi_\ell(\sigma)\vartheta(\sigma)\,\omega \end{aligned}$$

for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F')$.

Next we consider Case 2. $\sqrt{-D}$ ($\in K$) acts T-linearly on $\mathcal{M}$, so there exists some $t \in T$ such that $\eta' \circ \iota(\sqrt{-D}) = t \circ \eta'$ for any $\eta' \in \mathcal{M}$. We will show that $t \in Z$ (one should note that in Case 2, T has two $\mathbb{Q}$-subalgebras isomorphic to K, so it is not trivial that $t \in Z$). For any $\varphi \in \mathrm{End}^0_{\overline{\mathbb{Q}}}(J_f)$ and $\eta' \in \mathcal{M}$, we have

$$(\varphi \circ t) \circ \eta' = \varphi \circ (t \circ \eta') = \varphi \circ (\,\eta' \circ \iota(\sqrt{-D})\,) = (\varphi \circ \eta') \circ \iota(\sqrt{-D}) = t \circ (\varphi \circ \eta') = (t \circ \varphi) \circ \eta',$$

therefore $t \circ \varphi = \varphi \circ t$ in $\mathrm{End}^0_{\overline{\mathbb{Q}}}(J_f)$, hence $t \in Z$. This concludes that similarly with Case 1, there exists a $\mathbb{Q}$-algebra isomorphism $\kappa : K \xrightarrow{\sim} Z \subseteq T$ with the same property. Let $\gamma_1 : K \hookrightarrow H$ be the map induced by the inclusion $K \subseteq H$ and $\gamma_2 : K \hookrightarrow H$ be the other homomorphism. We define an isomorphism of $\mathbb{Q}$-algebras $\varepsilon : T \xrightarrow{\sim} H \oplus H$ by

$$z \; (\in Z) \longmapsto (\,\gamma_1(\,\kappa^{-1}(z)\,),\; \gamma_2(\,\kappa^{-1}(z)\,)\,), \qquad \theta(a) \; (\in \theta(H)\,) \longmapsto (a,\; a).$$

For $k = 1,\ 2$, we set

$$\chi_\ell^{(k)} : \mathrm{Gal}(\overline{\mathbb{Q}}/F') \xrightarrow[\chi_\ell]{} (T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times \xrightarrow[\varepsilon \otimes 1]{\sim} (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times \oplus (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times \xrightarrow[k\text{-th component}]{\text{projection to}} (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times.$$

These arguments imply the following:

Proposition 3.3. *Let the notations be as above. We regard $K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \subseteq T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ by injection $\kappa \otimes 1$.*
(1) *In Case 1, it holds that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F')$,*

$${}^\sigma\omega = \chi_\ell(\sigma)\vartheta(\sigma)\,\omega.$$

(2) *In Case 2, identifying $T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ with $(H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^{\oplus 2}$ by $\varepsilon \otimes 1$, it holds that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F')$,*

$${}^\sigma\omega = (\,\chi_\ell^{(1)}(\sigma)\gamma_1(\,\vartheta(\sigma)\,),\; \chi_\ell^{(2)}(\sigma)\gamma_2(\,\vartheta(\sigma)\,)\,)\,\omega,$$

where we denote $\gamma_k \otimes 1 : K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \hookrightarrow H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ by γ_k ($k = 1,\ 2$) for simplicity.

4 On relation between Eichler-Shimura theory and complex multiplication theory about J_f

In this section we will describe a relation between λ in $f = f_\lambda$ and the homomorphism corresponding to $\alpha_{E/F'}$ in higher dimensional case. The content of this section is essentially stated in the proof of Proposition 1.6 in [5] without detailed proof. We present the results in a slightly different form to be convenient to our purpose.

Firstly we consider Case 1. Then $L := \langle K, H\rangle$ ($\subseteq \mathbb{C}$) is a CM-field with $[L : \mathbb{Q}] = 2n$. We define an isomorphism of $\mathbb{Q}$-algebras $\iota' : L \xrightarrow{\sim} T = \mathrm{End}^0_K(J_f)$ by

$$a\ (\in K) \longmapsto \kappa(a)\ (\in Z), \qquad x\ (\in H) \longmapsto \theta(x).$$

Then $(J_f,\ \iota')$ is an abelian variety with complex multiplication defined over K in the sense of Shimura (see §19.7 in [6]). Since $\theta(H) \subseteq \mathrm{End}^0_{\mathbb{Q}}(J_f)$, the characteristic polynomial of any element of H acting on $H^0(J_f,\ \Omega^1_{/\mathbb{C}}) = H^0(J_f,\ \Omega^1_{/\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C}$ has $\mathbb{Q}$-rational coefficients. Therefore, by Lemma 1 in [7] (p. 38), the representation of H on $H^0(J_f,\ \Omega^1_{/\mathbb{C}})$ is equivalent to the regular representation of H over $\mathbb{Q}$. It is also proved that Z acts on $H^0(J_f,\ \Omega^1_{/\mathbb{C}})$ by scalar multiple. Let $(L,\ \{\varpi_1,\ \ldots,\ \varpi_n\})$ be the CM-type of $(J_f,\ \iota')$, then we have

$$\{\varpi_{1|H},\ \ldots,\ \varpi_{n|H}\} = \{\varpi \mid \varpi : H \hookrightarrow \mathbb{C}\}, \qquad \varpi_{i|K} = id_K \quad (i = 1,\ \ldots,\ n)$$

by changing the identification of K as subfield of $\mathbb{C}$ if necessary. Hence the reflex of $(L,\ \{\varpi_1,\ \ldots,\ \varpi_n\})$ is $(K,\ \{id_K\})$. Let $g' : K^\times_{\mathbb{A}} \longrightarrow L^\times_{\mathbb{A}}$ be the canonical map induced from the inclusion $K \subseteq L$. Similarly with case of E/F', the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ on $V_\ell(J_f)$ determines the homomorphism

$$\delta : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \longrightarrow (L \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times$$

and we define $\widetilde{\delta} : K^\times_{\mathbb{A}} \longrightarrow (L \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times$ by the same manner as defining $\widetilde{\vartheta}$. The theory of complex multiplication also implies the following:

Corollary 4.1. *For any $x \in K^\times_{\mathbb{A}}$, $\widetilde{\delta}(x) = (\alpha_{J_f/K}(x) g'(x)^{-1})_\ell$, where $\alpha_{J_f/K} : K^\times_{\mathbb{A}} \longrightarrow L^\times$ is the homomorphism corresponding to $\alpha_{E/F'}$ in higher dimensional case.*

Let $\{\mathfrak{p}_1,\ \ldots,\ \mathfrak{p}_s\}$ be the set of all bad primes of J_f/K. For every $\mathfrak{p}_k$ $(1 \leq k \leq s)$, we take the least positive integer t_k such that

$$x \in K^\times_{\mathfrak{p}_k} \subseteq K^\times_{\mathbb{A}}, \quad x - 1 \in \mathfrak{p}_k^{t_k} \Longrightarrow \alpha_{J_f/K}(x) = 1.$$

We set $\mathfrak{n} := \mathfrak{p}_1^{t_1} \cdots \mathfrak{p}_s^{t_s}$, $G_K(\mathfrak{n}) := \{x \in K^\times_{\mathbb{A}} \mid x_\infty = 1,\ x_{\mathfrak{p}_k} = 1\ \ (1 \leq k \leq s)\}$, $U_K := \{x \in K^\times_{\mathbb{A}} \mid x_{\mathfrak{p}} \in \mathcal{O}^\times_{K_{\mathfrak{p}}}$ for any finite prime $\mathfrak{p}\}$, and $U_K(\mathfrak{n}) := G_K(\mathfrak{n}) \cap U_K$. We consider the canonical isomorphism $G_K(\mathfrak{n})/U_K(\mathfrak{n}) \xrightarrow{\sim} I_K(\mathfrak{n})$ by which the class represented by $x \in G_K(\mathfrak{n})$ is sent to $il(x) \in I_K(\mathfrak{n})$. Since $U_K(\mathfrak{n}) \subseteq \mathrm{Ker}(\alpha_{J_f/K})$, we obtain the homomorphism

$$\widetilde{\alpha_{J_f/K}} : I_K(\mathfrak{n}) \longrightarrow L^\times$$

induced from $\alpha_{J_f/K}$. By the two properties of $\alpha_{J_f/K}$: (i) $x \in K^\times_\infty = \mathbb{C}^\times \subseteq K^\times_{\mathbb{A}} \Longrightarrow \alpha_{J_f/K}(x) = 1$; (ii) $x \in K^\times \subseteq K^\times_{\mathbb{A}} \Longrightarrow \alpha_{J_f/K}(x) = g'(x) = x$, it holds that

$$\alpha \in K^\times, \quad \alpha \equiv 1 \bmod^\times \mathfrak{n} \Longrightarrow \widetilde{\alpha_{J_f/K}}(\,(\alpha)\,) = \alpha.$$

It is clear that $\widetilde{\alpha_{J_f/K}} : I_K(\mathfrak{n}) \longrightarrow L^\times \subseteq \mathbb{C}^\times$ is primitive.

Proposition 4.2. *In Case 1, we have* $\lambda = \widetilde{\alpha_{J_f/K}}$ *and* $\mathfrak{m} = \mathfrak{n}$.

Next we investigate Case 2. Since J_f is defined over $\mathbb{Q}$, $\rho_{|K}$ ($\in \mathrm{Gal}(K/\mathbb{Q})$) acts on $T = \mathrm{End}^0_K(J_f)$. Identifying T with $H \oplus H$ by ε, this action corresponds to the automorphism of $H \oplus H$ defined by $(x,\ y) \longmapsto (y,\ x)$. Let ξ_1, ξ_2 be the elements of T which correspond to $(1,\ 0)$, $(0,\ 1)$ respectively. We take a positive integer r such that $r\xi_k \in \mathrm{End}_K(J_f)$ $(k = 1,\ 2)$ and set $\xi'_k := r\xi_k$. Then $C := \mathrm{Im}(\xi'_1)$ is an abelian subvariety of J_f defined over K. Since ${}^\rho\xi'_1 = \xi'_2$, we have ${}^\rho C = \mathrm{Im}(\xi'_2)$. So we can define an isogeny $\varphi : J_f \longrightarrow C \times {}^\rho C$ defined over K by $x \longmapsto (\xi'_1(x),\ \xi'_2(x))$ and this implies $J_f \sim_K C \times {}^\rho C$.

Lemma 4.3. *We have* $J_f \sim_{\mathbb{Q}} R_{K/\mathbb{Q}}(C) \sim_{\mathbb{Q}} R_{K/\mathbb{Q}}({}^\rho C)$*, where* $R_{K/\mathbb{Q}}(C)$ *denotes the Weil restriction from* K *to* $\mathbb{Q}$ *of* C.

To understand the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $V_\ell(J_f)$, it is sufficient to do so for that of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ on $V_\ell(C)$ by this lemma. Putting $R := \theta^{-1}(\mathrm{End}_{\mathbb{Q}}(J_f))$, we define a ring homomorphism $\iota'' : R \longrightarrow \mathrm{End}_K(C)$ by

$$a \longmapsto (C \ni x \mapsto (\theta(a))(x) \in C)$$

and denote $\iota''\otimes 1 : H = R\otimes_{\mathbb{Z}}\mathbb{Q} \longrightarrow \mathrm{End}^0_K(C)$ by the same notation ι''. In Case 2, $K \subseteq H$, so H is a CM-field. Then $(C,\ \iota'')$ is an abelian variety with complex multiplication defined over K. Let H_0 be the maximal real subfield of H and $(H,\ \{\tau_1,\ \ldots,\ \tau_{n_1}\})$ $(n_1 := \frac{n}{2})$ be the CM-type of $(C,\ \iota'')$. Since $H_0 \subseteq \mathrm{End}^0_K(C)$, the characteristic polynomial of any element of H_0 acting on $H^0(C,\ \Omega^1_{/\mathbb{C}})$ has K-rational coefficients. Since H_0 is totally real, its coefficients also lie in $\mathbb{R}$. So it has $\mathbb{Q}$-rational coefficients. It is also proved that $K \subseteq H$ acts on $H^0(C,\ \Omega^1_{/\mathbb{C}})$ by scalar mutiple because $\iota''(K)$ coincides with the center of $\mathrm{End}^0_{\overline{\mathbb{Q}}}(C) \cong M_{n_1}(K)$. Therefore we have

$$\{\tau_{1|H_0},\ \ldots,\ \tau_{n_1|H_0}\} = \{\tau \mid \tau : H_0 \hookrightarrow \mathbb{R}\}, \qquad \tau_{i|K} = id_K \quad (i = 1,\ \ldots,\ n_1)$$

by changing the identification of K as subfield of $\mathbb{C}$ if necessary. Hence the reflex of $(H,\ \{\tau_1,\ \ldots,\ \tau_{n_1}\})$ is $(K,\ \{id_K\})$. Let $g'' : K^\times_{\mathbb{A}} \longrightarrow H^\times_{\mathbb{A}}$ be the canonical map induced from $\gamma_1 : K \hookrightarrow H$. Similary with Case 1, we have

$$\delta' : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \longrightarrow (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times, \qquad \widetilde{\delta'} : K^\times_{\mathbb{A}} \longrightarrow (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times,$$

and the following:

Corollary 4.4. *For any* $x \in K^\times_{\mathbb{A}}$, $\widetilde{\delta'}(x) = (\alpha_{C/K}(x)g''(x)^{-1})_\ell$.

Let $\mathfrak{n}'$ be the one corresponding to $\mathfrak{n}$ in case of C/K. Then , as Case 1, we can define

$$\widetilde{\alpha_{C/K}} : I_K(\mathfrak{n}') \longrightarrow H^\times.$$

Proposition 4.5. *In Case 2, we have* $\lambda = \widetilde{\alpha_{C/K}}$ *and* $\mathfrak{m} = \mathfrak{n}'$.

5 Main results

Let $\beta_{E/F'} : F'^\times_{\mathbb{A}} \longrightarrow \mathbb{C}^\times$ be the Grössen-character of E/F'. (By definition, $\beta_{E/F'}(x) = (\alpha_{E/F'}(x) N_{F'/K}(x)^{-1})_\infty$.)

Theorem 5.1. *Let E be an elliptic curve with complex multiplication defined over an algebraic number field F ($\subseteq \mathbb{C}$) with $\mathrm{End}^0_{\overline{\mathbb{Q}}}(E) \cong K$ ($\subseteq \mathbb{C}$). Put $F' := \langle F, K \rangle$ ($\subseteq \mathbb{C}$). Then the following three conditions are equivalent:*
(1) *E is modular over F.*
(2) *There exists a Grössen-character $\gamma : K^\times_{\mathbb{A}} \longrightarrow \mathbb{C}^\times$ such that $\gamma \circ N_{F'/K} = \beta_{E/F'}$.*
(3) *All the points of E of finite order are rational over $\langle F', K_{ab} \rangle = \langle F, K_{ab} \rangle$.*

Proof. The equivalence of (2) and (3) is a special case of Theorem 4. p. 511 in [4].

We will prove that (1) implies (2). By assumption, there exists a normalized newform f of weight two (obtained by some $\lambda : I_K(\mathfrak{m}) \longrightarrow \mathbb{C}^\times$ as $f = f_\lambda$) such that $\mathrm{Hom}_{F'}(E, J_f) \neq \{0\}$. From f we define H as above. Firstly we consider Case 1. We define $\widetilde{\chi_\ell} : F'^\times_{\mathbb{A}} \longrightarrow (T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times$ from χ_ℓ by the same manner as defining $\widetilde{\vartheta}$ from ϑ in Section 3. By the commutative diagram

$$\begin{array}{ccc} F'^\times_{\mathbb{A}} & \xrightarrow{\text{norm}} & K^\times_{\mathbb{A}} \\ {\scriptstyle \text{reciprocity law}}\downarrow & & \downarrow{\scriptstyle \text{reciprocity law}} \\ \mathrm{Gal}(F'_{ab}/F') & \xrightarrow[\text{restriction}]{} & \mathrm{Gal}(K_{ab}/K), \end{array}$$

Proposition 2.4, Corollary 3.2, Proposition 3.3, and Corollary 4.1, we have that

$$\widetilde{\chi_\ell}(x) = \alpha_{E/F'}(x)^{-1} \alpha_{J_f/K}(N_{F'/K}(x)) \quad \text{for any } x \in F'^\times_{\mathbb{A}}.$$

(We identify L with T by ι'.) In Case 1, T is a field, so we have

$$\mathrm{Hom}_{F'}(E, J_f) \neq \{0\} \Longleftrightarrow \chi = 1 \Longleftrightarrow \widetilde{\chi_\ell} = 1 \Longleftrightarrow \alpha_{J_f/K} \circ N_{F'/K} = \alpha_{E/F'}.$$

We note that Proposition 4.2 is rephrased to that the map

$$G_K(\mathfrak{m}) \xrightarrow{il} I_K(\mathfrak{m}) \xrightarrow{\lambda} \mathbb{C}^\times$$

can be continuously extended to $K_{\mathbb{A}}^{\times}$ by the manner: any $x \in K^{\times}$ ($\subseteq K_{\mathbb{A}}^{\times}$) is mapped to 1 and this extended map, denoted by $\overline{\lambda}$, coincides with $\beta_{J_f/K}$. Then it holds that

$$\alpha_{J_f/K} \circ N_{F'/K} = \alpha_{E/F'} \Longleftrightarrow \overline{\lambda} \circ N_{F'/K} = \beta_{E/F'},$$

so we can take $\overline{\lambda}$ as γ in (2).

Next we consider Case 2. By the argument in the proof of Proposition 4.5, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$ on $V_\ell(J_f)$ corresponds to the homomorphism

$$\begin{array}{ccc} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & \xrightarrow{\Phi_\ell} & GL_2(H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) \\ \cup & & \cup \\ \mathrm{Gal}(\overline{\mathbb{Q}}/F') & \longrightarrow & (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^{\times} \oplus (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^{\times} \\ \Psi & & \Psi \\ \sigma & \longmapsto & (\,\delta'(\sigma),\ \delta'(\rho\sigma\rho)\,). \end{array}$$

By Proposition 2.4 and Proposition 3.3, we have that one of the following two statements holds:

(a) $\chi_\ell^{(1)}(\sigma) = \gamma_1(\,\vartheta(\sigma)\,)^{-1}\delta'(\sigma),\quad \chi_\ell^{(2)}(\sigma) = \gamma_2(\,\vartheta(\sigma)\,)^{-1}\delta'(\rho\sigma\rho)$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F')$;
(b) $\chi_\ell^{(1)}(\sigma) = \gamma_1(\,\vartheta(\sigma)\,)^{-1}\delta'(\rho\sigma\rho),\quad \chi_\ell^{(2)}(\sigma) = \gamma_2(\,\vartheta(\sigma)\,)^{-1}\delta'(\sigma)$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F')$.

We will prove that (b) is impossible. For this we assume that (b) holds. For $k = 1,\ 2$, we define $\widetilde{\chi_\ell^{(k)}}$ similary with $\widetilde{\chi_\ell}$. If $\sigma_{|F'_{ab}} = [x,\ F']$ ($[x,\ F']$ denotes the image of $x \in F_{\mathbb{A}}'^{\times}$ by the reciprocity law of F'), then we have $\rho\sigma\rho_{|K_{ab}} = [\rho(\,N_{F'/K}(x)\,),\ K]$ by the class field theory. Therefore, for any $x \in F_{\mathbb{A}}'^{\times}$, we have

$$\begin{aligned} \widetilde{\chi_\ell^{(1)}}(x) &= \gamma_1(\widetilde{\vartheta}(x)^{-1})\widetilde{\delta}'(\,\rho(\,N_{F'/K}(x)\,)\,) \\ &= \gamma_1(\,\alpha_{E/F'}(x)\,)^{-1}\gamma_1(\,(\,N_{F'/K}(x)\,)_\ell)\alpha_{C/K}(\,\rho(\,N_{F'/K}(x)\,)\,)\gamma_1(\,(\rho(\,N_{F'/K}(x)\,)\,)_\ell)^{-1}. \end{aligned}$$

Since $\gamma_1 \circ \rho = \gamma_2$, this is rephrased to that

$$\frac{\gamma_1(\,(\,N_{F'/K}(x)\,)_\ell)}{\gamma_2(\,(\,N_{F'/K}(x)\,)_\ell)} = \frac{\widetilde{\chi_\ell^{(1)}}(x)\alpha_{E/F'}(x)}{\alpha_{C/K}(\,\rho(\,N_{F'/K}(x)\,)\,)}.$$

We can take a transcendental element π of $\mathbb{Q}_\ell$ over $\mathbb{Q}$ and put $x_0 := 1\otimes 1+\sqrt{-D}\otimes\pi \in (K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^{\times} \subseteq (F' \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^{\times} \subseteq F_{\mathbb{A}}'^{\times}$. Now we suppose that ℓ splits completely in H. Since $K \subseteq H$, we can view $K \subseteq \mathbb{Q}_\ell$. By the isomorphism

$$\Bigl(\prod_{\substack{j:H\hookrightarrow\mathbb{Q}_\ell \\ j(\sqrt{-D})=\sqrt{-D}}} j\otimes 1\Bigr)\oplus\Bigl(\prod_{\substack{r:H\hookrightarrow\mathbb{Q}_\ell \\ r(\sqrt{-D})=-\sqrt{-D}}} r\otimes 1\Bigr) : H\otimes_{\mathbb{Q}}\mathbb{Q}_\ell \xrightarrow{\sim} \underbrace{(\mathbb{Q}_\ell \oplus \cdots \oplus \mathbb{Q}_\ell)}_{\frac{n}{2}}\oplus\underbrace{(\mathbb{Q}_\ell \oplus \cdots \oplus \mathbb{Q}_\ell)}_{\frac{n}{2}},$$

the element

$$\frac{\gamma_1(\,(\,N_{F'/K}(x_0)\,)_\ell)}{\gamma_2(\,(\,N_{F'/K}(x_0)\,)_\ell)} = \frac{\gamma_1(x_0^d)}{\gamma_2(x_0^d)}$$

is mapped to

$$\left(\left(\frac{1+\pi\sqrt{-D}}{1-\pi\sqrt{-D}}\right)^d, \ldots, \left(\frac{1+\pi\sqrt{-D}}{1-\pi\sqrt{-D}}\right)^d, \left(\frac{1-\pi\sqrt{-D}}{1+\pi\sqrt{-D}}\right)^d, \ldots, \left(\frac{1-\pi\sqrt{-D}}{1+\pi\sqrt{-D}}\right)^d\right),$$

where $d = [F' : K]$. Putting

$$\xi := \frac{\widetilde{\chi_\ell^{(1)}}(x_0)\alpha_{E/F'}(x_0)}{\alpha_{C/K}(\rho(N_{F'/K}(x_0)))} \in H^\times \subseteq (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times$$

and taking $j : H \hookrightarrow \mathbb{Q}_\ell$ with $j(\sqrt{-D}) = \sqrt{-D}$, we have that

$$\left(\frac{1+\pi\sqrt{-D}}{1-\pi\sqrt{-D}}\right)^d = j(\xi) \text{ in } \mathbb{Q}_\ell.$$

We note that $j(\xi)$ is algebraic over $\mathbb{Q}$. So we have that

$$\pi = \frac{\sqrt[d]{j(\xi)} - 1}{\sqrt{-D}(1 + \sqrt[d]{j(\xi)})} \in \overline{\mathbb{Q}}.$$

This is a contradiction. Hence we have proved that (a) holds.

We set

$$\iota''' : H \longrightarrow \mathrm{End}_K^0({}^\rho C), \qquad a \longmapsto \theta(\rho(a))_{|{}^\rho C}.$$

Then $({}^\rho C,\ \iota''')$ is an abelian variety with complex multiplication defined over K which has the same CM-type with $(C,\ \iota'')$. As case of $(C,\ \iota'')$, we have $\alpha_{{}^\rho C/K} : K_{\mathbb{A}}^\times \longrightarrow H^\times$. Since $\alpha_{{}^\rho C/K} = \rho \circ \alpha_{C/K} \circ \rho$, it holds that

$$\text{(a)} \iff \widetilde{\chi_\ell^{(1)}}(x) = \alpha_{E/F'}(x)^{-1}\alpha_{C/K}(N_{F'/K}(x)),\ \ \widetilde{\chi_\ell^{(2)}}(x) = \rho(\alpha_{E/F'}(x)^{-1}\alpha_{{}^\rho C/K}(N_{F'/K}(x))) \quad \text{for any } x \in F_{\mathbb{A}}'^{\times}.$$

Therefore we have

$$\begin{aligned} \mathrm{Hom}_{F'}(E,\ J_f) \neq \{0\} &\iff \chi^{(1)} = 1 \text{ or } \chi^{(2)} = 1 \iff \chi_\ell^{(1)} = 1 \text{ or } \chi_\ell^{(2)} = 1 \\ &\iff \alpha_{C/K} \circ N_{F'/K} = \alpha_{E/F'} \text{ or } \alpha_{{}^\rho C/K} \circ N_{F'/K} = \alpha_{E/F'}. \end{aligned}$$

Set $\lambda' := \rho \circ \lambda \circ \rho : I_K(\rho(\mathfrak{m})) \longrightarrow \mathbb{C}^\times$. As Case 1, we can construct a Grössen-character $\overline{\lambda}$ (resp. $\overline{\lambda'}$) of $K_{\mathbb{A}}^\times$ from λ (resp. λ'). Then we have

$$\alpha_{C/K} \circ N_{F'/K} = \alpha_{E/F'} \text{ or } \alpha_{{}^\rho C/K} \circ N_{F'/K} = \alpha_{E/F'} \iff \overline{\lambda} \circ N_{F'/K} = \beta_{E/F'} \text{ or } \overline{\lambda'} \circ N_{F'/K} = \beta_{E/F'}.$$

Hence we can take $\overline{\lambda}$ or $\overline{\lambda'}$ as γ in (2).

Finally we will prove that (2) implies (1). By Lemma 2.2, it is sufficient to show that there exists a normalized newform $f = f_\lambda$ of weight two constructed from some $\lambda : I_K(\mathfrak{m}) \longrightarrow \mathbb{C}^\times$ such that $\mathrm{Hom}_{F'}(E,\ J_f) \neq \{0\}$.

Claim. *Let γ be as in (2) and $\mathfrak{n}_0$ be the conductor of γ. As defining $\widetilde{\alpha_{J_f/K}}$ from $\alpha_{J_f/K}$ in Section 4, we can also define $\tilde{\gamma} : I_K(\mathfrak{n}_0) \longrightarrow \mathbb{C}^\times$ from γ. Then it holds that for any $x \in K^\times$ s.t. $x \equiv 1 \bmod^\times \mathfrak{n}_0$,*

$$\tilde{\gamma}((x)) = x.$$

By Claim, from $\tilde{\gamma}$ we can construct a normalized newform $f = f_{\tilde{\gamma}}$ of weight two. Then the arguments in the proof of the statement: (1) $\Rightarrow$ (2) imply that

$$\gamma \circ N_{F'/K} = \beta_{E/F'} \iff \begin{cases} \alpha_{J_f/K} \circ N_{F'/K} = \alpha_{E/F'} & (\text{if } K \not\subseteq H) \\ \alpha_{C/K} \circ N_{F'/K} = \alpha_{E/F'} & (\text{if } K \subseteq H) \end{cases}$$
$$\Longrightarrow \mathrm{Hom}_{F'}(E,\, J_f) \neq \{0\}.$$

So we have proved that (2) $\Rightarrow$ (1). □

Theorem 5.2. *Let E/F, K, F', and $\beta_{E/F'}$ be as in Theorem 5.1. Assume that the condition (2) in Theorem 5.1 holds. Let $\mathfrak{m}$ be the conductor of γ and set*

$$f(z) = f_{\tilde{\gamma}}(z) := \sum_{\substack{\mathfrak{a} \in I_K(\mathfrak{m}) \\ \mathfrak{a}:\,\text{integral}}} \tilde{\gamma}(\mathfrak{a}) q^{N(\mathfrak{a})} = \sum_{m \geq 1} a_m q^m \quad (q = e^{2\pi i z}).$$

Put $H := \mathbb{Q}(a_m | m \geq 1)$. Then we have the followings:
(1) For any normalized newform g of weight two, $\mathrm{Hom}_F(E,\, J_g) \neq \{0\}$ if and only if there exists some γ as above such that $g = f_{\tilde{\gamma}}$.
(2) Case 1: $K \not\subseteq H$. Then we have

$$J_f \sim_F \underbrace{E \times \cdots \times E}_{n} \quad (n = \dim J_f = [H : \mathbb{Q}]).$$

Case 2: $K \subseteq H$.
(a) If $\gamma = \rho \circ \gamma \circ \rho$ on $P := K^\times N_{F'/K}(F'^{\times}_{\mathbb{A}})$, then we have

$$J_f \sim_F \underbrace{E \times \cdots \times E}_{n}.$$

(b) If $\gamma \neq \rho \circ \gamma \circ \rho$ on P, then we have that $F = F'$ and there exists an abelian variety A of dimension $\frac{n}{2}$ defined over K such that

$$J_f \sim_F \underbrace{E \times \cdots \times E}_{\frac{n}{2}} \times A_{/F}, \qquad \mathrm{Hom}_F(E,\, A_{/F}) = \{0\}.$$

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