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String Equation and One Dimensional Quasiperiodic Dynamical Systems

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This paper is the résumé of the paper [Ya0]. Most of proofs of Theorems, Propositions, Lemmas and so on are omitted. Only outlines of proofs wil be shown, We refer [Ya0] for those proofs.

1 Introduction

Let D be a noncylindrical domain in (x,t)-plane with time-quasiperiodic boundaries defined by

$$a_1(t) < x < a_2(t), \quad t \in R^1.$$

Here the given functions $a_i(t)$, $i=1,2$, are quasiperiodic functions.

Initial Boundary Value Problem

We shall consider IBVP for a linear wave equation in D

$$\begin{cases} (\partial_t^2 - \partial_x^2)u(x,t) = h(x,t), \quad (x,t) \in D, \\ u(a_1(t),t) = r_1(t), \quad u(a_2(t),t) = r_2(t), \quad t \in R^1, \\ u(x,0) = \phi(x), \ \ \partial_t u(x,0) = \psi(x), \ \ x \in [a_1(0), a_2(0)]. \end{cases} \tag{1.1}$$

Here $r_i(t)$, $i = 1, 2$, and $h(x,t)$ are quasiperiodic functions in t. We assume that

$$|a_i'(t)| < 1 \quad \text{for all } t \in R^1. \tag{1.2}$$

This means that the velocity of boundaries in x-direction is less than the eigen-velocity. Then shock waves do not appear.

IBVP (1.1) describes some physical models like as the motions of the string with time-quasiperiodically oscillating ends ([Ya1][Ya3]), one-dimensional optical resonator with a quasiperiodically moving wall ([L-P][P-L-V]) and so on.

Conjecture and Some Counter Examples

We may generally expect that

(C) *every solution of* IBVP (1.1) *is quasiperiodic in* t.

There are counter works by Cooper [C] and Yamaguchi [Ya4]. Cooper showed that in a simpler IBVP

$$\begin{cases} (\partial_t^2 - \partial_x^2)u(x,t) = 0, & (x,t) \in D, \\ u(0,t) = 0, \quad u(a(t),t) = 0, & t \in R^1, \\ u(x,0) = \phi(x), \quad \partial_t u(x,0) = \psi(x), & x \in [a_1(0), a_2(0)], \end{cases} \tag{1.3}$$

where $a(t)$ is a periodic function, the solutions are unbounded in t under the periodicity of reflected characteristics. In [Ya4], for a family of some quasiperiodic h each solution of IBVP

$$\begin{cases} (\partial_t^2 - \partial_x^2)u(x,t) = h(x,t), & (x,t) \in (0,\pi) \times R^1, \\ u(0,t) = u(0,t) = 0, & t \in R^1, \\ u(x,0) = \phi(x), \quad \partial_t u(x,0) = \psi(x), & x \in [0,\pi] \end{cases} \tag{1.4}$$

is unbounded in t. In this case each h has the property that the Diophantine order of its basic frequencies is large and the differentiability in (x,t) is small. Contrary to [Ya4], it is shown in [Ya7] that every solution of IBVP (1.4) is time-quasiperiodic if the differentiability of h is suitably larger than the Diophantine order of the basic frequencies of h. Thus the Diophantine condition is necessary in order that the solutions of IBVP (1.1) are time-quasiperiodic.

In this paper *we shall give general conditions under which conditions every solution of IBVP* (1.1) *is quasiperiodic in t.*

1D Periodic DS and Rotation Numbers

Let $a_i(t)$, $i = 1, 2$, be periodic functions with the rational ratio of the periods. Then a composed function

$$A = A_1^{-1} \circ A_2, \quad A_i = (I + a_i) \circ (I - a_i)^{-1}, \quad i = 1, 2, \tag{1.5}$$

is a periodic dynamical system (DS). Here I is the identity, f^{-1} is the inverse of f, $f \circ g$ is the composition of f and g and f^n is the composed iterates of f. It is well-known [H] that for periodic $(A - I)(x)$ the rotation number of A,

$$\lim_{n\to\infty} \frac{(A^n - I)(x)}{n},$$

exists for each $x \in R^1$ and is independent of x. A and its rotation number are the essential notions to describe periodic and quasiperiodic property of the solutions of IBVP and BVP with periodically oscillating boundaries. We could apply the Herman-Yoccoz reduction theorem to A.

In [Ya1,Ya2,Ya3] we deal with the case where $h(x,t)$ vanishes identically, and all data $a_i(t)$ and $r_i(t)$, $i = 1, 2$, are periodic, where the ratio of the periods of $a_i(t)$ is a rational number. It was shown that if the periods of $a_i(t)$, $r_i(t)$ and the rotation number ω of A satisfy some Diophantine approximation inequality, every solution is quasiperiodic in t with basic periods $(\omega, \alpha_1, \alpha_2, 1)$. In [Ya2] it was shown that the properties of A and the reflected characteristics determine the periodicity of the solutions of IBVP (1.1)-(1.3) with $a_1(t) = r_1(t) = r_2(t) = h(x,t) \equiv 0$.

1D Quadiperiodic DS and Upper (Lower) Rotaion Numbers

In general, the mapping A is defined for $a_i(t)$ satisfying $a_1(t) < a_2(t)$ and $|a_i'(t)| < 1$, even if $a_i(t)$ are not necessarily periodic.

All the above results are essentially due to the *periodicity* of $A - I$ that assures the existence of the rotation number ([Ya1,Ya2,Ya3]) and the periodic reflected characteristics ([Ya2]). However, for example, if the periods of $a_1(t)$

and $a_2(t)$ have *an irrational ratio*, $A - I$ is not periodic but quasiperiodic with two basic periods. In this case several difficult problem arise. In this paper, we shall be interested in such cases. We shall treat IBVP (1.1) under more general condition that both $a_i(t)$, $i = 1, 2$, are quasiperiodic functions. In this case naturally $A - I$ is quasiperiodic and A is a quasiperiodic DS

$$A(x) = x + g(x)$$

with quasiperiodic term $g(x)$. It is not known that for the quasiperiodic DS the rotation number exists for all x. As a matter of fact, in this paper we shall see that the existence of the rotation number is not necessary. Instead, we shall introduce a more weak notion *upper and lower rotation number of f at every point x* (see section 2)

$$\limsup_{n\to\infty} \frac{(A^n - I)(x)}{n}, \quad \liminf_{n\to\infty} \frac{(A^n - I)(x)}{n}.$$

The upper (lower) rotation numbers have several important properties lile as *semi-invariant property under conjugation.*

Reduction Theorem for Quasiperiodic DS

The following holds : Assume that an upper (lower) rotation number ω of f and the basic frequencies of the quasiperiodic term g satisfy the Diophantine condition. Then the nearly affine mapping f written in the form $x + \omega + q(x)$ is conjugate to an affine mapping $x + \omega$, provided that q is small enough. As a consequence, it will be shown that under the same Diophantine condition the rotation number of f exists and coinsides with the upper (lower) rotation number (Corollary of the Reduction Theorem in section 3). The main tool we shall use here to show the above reduction theorem is the rapidly convergent iteration method based on the Newton iteration method [S-M], instead of the Herman-Yoccoz theory [H,Yoc] used in case of the periodic $A - I$ in [Ya1,Ya3]. Then we shall show that under the Diophantine conditions on an upper (lower) rotation number of A and the basic periods of a_i, r_i, h, every solution of IBVP (1.1) is quasiperiodic in t and x. In this case the rotation number of A exists and coincides with the upper (lower) rotation number of A. The reduction problem is also treated by [P-L-V,L-P] in the quite different point of view.

Note that our results are obtained for $a_i(t)$ with the small perturbation forms, different from those of [Ya1]-[Ya3].

Domain Transformation

[Ya1,Ya2,Ya3] dealt with homogeneous string equations. In order to treat a nonhomogeneous equation in (1.1) with $h(x,t) \not\equiv 0$, we shall introduce the useful domain transformation of the noncylindrical domain $\bar{D}$ to a cylindrical domain $[0,\omega/2] \times R^1$, where ω is the upper (lower) rotation number of A. It is remarkable that different from other domain transformations which change the noncylindrical domain to a cylindrical domain, our transformation preserves the d'Alembert operator and does not produce any lower order differential operators. This will be constructed by using the conjugate function of the Reduction Theorem. In case where $a_1(t)$ vanishes identically and $a_2(t)$ is periodic, in [Ya-Yos] we have already constructed such d'Alembertian-preserving transformation of a time-periodic one-sided noncylindrical domain onto the cylindrical domain. And using this transformation, we treated IBVP for a nonhomogeneous string equation ([Ya-Yos]). In [Ya5] the above domain transformation is generalized to a time-periodic both sided noncylindrical domain, and IBVP for a 3-dimensional radially symmetric wave equation is studied. Also [Ya6] treated the periodic solutions of nonlinear string equation with periodic nonlinear term.

2 1D Quasiperiodic DS and its Upper and Lower Rotation Numbers

Definition. A function $g(t)$, $t \in R^1$, is called *quasiperiodic with basic frequencies* $\beta = (\beta_1, \cdots, \beta_m) \in R^m$ (*briefly* $2\pi/\beta$*-q.p.*) if there exists a continuous function $\hat{g}(\theta), \theta = (\theta_1, \cdots, \theta_m) \in R^m$, that is 2π-periodic in each θ_i such that $g(t) = \hat{g}(\beta t)$ holds. $\hat{g}(\theta)$ is called a corresponding function of g and $2\pi/\beta = (2\pi/\beta_1, \cdots, 2\pi/\beta_m)$ is called the basic periods of g. Without loss of generality, basic frequencies $\beta_1, \cdots, \beta_m$ of any q.p. functions are always assumed to be rationally independent.

We consider a monotone increasing mapping of R^1 to R^1

$$f(x) = x + g(x),$$

where $g(x)$ is a $2\pi/\beta$-q.p. function. We denote the set of such functions f by D_β. D_β is a group with respect to the operation of the composition of functions.

Definition. *A rotation number* for any monotone increasing mapping f with continuous periodic g is defined by

$$\rho = \rho(f) = \lim_{n\to\infty} \frac{f^n(x) - x}{n}.$$

In periodic DS the above limit ρ exists, is independent of x and has conjugacy invariant property

$$\rho(\phi \circ f \circ \phi^{-1}) = \rho(f)$$

for every periodic and continuous DS ϕ.

Definition. *An upper (lower) rotation number $\bar{\rho}(f)(x)$ $(\underline{\rho}(f)(x))$ of $f \in D_\beta$ at $x \in R^1$* is defined by

$$\bar{\rho}(f)(x) = \limsup_{n\to\infty} \frac{f^n(x) - x}{n},$$

$$\underline{\rho}(f)(x) = \liminf_{n\to\infty} \frac{f^n(x) - x}{n}.$$

If $\bar{\rho}(f)(x) = \underline{\rho}(f)(x)$ holds, then the rotation number $\rho(f)(x)$ exists and vice versa, and $\bar{\rho}(f)(x) = \underline{\rho}(f)(x) = \rho(f)(x)$ holds. The above superior (inferior) limit exists for each $x \in R^1$.

Proposition 2.1 *Consider $f \in D_\beta$. For any $x \in R^1$ and any $H \in D_\beta$ there exists $y \in R^1$ such that*

$$\bar{\rho}(H^{-1} \circ f \circ H)(y) = \bar{\rho}(f)(x).$$

Especially, if f has a rotation number $\rho(f)$ independent of x, then

$$\rho(H^{-1} \circ f \circ H) = \rho(f)$$

holds.

Remark 2.1 For any $\omega \in R^1$ there exist $f_\omega \in D_\beta$ that has the rotation number ω.

For any $x_0 \in R^1$ denote $\bar{\rho}(f)(x_0)$ by $\omega = \omega(x_0)$. Then we rewrite f

$$f(x) = x + \omega + q(x). \tag{2.1}$$

Proposition 2.2 *For any $f \in D_\beta$ of the form* (2.1)

$$\inf_{x\in R^1} |q(x)| = 0$$

holds.

3 Reduction Problem of 1D Quasiperiodic DS

Consider $Q \in D_\beta$

$$Q(x) = x + g(x).$$

Here $g(x)$ is a $2\pi/\beta$-q.p. function.

We assume that $\hat{g}(\theta)$ is real analytic defined in a strip $|\Im\theta_i| \le r$, $i = 1, \cdots, m$. Here $\Im z$ is the imaginery part of $z \in C^1$. Let a_0 be a point in R^1 and denote $\bar{\rho}(Q)(a_0)$ by $\omega = \omega(a_0)$. Then we rewrite Q of $\{x \in C^1 : |\Im x| \le \tilde{r}\}$ to C^1

$$x_1 = Q(x) = x + \omega + q(x). \tag{3.1}$$

<u>**Reduction Problem**</u>

By a suitable transformation of the variable x to ξ

$$x = H(\xi) = \xi + h(\xi), \tag{3.2}$$

where h is a $2\pi/\beta$-q.p. function and $\hat{h}(\theta)$ is a real analytic function, reduce (3.1) *to an affine mapping*

$$\xi_1 = H^{-1} \circ Q \circ H(\xi) = R(\xi) = \xi + \omega. \tag{3.3}$$

<u>**Notation.**</u> Let C^m be the m-dimensional complex Euclidean space. For $\theta = (\theta_1, \cdots, \theta_m) \in C^m$ we set $|\Im\theta| = \max_{1\le j\le m} |\Im\theta_i|$. Let Π_r and $\hat{\Pi}_r$ be sets $\{\theta \in C^m : |\Im\theta| \le r\}$ and $\{\theta \in C^m : |\Im\theta| < r\}$ (resp.). Let $f(\theta)$ be 2π-periodic in each θ_i and real analytic in Π_r. Set $\partial_\theta f(\theta) = (\partial_{\theta_1} f(\theta), \cdots, \partial_{\theta_m} f(\theta))$. We define the norms

$$|f|_r = \max_{\theta\in\Pi_r} |f(\theta)|, \quad |\partial_\theta f|_r = \max_{1\le j\le m} \max_{\theta\in\Pi_r} |\partial_{\theta_j} f(\theta)|.$$

For $\beta = (\beta_1, \cdots, \beta_m)$ set $\| \beta \| = \max_{1\le i\le m} |\beta_i|$. For $k = (k_1, \cdots, k_m) \in Z^m$ set $|k| = |k_1| + \cdots + |k_m|$ and $(k, \beta) = k_1\beta_1 + \cdots + k_m\beta_m$.

Consider a mapping Q of (3.1).

(C) There exists a point $a_0 \in R^1$ such that $\omega = \omega(a_0) = \bar{\rho}(Q)(a_0)$ and $\beta = (\beta_1, ..., \beta_m)$ satisfy the Diophantine condition : There exists a positive constant C_0 depending on β such that

$$|(k, \beta) + \pi l/\omega| > \frac{C_0}{|k|^{m+1}}$$

holds for all $k \in Z^m \setminus \{0\}$ and all $l \in Z$.

Reduction Theorem. *Consider* (3.1) *with* $\omega = \bar{\rho}(Q)(a_0)$, *where* $q(x)$ *is a* $2\pi/\beta$*-q.p. function with* $\hat{q}(\theta)$ *real analytic in* $\mathring{\Pi}_r$ *and continuous in* Π_r. *Assume* (C). *Then there exists a constant* $M^0 > 0$ *dependent on* C_0, r *such that if* $|\hat{q}|_r \leq M^0$ *holds, then* (3.1) *is reduced to the affine mapping* (3.3) *by* (3.2) *with a* $2\pi/\beta$*-q.p. term* $h(\xi)$ *with* $\hat{h}(\theta)$ *real analytic in* $\mathring{\Pi}_{r/2}$.

Outline of the proof of this theorem will be shown in section 7.

Corollary. *Consider* (3.1) *with* $\omega = \bar{\rho}(Q)(a_0)$. *Under the same assumptions of the reduction theorem* (3.1) *has a rotation number independent of* $x \in R^1$. *In other words,* $\rho(Q) = \bar{\rho}(Q)(x) = \omega(x)$ *holds for any* $x \in R^1$.

4 Quasiperiodic Solutions of IBVP (1.1)

(C1) $a_i(t)$, $i = 1, 2$, are η-q.p. functions, where $\eta \in R^m$. $\hat{a}_i(\theta)$ are real analytic, and satisfy $0 < \inf_{\theta \in R^m} \hat{a}_2(\theta) - \sup_{\theta \in R^m} \hat{a}_1(\theta)$ and $|\hat{a'}_i(\theta)| < 1$ for $\theta \in R^m$. $a_i(t)$ satisfy $a_i'(0) = a_i''(0) = 0$, $i = 1, 2$, $a_1(0) = 0$.

(C2) $r_i(t)$, $i = 1, 2$, are α_i-q.p. functions, where $\alpha_i \in R^{m_i}$ (resp.). $\hat{r}_i(\theta)$ are C^∞-function satisfying $\int_{T^{m_i}} \hat{r}_i(\theta) d\theta = 0$. $r_i(t)$ satisfy $r_i(0) = r_i'(0) = r_i''(0) = 0$, $i = 1, 2$.

(C3) $h(x, t)$ is a μ-q.p. function, where μ belongs to R^p. $\hat{h}(x, \theta)$ is of C^∞ in D, and the support of h is contained in the cylinder $W = (\sup_{t \in R^1} a_1(t), \inf_{t \in R^1} a_2(t)) \times R^1$.

Remark 4.1 (1) $a_i'(0) = a_i''(0) = 0$, $i = 1, 2$, $a_1(0) = 0$ in (C1) and $r_i(0) = r_i'(0) = r_i''(0) = 0$, $i = 1, 2$ in (C2) are compatibility conditions with the latter part of (C6) below.

(2) It follows from the Weyl Theorem that

$$\sup_{t\in R^1} a_1(t) = \sup_{\theta\in R^m} \hat{a}_1(\theta), \quad \inf_{t\in R^1} a_2(t) = \inf_{\theta\in R^m} \hat{a}_2(\theta).$$

(3) For the same reason as (2) it follows immediately that

$$|a_i'(t)| \leq \sup |a_i'(t)| = \sup |\hat{a'}_i(\theta)| < 1$$

for all $t \in R^1$.

By $|a_i'(t)| < 1$, A is well-defined by (1.5). Then $A(x)$ belongs to D_β. Let ω be an upper rotation number of A. Then $A(x)$ is represented by

$$A(x) = x + \omega + q(x).$$

Here $q(x)$ is an η-q.p. function. $\hat{q}(\theta)$ is real analytic in $\{\theta \in C^m; |\Im\theta_i| \leq r_0, i = 1, ..., m\}$ for a constant $r_0 > 0$.

Remark 4.2 Without loss of generality we can assume that the basic periods of $a_1(t)$ and $a_2(t)$ are the same.

The following proposition is important. It assures the existence of the infinite number of the boundary functions $a_1(t)$, $a_2(t)$ that satisfy both *the analytical condition* (C1) and *the number-theoretic conditions* (C4) and (C5).

Proposition 4.1 *Let ω be any positive number. Then for any small ε there exist an infinite number of real analytic $a_1(t)$ and $a_2(t)$ satisfying* $\inf_t a_2(t) > \sup_t a_1(t)$ *such that A has the rotation number ω and $q(x)$ satisfies $|\hat{q}|_r < \varepsilon$. The set of such functions $a_1(t)$, $a_2(t)$ has a continuum cardinal number.*

The proof is seen in [Ya0].

Remark 4.3 If $h(x,t)$ identically vanishes, then the condition $0 < \inf_t a_2(t) - \sup_t a_1(t)$ in (C1) is not necessary.

For simplicity we set

$$\begin{aligned} \beta &= 2\pi/\eta = (2\pi/\eta_1, ..., 2\pi/\eta_m), \\ \lambda_i &= 2\pi/\alpha_i = (2\pi/\alpha_i^1, ..., 2\pi/\alpha_i^{m_i}), \\ \gamma &= 2\pi/\mu = (2\pi/\mu_1, ..., 2\pi/\mu_p), \end{aligned}$$

where $\alpha_i = (\alpha_i^1, ..., \alpha_i^{m_i})$ and $\mu = (\mu_1, ..., \mu_p)$.

The following Diophantine condition is essential in order that each solution is q.p. in t.

(C4) β, λ_1, λ_2, γ and ω satisfy the Diophantine condition : There exists a constant $C > 0$ depending on β, λ_1, λ_2, γ and ω such that

$$|(k_1, \lambda_1) + (k_2, \lambda_2) + (k, \beta) + (j, \gamma) + \pi l/\omega| > \frac{C}{(|k_1| + |k_2| + |k| + |j|)^{m+p+m_1+m_2+1}}$$

holds for all $(k_1, k_2, k, j) \in Z^{m_1+m_2+m+p} \setminus \{0\}$ and all $l \in Z$.

Remark 4.4 It is well-known in number theory that almost all vector

$$(\lambda_1, \lambda_2, \beta, \gamma, \omega) \in R^{m_1+m_2+m+p+1}$$

satisfy (C4). "Almost all" means the Lebesgue measure sense. We can construct such vectors as solutions of algebraic equation of order $m_1 + m_2 + m + p + 1$. For the construction, see Appendix in [Ya4].

(C5) β and ω satisfy the Diophantine condition : There exists a positive constant C_0 depending on β such that

$$|(k, \beta) + \pi l/\omega| > \frac{C_0}{|k|^{m+1}}$$

holds for all $k \in Z^m \setminus \{0\}$ and all $l \in Z$.

Remark 4.5 In section 7 it will be shown that for suitable $a_i(t)$, $r_i(t)$ and $h(x,t)$ not satisfying (C4) or (C5), every solutions of IBVP (1.1) grows up as a time sequence $\{t_j\}$ tends to infinity.

(C6) The initial data ϕ and ψ are of C^∞-class in $(a_1(0), a_2(0))$, and of C^2-class and C^1-class (resp.) in $[a_1(0), a_2(0)]$. ϕ, ϕ'' and ψ vanish at $x = a_1(0)$ and $x = a_2(0)$.

Theorem 4.1 *Assume* (C1)-(C6). *Then there exists a constant* $\epsilon > 0$ *dependent on* C_0, β *and* r_0 *such that if* $|\hat{q}|_{r_0} < \epsilon$ *holds, IBVP* (1.1) *has a unique* C^2*-solution* u *that is* $(\alpha_1, \alpha_2, \eta, \omega)$*-q.p. in* t *and* x. u *is represented by* $u_0 + u_1 + u_2 + u_3$ *satisfying the following properties :*

1. *Solutions of BVP*

   (a) u_0 *satisfies*

   $$\partial_t^2 u - \partial_x^2 u = 0, \quad (x,t) \in R^2,$$

   $$u(a_1(t),t) = u(a_2(t),t) = 0, \quad t \in R^1.$$

   (b) u_1 *satisfies*

   $$\partial_t^2 u - \partial_x^2 u = 0, \quad (x,t) \in R^2,$$

   $$u(a_1(t),t) = r_1(t), \quad u(a_2(t),t) = 0, \quad t \in R^1.$$

   (c) u_2 *satisfies*

   $$\partial_t^2 u - \partial_x^2 u = 0, \quad (x,t) \in R^2,$$

   $$u(a_1(t),t) = 0, \quad u(a_2(t),t) = r_2(t), \quad t \in R^1.$$

   (d) u_3 *satisfies*

   $$\partial_t^2 u - \partial_x^2 u = \tilde{h}(x,t), \quad (x,t) \in R^2,$$

   $$u(a_1(t),t) = u(a_2(t),t) = 0, \quad t \in R^1.$$

   Here $\tilde{h}(x,t)$ *is an extension of* $h(x,t)$ *defined in* D *to* R^2 *seen in Remark* 6.2.

2. *Regularity*

   (a) u_0 *is of* C^2*-class in* $(x,t) \in R^2$ *and of* C^∞*-class in* $(x,t) \in R^2 \setminus S$, *where* $S = \{(x,t) \in R^2\,;\, x+t = A_1 \circ A^n(\mu),\ -x+t = A^n(\mu),\ \mu = 0, a_2(0),\ n \in Z\}$.

   (b) $u_i\ (i = 1,2,3)$ *is of* C^∞*-class in* $(x,t) \in R^2$.

3. *Quasiperiodicity*

   (a) u_0 *is* (ω,η)*-q.p. in both* t *and* x.

   (b) u_i, $i = 1,2$, *is* (α_i,η)*-q.p.(resp.) in both* t *and* x.

   (c) u_3 *is* (μ,η)*-q.p.(resp.) in* t.

The outline of this theorem will be given in sections 5 and 6.

5 IBVP for Homogeneous Wave Equation

Consider IBVP for a homogeneous wave equation in D

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) = 0, \quad (x,t) \in D, \tag{5.1}$$

$$u(a_1(t),t) = r_1(t), \quad u(a_2(t),t) = r_2(t), \quad t \in R^1, \tag{5.2}$$

$$u(x,0) = \phi(x), \ \partial_t u(x,0) = \psi(x), \quad x \in [a_1(0), a_2(0)]. \tag{5.3}$$

By $|a_i'(t)| < 1$, $i = 1,2$ in (C1) we have the d'Alembert representation formula of solutions of (5.1)

$$u(x,t) = f(-x+t) + g(x+t). \tag{5.4}$$

We shall show that f and g are determined so that (5.4) may satisfy (5.2) and (5.3). First, from (5.2) we have

$$f(-a_i(t)+t) + g(a_i(t)+t) = r_i(t), \quad i = 1,2. \tag{5.5}$$

From (5.4) and (5.5) we obtain

$$u(x,t) = f(-x+t) - f \circ A_1^{-1}(x+t) + r_1 \circ (I+a_1)^{-1}(x+t)$$

and by setting $\tau = (I - a_2)^{-1}(t)$

$$f \circ A(\tau) - f(\tau) = r_1 \circ (I+a_1)^{-1} \circ A_2(\tau) - r_2 \circ (I-a_2)^{-1}(\tau). \tag{5.6}$$

1. *Construction of $u_1(x,t)$ and $u_2(x,t)$*

The following lemmas shall be used to solve (5.6) for the given functions a_i and r_i, $i = 1,2$.

Lemma 5.1 *Let $R(x)$ be a q.p. function with basic frequencies $\Xi = (\Xi_1, ..., \Xi_\nu)$ whose corresponding function $\hat{R}(\theta)$ is of C^∞ and satisfies $\int_{T^\nu} \hat{R}(\theta)d\theta = 0$. Let γ be a constant in $R^1 \backslash \{0\}$. Assume that $\Xi \in R^\nu$ and γ satisfy the Diophantine condition : There exist constants $C = C(\Xi,\gamma) > 0$ and $\tau \geq 1$ such that the following inequality*

$$|(k,\Xi) - (\pi/\gamma)l| > \frac{C}{|k|^{\nu+\tau}}$$

holds for all $k \in Z^\nu \setminus \{0\}$ and all $l \in Z$. Then a functional equation

$$G(x+\gamma) - G(x) = R(x)$$

has a q.p. solution $G(x)$ with basic frequencies Ξ. $\hat{G}(\theta)$ is of C^∞. $G(x)$ is the only q.p. solution with basic frequencies Ξ which satisfies $\int_{T^\nu} \hat{G}(\theta)d\theta = 0$.

This lemma and its proof are seen in [Ya1], Lemma 2.9.

Lemma 5.2 *Let $a(t)$ and $b(t)$ be $2\pi/\alpha$-q.p. and $2\pi/\beta$-q.p. (resp.) whose corresponding functions are of C^∞, where $\alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_n$ are rationally independent. Then a composed function $a \circ (I+b)(t)$ is $(2\pi/\alpha, 2\pi/\beta)$-q.p. Its corresponding function $F(\theta_1, \cdots, \theta_m, \Theta_1, \cdots, \Theta_n)$ is of the form*

$$\hat{a}(\theta_1 + \alpha_1 \hat{b}(\Theta), \cdots, \theta_m + \alpha_m \hat{b}(\Theta))$$

of C^∞. Moreover it holds

$$\int_{T^m} F(\theta, \Theta)d\theta = \int_{T^m} \hat{a}(\theta)d\theta.$$

This lemma is shown in [Ya0].

Since by (C1) and (C4) A satisfies the assumptions of the Reduction Theorem, it follows that there exists a real analytic function $H(\xi) = \xi + h(\xi)$, where $h(\xi)$ is an η-q.p. function, such that

$$H^{-1} \circ A \circ H(\xi) = \xi + \omega. \tag{R}$$

Consider functional equations

$$g_i(\xi + \omega) - g_i(\xi) = \tilde{r}_i(\xi), \quad i = 1, 2,$$

where

$$\tilde{r}_1(\xi) = r_1 \circ (I + a_1)^{-1} \circ A_2 \circ H(\xi),$$
$$\tilde{r}_2(\xi) = -r_2 \circ (I - a_2)^{-1} \circ H(\xi).$$

Since a_i and r_i are α_i-q.p. and η-q.p. (resp.), it follows from Lemma 5.2 that $\tilde{r}_i(\xi), i = 1, 2$, are (α_i, η)-q.p. whose corresponding functions are of C^∞ class and have the 0-mean value. By this fact and the Diophantine inequality in (C3), it follows from Lemma 5.1 that each equation has a unique (α_i, η)-q.p. solution g_i whose corresponding function is of C^∞. Then $f_i(\tau) = g_i \circ H^{-1}(\tau), i = 1, 2$, are of C^∞ in R^1 and are the solutions of Eq.s (resp.)

$$f_i \circ A(\tau) - f_i(\tau) = \tilde{r}_i \circ H^{-1}(\tau), \quad i = 1, 2.$$

We set

$$u_1(x,t) = f_1(-x+t) - f_1 \circ A_1^{-1}(x+t) + r_1 \circ (I + a_1)^{-1}(x+t), \tag{5.7}$$
$$u_2(x,t) = f_2(-x+t) - f_2 \circ A_1^{-1}(x+t). \tag{5.8}$$

Then u_1 and u_2 are of C^∞ in $(x,t) \in R^2$ and satisfy

$$\begin{cases} \partial_t^2 u_i - \partial_x^2 u_i = 0, \quad (x,t) \in R^2, \\ u_1(a_1(t), t) = r_1(t), \ u_1(a_2(t), t) = 0, \\ u_2(a_1(t), t) = 0, \ u_2(a_2(t), t) = r_2(t). \end{cases}$$

Since $g_i(\xi)$ and $h(\xi)$ are (α_i, η)-q.p. and η-q.p. (resp.), it follows from Lemma 5.2 that f_i is (α_i, η)-q.p. Hence $u_i(x,t)$ is (α_i, η)-q.p. (resp.) both in x and t. Thus the assertions 1.(b),1.(c), 2.(b) and 3.(b) of Theorem 4.1 are proved.

2. *Construction of* $u_0(x,t)$

We shall construct a unique ω-periodic function $g_0(\xi)$ so that u_0 defined by

$$u_0(x,t) = f_0(-x+t) - f_0 \circ A_1^{-1}(x+t), \quad f_0 = g_0 \circ H^{-1}$$

may satisfy the assertions 1.(a), 2.(a) and 3.(a) of Theorem 4.1.

Let γ_1 and γ_2 be equal to $A_1^{-1}(a_2(0))$ and $-a_2(0)$ (resp.). For ϕ_0 and ψ_0 satisfying (C5) define $\hat{\phi}_0(x)$ and $\hat{\psi}_0(x)$ by

$$\hat{\phi}_0(x) = \begin{cases} \phi_0 \circ A_1(x) & (0 \le x \le \gamma_1) \\ -\phi_0(-x) & (\gamma_2 \le x < 0), \end{cases}$$

$$\hat{\psi}_0(x) = \begin{cases} \psi_0 \circ A_1(x) A_1'(x) & (0 \le x \le \gamma_1) \\ -\psi_0(-x) & (\gamma_2 \le x < 0). \end{cases}$$

Define $\hat{f}_0(x)$ and $\hat{g}_0$ by

$$\hat{f}_0(x) = -(1/2)\left(\hat{\phi}_0(x) + \int_0^x \hat{\psi}_0(\eta)d\eta\right), \quad \hat{g}_0(\xi) = \hat{f}_0 \circ H(\xi).$$

$\hat{g}_0(\xi)$ is defined in $[\xi_2, \xi_1]$, where ξ_i, $i = 1, 2$, is the solution of equation $H(\xi_i) = \gamma_i$.

Lemma 5.3 *Let ϕ_0 and ψ_0 satisfy* (C6). *Then there exists an ω-periodic function $g_0(\xi)$ such that*

(1) $g_0(\xi) = \hat{g}_0(\xi)$ *for* $\xi \in [\xi_2, \xi_1]$

(2) $g_0(\xi)$ *is of C^2 class in R^1*

(3) $g_0(\xi)$ *is of C^∞ class in $R^1 \backslash W$, where W is a set $\{\xi_i + n\omega; i = 0, 1, n \in Z\}$ and $\xi_0 = H^{-1}(0)$.*

To show this lemma we prepare the following lemma. See [Ya0] for the proof.

Lemma 5.4 *The followings hold.*

(a) *$\hat{\phi}_0$ and $\hat{\psi}_0$ are of C^2 and C^1 class (resp.) in $[\gamma_2, \gamma_1]$ and of C^∞ class in $(\gamma_2, 0) \cup (0, \gamma_1)$.*

(b) *$\hat{f}_0(x)$ is of C^∞ class in $(\gamma_2, 0) \cup (0, \gamma_1)$, and of C^2 class in $[\gamma_2, \gamma_1]$, and satisfies $\hat{f}_0(\gamma_2) = \hat{f}_0(\gamma_1)$.*

(c) *$\hat{g}_0(x)$ is of C^∞ class in $(\xi_2, \xi_0) \cup (\xi_0, \xi_1)$, and of C^2 class in $[\xi_2, \xi_1]$, and satisfies $\hat{g}_0(\xi_2) = \hat{g}_0(\xi_1)$.*

(d) *$\xi_1 - \xi_2 = \omega$.*

Using Lemma 5.3, we obtain the following lemma.

Lemma 5.5 *Let ϕ_0 and ψ_0 satisfy* (C5). *Let $g_0(\xi)$ be the ω-periodic function in Lemma* 5.3. *Let $f_0(x)$ be defined by $f_0(x) = g_0 \circ H^{-1}(x)$. Then*

$$u_0(x,t) = f_0(-x+t) - f_0 \circ A_1^{-1}(x+t) \tag{5.9}$$

is the solution of IBVP

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0, \ \ (x,t) \in (a_1(t), a_2(t)) \times R^1, \\ u(a_1(t),t) = u(a_2(t),t) = 0, \ \ t \in R^1, \\ u(x,0) = \phi_0(x), \ \ \partial_t u(x,0) = \psi_0(x), \ \ x \in [a_1(0), a_2(0)], \end{cases} \tag{5.10}$$

and of C^2 in R^2 and of C^∞ in $R^2 \setminus S$, where

$$S = \{(x,t) \in R^2; x+t = A_1 \circ A^n(\mu), -x+t = A^n(\mu), \\ \mu = -a_2(0), a_1(0), n \in Z\}.$$

u_0 is (ω, η)-q.p. both in t and x.

We also have the following lemma. See [Ya0] for the proof.

Lemma 5.6 *Let $u_i, i = 1,2$, be the functions defined by* (5.7) *and* (5.8). *Let $u_i(x,0)$ and $\partial_t u_i(x,0), i = 1,2$, be denoted by $\phi_i(x)$ and $\psi_i(x)$, $i = 1,2$ (resp.). Then the restrictions of ϕ_i and $\psi_i, i = 1,2$ to $[a_1(0), a_2(0)]$ satisfy* (C6).

6 BVP for Nonhomogeneous Wave Equation

We consider BVP for a nonhomogeneous wave equation

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) = h(x,t), \ \ (x,t) \in D, \tag{1.1}$$

$$u(a_1(t),t) = 0, \quad u(a_2(t),t) = 0, \ \ t \in R^1. \tag{1.2'}$$

Now let H be the function in (R) in section 5. Let X be a mapping of R^2 to R^2 defined by

$$\begin{cases} y = \left(H^{-1} \circ A_1^{-1}(-x+t) - H^{-1}(x+t)\right)/2, \\ s = \left(H^{-1} \circ A_1^{-1}(-x+t) + H^{-1}(x+t)\right)/2. \end{cases} \tag{6.1}$$

Such transformations were considered in [Ya5, Ya-Yo] in case where $A(t)$ is a periodic DS. Without any difficulty, we are able to extend the transformations for periodic DSs to one for quasiperiodic DS $A(t)$ due to the Reduction Theorem in section 3. Similarly to Propositions 4.1 and 4.2 in [Ya5], we are able to show the following Proposition 6.1. We assume

(C1') $a_i(t)$, $i = 1, 2$, are η-q.p. functions, where η belongs to R^m. $\hat{a}_i(\theta)$ are of C^2 in T^m. $a_i(t)$ satisfy $0 < \inf_{t\in R^1} a_2(t) - \sup_{t\in R^1} a_1(t)$ and $|a_i'(t)| < 1$ for $t \in R^1$.

(R1) A is reducible in the following sense : there exists a conjugate function $H \in D_\beta$, where $\hat{H}$ and $(H^{-1})\hat{}$ are of C^2 in T^m, such that

$$H^{-1} \circ A \circ H(x) = x + \omega. \tag{R}$$

Remark 6.1 Assume (C1) and (C5). Then it follows from the Reduction Theorem that there exists a constant $\varepsilon > 0$ such that for q with $|q|_r < \varepsilon$ A is reducible by a real analytic conjugate function H. Hence (C1') and (R1) are satisfied.

Proposition 6.1 *Assume* (C1') *and* (R1). *X is the bijection of $\bar{D}$ to $\bar{E}$, and maps the boundaries $x = a_1(t)$ and $x = a_2(t)$ onto the boundaries $y = 0$ and $y = \omega/2$ (resp.) bijectively.*

Let $u(x,t)$ be of C^2 in $(x,t) \in R^2$ and $v(y,s)$ be defined by $u(X^{-1}(y,s))$. Then

$$(\partial_t^2 - \partial_x^2)u(x,t) = K(y,s)(\partial_s^2 - \partial_y^2)v(y,s),$$

where $K(y,s)$ is defined by

$$(H^{-1})' \circ H(y+s)(H^{-1})' \circ H(-y+s)(A_1^{-1})' \circ A_1 \circ H(y+s).$$

$K(y,s)$ is η-q.p. in s and $\hat{K}(y,\theta)$ is real analytic.

We apply X to BVP (1.1)-(1.2'). Then we obtain BVP in the cylindrical domain E :

$$\begin{cases} \partial_s^2 v(y,s) - \partial_y^2 v(y,s) = g(y,s), & (y,s) \in E, \\ v(0,s) = v(\omega/2,s) = 0, & s \in R^1. \end{cases} \tag{6.2}$$

Here $v(y,s) = u \circ X^{-1}(y,s)$ and $g(y,s) = (1/K(y,s))h \circ X^{-1}(y,s)$. $g(y,s)$ is (η,μ)-q.p. and $\hat{g}(y,\theta)$ is real analytic in Π_r. We have the following proposition.

(C4') β, γ and ω satisfy the Diophantine condition : There exists a positive constant C depending on β, γ and ω such that

$$|(k,\beta) + (j,\gamma) + \pi l/\omega| > \frac{C}{(|k| + |j|)^{m+p+1}}$$

holds for all $(k,j) \in Z^{m+p} \setminus \{0\}$ and all $l \in Z$.

Proposition 6.2 *Assume* (C1), (C3) *and* (C4'). *Then BVP* (6.2) *has a* (η, μ)*-q.p. solution* $v(y,s)$. $\hat{v}(y,\theta)$ *is of* C^∞ *in* $(0,\omega/2) \times T^{m+p}$.

It follows from Proposition 6.2 that $u_3(x,t) = v \circ X(x,t)$ is a (η,μ)-q.p. solution of BVP (1.1)-(1.2') that is of C^∞ in D.

Lemma 6.1 *Let* $u_3(x,0)$ *and* $\partial_t u_3(x,0)$ *be denoted by* $\phi_3(x)$ *and* $\psi_3(x)$ *(resp.). Then* ϕ_3 *and* ψ_3 *satisfy* (C6).

Thus we have obtained the following proposition.

Proposition 6.3 *Assume* (C1), (C3) *and* (C4'). *Then BVP* (1.1)-(1.2') *has a* (η,μ)*-q.p. solution* $u_3(x,t)$ *of* C^∞ *in* D. *Moreover* $\phi_3(x) = u_3(x,0)$ *and* $\psi_3(x) = \partial_t u_3(x,0)$ *satisfy* (C6).

Remark 6.2 From the proof of the above proposition the solution is extended to $R_x^1 \times R_t^1$ if we extend $h(x,t)$ to $R_x^1 \times R_t^1$ as $\tilde{h}(x,t) = g \circ X(x,t)$.

We shall outline the proof of Theorem. Let ϕ_0 and ψ_0 be defined by

$$\phi_0 = \phi - (\phi_1 + \phi_2 + \phi_3), \quad \psi_0 = \psi - (\psi_1 + \psi_2 + \psi_3),$$

where ϕ, ψ are the initial values in (1.1) satisfying (C6), and $\phi_i, \psi_i, i = 1,2$ are defined in Lemma 5.6 : $\phi_i(x) = u_i(x,0)$, $\psi_i(x) = \partial_t u_i(x,0)$. Then since ϕ_i and $\psi_i, i = 1,2$ and ϕ_3 and ψ_3 satisfy (C6) from Lemma 5.6 and Proposition 6.3 (resp.), Then ϕ_0 and ψ_0 also satisfy (C6). Therefore it follows from Lemma 5.4 that u_0 defined by (5.9) is the (ω,η)-q.p. solution of IBVP (5.10) and satisfies the regularity conditions 2.(a) of Theorem. Clearly $u = u_0 + u_1 + u_2 + u_3$ is the unique solution of IBVP (1.1). Thus we have proved the theorem.

7 Quasiperiodic Solutions by the Superposition of Unbounded waves

As we have seen in the previous sections, the solutions of IBVP (1.1) are represented as the superposition of the forward waves and the backward waves that are quasiperiodic in t. The quasiperiodicity of solutions is shown, provided that the Diophantine condition and the differentiability of a_i, r_i, h

are supposed. In this section we shall construct r_i so that every quasiperiodic solution of IBVP (1.1) is the superposition of the forward wave and the backward wave that are *sequentially time-unbounded.* The order of the growth rate of the waves depends on the differentiability of r_i and the order of the Diophantine order. As we stated in section 1, in [Ya4] for the fixed end case, we have already constructed $h(x,t)$ so that every solution of IBVP (1.4) may be time-unbounded. Hence in this section we shall treat the case where $h(x,t)$ identically vanishes. By the similar number-theoretic arguments to [Ya4], we shall take appropriate basic frequencies, and then by use of the basic frequencies we shall construct $r_i(t)$ as lacunary Fourier series for which every solution of IBVP (1.1) is the superposition of sequentially time-unbounded waves. In this section we shall assume that $a_i(t)$, $i = 1,2$, are periodic functions with the same period 1.

Consider IBVP for a linear homogeneous wave equation :

$$\begin{cases} \partial_t^2 u(x,t) - \partial_x^2 u(x,t) = 0, & (x,t) \in D, \\ u(a_1(t),t) = r_1(t), \quad u(a_2(t),t) = r_2(t), & t \in R^1, \\ u(x,0) = \phi(x), \ \ \partial_t u(x,0) = \psi(x), & x \in [a_1(0), a_2(0)]. \end{cases} \tag{7.1}$$

Here $a_i(t)$, $i = 1,2$, are C^∞ periodic functions satisfying (1.2), and $r_i(t)$, $i = 1,2$, are q.p. with basic frequencies $\lambda = (\lambda_1, \cdots, \lambda_m)$ given later. For simplicity we take the periods of $a_i(t)$ as 1. The initial data ϕ, ψ satisfy (C6) in section 4.

First we shall change IBVP (7.1) to IBVP that gives boundary values at fixed x in the same way in section 6.

Consider $A(t)$ in (1.5). Since $a_i(t)$ are periodic, it follows from (1.2) that A is a DS in $D(T^1)$, where $D(T^1)$ is the set of all 1D 1-periodic DSs. Let the rotation number of A be ω. ω is positive ([Ya1]).

Assume the following condition on ω.

(AH) ω satisfies the Diophantine condition : There exist positive constants c and δ such that the Diopahantine inequality holds

$$|q\omega - p| \geq \frac{c}{q^\delta}$$

for any $p \in N$, $q \in N$.

It is well-known that the set of ω that satisfy (AH) for $\delta > 1$ is of full measure in R^1_+.

By (AH) we are able to apply the Herman-Yoccoz Theorem to $A(t)$ ([Ya3]). Then we obtain

$$H^{-1} \circ A \circ H(\xi) = \xi + \omega$$

for $H \in D(T^1)$. Then by the same argument as that of section 6, using the domain transformation X of R^2 onto R^2 defined by

$$\begin{cases} y = \left(H^{-1} \circ A_1^{-1}(-x+t) - H^{-1}(x+t)\right)/2, \\ s = \left(H^{-1} \circ A_1^{-1}(-x+t) + H^{-1}(x+t)\right)/2, \end{cases} \tag{6.1}$$

we transform D onto $K = (0, \omega/2) \times R^1$. By this transformation IBVP (7.1) becomes

$$\begin{cases} \partial_t^2 v(y,s) - \partial_x^2 v(y,s) = 0, \quad (y,s) \in K, \\ v(0,s) = \rho_1(s), \quad v(\omega/2, s) = \rho_2(s), \quad s \in R^1, \\ v(y,0) = \phi_1(y), \quad \partial_t v(y,0) = \psi_1(y), \quad y \in (0, \omega/2), \end{cases} \tag{7.2}$$

where $v(y,s) = u \circ X^{-1}(y,s)$, $\rho_i(s)$ are q.p. functions with basic frequencies $(1, \lambda)$ whose corresponding function $\hat{\rho}_i(\theta)$ have the same smoothness as $\hat{r}_i$ in T^{m+1}, and ϕ_1, ψ_1 are C^∞ in $(0, \omega/2)$ and C^2 in $[0, \omega/2]$ and satisfy suitable compatibility conditions.

In order to show our assertion we shall need some number-theoretic arguments. We shall prepare some lemmas.

Lemma 7.1 *Let a be any natural number. Then there exist countably many a-dimensional real vectors $\zeta = (\zeta_1, ..., \zeta_a)$; $\zeta_i > 0$, $i = 1, ..., a$, with the following property : There exist positive constants C_0 and C_1, and a sequence $\{k_j\} \subset Z^a \setminus \{0\}$ with $|k_j| \to \infty\ (j \to \infty)$ such that*

$$\frac{C_0}{|k_j|^a} \le |(k_j, \zeta)| \le \frac{C_1}{|k_j|^a} \tag{7.3}$$

holds.

This lemma is simply proved by using Theorem VI and Theorem VIII in [Ca], p.13 and p.15 (resp.).

First we shall give the basic frequencies of $r_i(t)$, $i = 1, 2$, by Lemma 7.1. Let a be equal to $m+1$ and $(\zeta_1, \cdots, \zeta_{m+1})$ be the vector given in Lemma 7.1. We take $\lambda = (\lambda_1, \cdots, \lambda_m) = (\zeta_1/\zeta_{m+1}, \cdots, \zeta_m/\zeta_{m+1})$ as the bacic frequencies of $r_i(t)$. Set $\Lambda = (\lambda, 1) \in R^{m+1}$. Then it follows from Lemma 7.1 that there exists a sequence $\{k_j\} \subset Z^{m+1} \setminus \{0\}$ such that

$$\frac{C_2}{|k_j|^{m+1}} \leq |(k_j, \Lambda)| \leq \frac{C_3}{|k_j|^{m+1}}$$

holds for any j. Here C_2 and C_3 are positive constants equal to C_0/ζ_{m+1} and C_1/ζ_{m+1} (resp.).

Remark 7.1 The real vectors $(\zeta_1, \cdots, \zeta_{m+1})$ satisfying (7.3) are constructed as algebraic solutions of some algebraic equations of degree $m+1$. There are infinitely many vectors in R^{m+1} that satisfy (7.3). See [Ca] and [Ya4].

The following lemma is shown similarly to Proposition 2.4 in [Ya4], p.488.

Lemma 7.2 *There exists a subsequence of the above $\{k_j\}$ with the following properties. We again write the subsequence by $\{k_j\}$.*

1. $0 < (k_j, \Lambda) \leq 2\pi/3\omega$ *for any* $j \in N$.
2. $\{(k_j, \Lambda)\}$ *is monotone decreasing* : $(k_{j+1}, \Lambda) \leq (k_j, \Lambda)$.
3. *There exists a positive constant* $M < 1$ *such that* $|k_j| \leq M|k_{j+1}|$ *holds for any* $j \in N$.

Let $\{k_j\}$ be the sequence given in Lemma 7.2. We define $\hat{f}(\theta)$ by a lacunary Fourier series

$$\hat{f}(\theta) = \sum_{j=1}^{\infty} f_j \cos(k_j, \theta).$$

Let the Fourier coefficients f_j satisfy the following : There exists positive constasnts $c_i = c_i(f)$, $i = 1, 2$, such that

$$\frac{c_1}{|k_j|^N} \leq f_j \leq \frac{c_2}{|k_j|^N}$$

for a given $N \in Z_+$. We denote the set of such functions $\hat{f}(\theta)$ by $\hat{Q}^N$ and the set of q.p. functions $f(t)$ defined by $f(t) = \hat{f}(\Lambda t)$ by Q^N_Λ. Clearly Q^N_Λ is a subset of D_Λ.

Lemma 7.3 *Consider a functional equation*

$$F(t+\omega) - F(t) = G(t), \quad t \in R^1, \tag{7.4}$$

where $G(t)$ *is an element of* Q_Λ^N *with* $\int_{T^{m+1}} \hat{G}(\theta)d\theta = 0$. *Let* $m+1 > N$. *Then there exists a solution* $F(t)$ *of* (7.4) *of the form*

$$F(t) = S(t) + G(t)/2$$

that is unique except the difference of constants. $S(t)$ *satisfies the following : There exist positive constants* C, $\tilde{C}$ *and a sequence* $\{t_\nu\}$ *with* $t_\nu \to \infty$ $(\nu \to \infty)$ *such that*

$$C\, t_\nu^{1-N/(m+1)} \leq S(t_\nu) \leq \tilde{C}\, t_\nu^{1-N/(m+1)}, \quad \nu \in N.$$

The proof of this lemma is done in the similar way to that of Theorem 3.1 in [Ya4], p.490-p.495.

Now we show the quasiperiodicity of the solution $v(y,s)$ of IBVP (7.2). v is represented by the superposition of the forward wave and the backward wave by the d'Alembert formula (See section 5)

$$v(y,s) = f(-y+s) - f(y+s) + \rho_1(y+s),$$

where $f(s)$ satisfies

$$f(s+\omega) - f(s) = \rho_1(s+\omega) - \rho_2(s+\omega/2) \equiv \rho(s). \tag{7.5}$$

$\rho(s)$ is q.p. with basic frequencies Λ. We take ρ as an element of Q_Λ^N. We apply Lemma 7.3 to (7.5). Then there exists a solution $f(s)$ of the form

$$f(s) = S(s) + \rho(s)/2,$$

where $S(s)$ satisfies the following : There exist a sequence $\{s_\nu\}$, $s_\nu \to \infty$, and positive constants C, $\tilde{C}$ such that

$$C\, s_\nu^{1-N/(m+1)} \leq S(s_\nu) \leq \tilde{C}\, s_\nu^{1-N/(m+1)}$$

holds.

Next we shall show the quasiperiodicity of the solution v. Since we have

$$\begin{aligned} f(-y+s) - f(y+s) = &\big(S(-y+s) - S(y+s)\big) \\ &+ \big(\rho(-y+s) - \rho(y+s)\big)/2, \end{aligned}$$

and the latter part is q.p. in t, we have only to estimate the former part :

$$\begin{aligned}|S(-y+s)-S(y+s)| &\leq \sum_{j=1}^{\infty} \frac{G_j \cos(k_j,\Lambda_1)}{2\ \sin(k_j,\Lambda_1)} \\ &\times \left|\sin(k_j,\Lambda)(-y+s)-\sin(k_j,\Lambda)(y+s)\right| \\ &\leq \sum_{j=1}^{\infty} G_j \frac{1}{\sin(k_j,\Lambda_1)} \left|\sin(k_j,\Lambda)y\cos(k_j,\Lambda)s\right| \\ &\leq 2C_3 \sum_{j=1}^{\infty} \frac{1}{|k_j|^N} \\ &< +\infty.\end{aligned}$$

This means the quasiperiodicy of $S(-y+s)-S(y+s)$. Therefore our assertion is proved.

Since from the first identity of (6.1) t is written of the form $s+w(y,s)$ with $w(y,s)$ periodic in y, s, t tends to $+\infty$ as s tends to $+\infty$. Since we have, from (6.1),

$$u(x,t) = v \circ X(x,t),$$

it follows that $u(x,t)$ also is the superposition of the sequentially time-unbounded waves. Hence we have seen the conclusion.

8 Outline of Proof of Reduction Theorem

We shall deal with

$$x_1 = Q(x) = x + \omega + q(x), \tag{3.1}$$

and by a suitable transformation

$$x = H(\xi) = \xi + h(\xi) \tag{3.2}$$

reduce (3.1) to the affine mapping

$$\xi_1 = H^{-1} \circ Q \circ H(\xi) = R(\xi) = \xi + \omega. \tag{3.3}$$

Let κ and α be constants in $(0,1)$ and $(1,\infty)$ (resp.). We set $r_0 = r$ and

$$a_\alpha = \sum_{s=1}^{\infty}(1/s^\alpha), \quad d_0 = \min(r_0/(4a_\alpha), 1/(2a_\alpha)),$$
$$d_s = d_0/s^\alpha, \; s = 1, 2, \cdots .$$

We take α so as to satisfy $r_0 > 2d_0$. We define sequences $\{r_s\}, \{\rho_s\}, \{\zeta_s\}$ and $\{M_s\}$ by

$$r_{s+1} = r_s - 2d_s, \quad \rho_s = r_s - d_s, \quad \zeta_s = r_s - d_s/2, \quad M_s = M^{(1+\kappa)^s}$$

for every $s = 0, 1, \cdots$ and $M = |\hat{q}|_r$. For later use we note that

$$r_{s+1} - \rho_s = -d_s, \quad \zeta_{s+1} = \rho_{s+1} + d_{s+1}/2, \quad \rho_s - \zeta_{s+1} = d_s + d_{s+1}/2.$$

In this section all C are positive constants dependent on some or all of $m, \beta, \omega, \alpha, r$.

Consider a sequence of mappings $\{Q_s\}$ of the form

$$(T)_s \qquad x_{1s} = Q_s(x_s) = x_s + \omega + q_s(x_s), \; s = 0, 1, \cdots , \tag{8.1}$$

where q_s is a $2\pi/\beta$-q.p. function and ω is the upper rotation number $\bar{\rho}(Q)(a_0)$, the same constant in (3.1), and for $s = 0$ we set $x_0 = x, x_{10} = x_1, Q_0 = Q$ and $q_0 = q$, where x_1, x, Q, q are seen in (3.1). We shall successively construct a sequence $\{H_s\}$ of transformations in D_β of the variables x_s

$$x_s = H_s(x_{s+1}) = (I + h_s)(x_{s+1}), \; s = 0, 1, \cdots ,$$

where h_s is a $2\pi/\beta$-q.p. function and I is an identity, so that the mappings $(T)_s$ may become closer and closer to the affine mapping (3.3) by the successive transformations. It should be noted that in this process we shall keep the upper rotation number of Q_s fixed i.e., for every $s = 0, 1, \cdots$ there exists $a_s \in R^1$ such that $\bar{\rho}(Q_s)(a_s) = \bar{\rho}(Q)(a_0) = \omega$. This is assured by Proposition 2.1.

We assume

(A_s) $\hat{q}_s(\theta)$ is real analytic in $\hat{\Pi}_{\rho_s}$, continuous in Π_{ρ_s} and satisfies

$$|\hat{q}_s|_{\rho_s} \le M_s.$$

The following proposition is fundamental to prove Reduction Theorem.

Proposition 8.1 *Consider a mapping* (8.1). *Assume that* (C) *holds. Then there exists a positive constant* $M^0 = M^0(\kappa, \alpha, \|\beta\|, r, m, C_0)$ *independent of* s *such that for any* $M \in [0, M^0)$, *under the assumption* (A_s) *the mapping* $(T)_s$ *is transformed to*

$$(T)_{s+1} \qquad x_{1s+1} = Q_{s+1}(x_{s+1}) = x_{s+1} + \omega + q_{s+1}(x_{s+1}) \tag{8.2}$$

by a transformation with a $2\pi/\beta$*-q.p. term* h_s

$$x_s = H_s(x_{s+1}) = (I + h_s)(x_{s+1}) \tag{8.3}$$

with the following properties :

1. q_{s+1} *is a* $2\pi/\beta$*-q.p. function, and* $\hat{q}_{s+1}$ *is real analytic in* $\hat{\Pi}_{\rho_{s+1}}$, *continuous in* $\Pi_{\rho_{s+1}}$ *and satisfies*

$$|\hat{q}_{s+1}|_{\rho_{s+1}} \le M_{s+1} = M_s^{1+\kappa}. \tag{8.4}$$

2. h_s *is a solution of a functional equation*

$$h_s(x + \omega) - h_s(x) = q_s(x) - \nu_s, \tag{8.5}$$

where q_s *is expanded into the Fouriuer series*

$$q_s(x) = \sum_{n \in Z^m} q_s^n e^{i(n,\beta)x}, \quad q_s^n = (1/2\pi)^m \int_{T^m} \hat{q}_s(\theta) e^{-i(n,\theta)} d\theta, \tag{8.6}$$

and ν_s *is given by*

$$\nu_s = q_s^0 = (1/2\pi)^m \int_{T^m} \hat{q}_s(\theta) d\theta. \tag{8.7}$$

$\hat{h}_s$ *is real analytic in* $\hat{\Pi}_{r_{s+1}}$, *continuous in* $\Pi_{r_{s+1}}$ *and satisfies*

$$\begin{aligned} |\hat{h}_s|_{r_{s+1}} &\le M_s^{(5\kappa+3)/8}/10 \le M_s^{\kappa}/10, \\ |D_\theta \hat{h}_s|_{r_{s+1}} &\le M_s^{(5\kappa+3)/8}/10 \le M_s^{\kappa}/10. \end{aligned} \tag{8.8}$$

In the following lemmas in this section M is taken suitably small.

Outline of Proof of Proposition 8.1.

Proof of Proposition 8.1 shall be done by several steps.

1. *Estimates of* $\hat{h}_s$ *and* $\partial_\theta \hat{h}_s$

We show (8.8). Setting

$$h_s(x) = \sum_{n \in Z^m} h_s^n e^{i(n,\beta)x},$$

we obtain, from (8.5) with (8.6) and (8.7),

$$h_s^n = q_s^n/(e^{i(n,\beta)\omega} - 1)$$

for every $n \neq 0$. Since h_s^0 is arbitrary, we take $h_s^0 = 0$. For $\theta \in \hat{\Pi}_{r_{s+1}}$ we have

$$|\hat{h}_s(\theta)| \leq \sum |h_s^n| e^{|n||\Im\theta|} \leq \sum |h_s^n| e^{|n| r_{s+1}},$$

where $|n| = |n_1| + \cdots + |n_m|$ for $n = (n_1, \cdots, n_m)$. By (C) we have

$$\begin{aligned} |e^{i(n,\beta)\omega} - 1| &\geq \frac{2}{\pi} |(n,\beta)\omega - \pi l| \\ &\geq \frac{2|\omega| C_0 \pi}{|n|^{m+1}} \end{aligned}$$

for a suitable $l \in Z$. Since $|q_s^n| \leq M_s \exp(-|n|\rho_s)$ for $n \in Z^m$, we have

$$\begin{aligned} |\hat{h}_s(\theta)| &\leq (\pi/2|\omega|C_0) \sum |n|^{m+1} M_s\, e^{-|n|\rho_s}\, e^{|n| r_{s+1}} \\ &= (\pi/2|\omega|C_0)\, M_s \sum |n|^{m+1} e^{-|n| d_s}. \end{aligned}$$

Since

$$\sum_{n \in Z^m} |n|^k e^{-|n|d} \leq C_m d^{-(k+m+1)}$$

holds for any positive d, it follows that

$$|\hat{h}_s(\theta)| \leq C M_s / d_s^{2m+2}.$$

Take M sufficiently small. Then we have

$$|\hat{h}_s(\theta)| \leq M_s^{(5\kappa+3)/8}/10 \leq M_s^{\kappa}/10$$

in $\Pi_{r_{s+1}}$. Hence the first inequality of (8.8) is proved. Similarly we can obtain the second inequality. It is clear to show that $h_s(x)$ is real-valued for each real x. In fact, $h_s^{-n} = \overline{h_s^n}$ holds.

2. *Estimate of* $\hat{q}_{s+1}$

We denote H_s^{-1} by $I + g_s$. Clearly $g_s = -h_s \circ H_s^{-1}$.

Lemma 8.1 *g_s is $2\pi/\beta$-q.p., and $\hat{g}_s$ is real analytic in $\Pi_{\zeta_{s+1}}$.*

This lemma proved in [S-M], p.261-p.263.

Lemma 8.2 *For any $\Theta \in \Pi_{\zeta_{s+1}}$ there exists $\theta \in \Pi_{r_{s+1}}$ such that*

$$\Theta = \theta + \beta \hat{h}_s(\theta).$$

For the proof of the lemma, see [Ya0].

From (8.1), (8.2), (8.3) and (8.5) we have

$$\begin{aligned} q_{s+1}(x_{s+1}) = (I + h_s)^{-1}\Big((I + h_s)(x_{s+1} + \omega) + \nu_s \\ + [q_s \circ (I + h_s)(x_{s+1}) - q_s(x_{s+1})]\Big) - (x_{s+1} + \omega). \end{aligned} \tag{8.9}$$

Note that $q_{s+1}(x)$ is real-valued for real x. For brevity set $x = x_{s+1}$ and

$$J_0 = (I + h_s)(x + \omega), \quad J_1 = q_s \circ (I + h_s)(x) - q_s(x), \quad J = J_1 + \nu_s.$$

Using the mean-value theorem, we obtain

$$q_{s+1}(x) = J \int_0^1 \left((I + h_s)^{-1}\right)' (J_0 + tJ)dt. \tag{8.10}$$

First we estimate J.

Lemma 8.3 *J satisfies*

$$|\hat{J}|_{\rho_{s+1}} \leq (1/5)M_s^{\kappa+1}.$$

Lemma 8.4 *The inverse of $I + h_s$ satisfies*

$$\left|((I + h_s)^{-1})'(\theta)\right|_{\zeta_{s+1}} \leq 4/3.$$

For the proofs of these lemmas, see [Ya0].

From (8.9) and Lemma 8.2, $\hat{q}_{s+1}$ is real analytic in $\Pi_{\rho_{s+1}}$. We estimate $\hat{q}_{s+1}$. Using (8.10), Lemmas 8.3 and 8.4, we have

$$|\hat{q}_{s+1}|_{\rho_{s+1}} \leq (M_s^{1+\kappa}/5)(4/3) \leq M_s^{1+\kappa} = M_{s+1}.$$

This proves (8.4). Thus Proposition 8.1 is proved.

Now using Proposition 8.1, we shall prove the Reduction Theorem. Set

$$G_s = H_1 \circ H_2 \circ \ldots \circ H_s, \quad F_s = G_s - I.$$

Lemma 8.5 *The sequence $\{\hat{F}_s\}$ converges uniformly to a function $\hat{F}$ in $\{|\Im\theta| \leq r/2\}$.*

Let H be the limit of the sequence $\{G_s\}$. Since

$$G_s^{-1} \circ Q \circ G_s(x) = x + \omega + q_s(x)$$

and $q_s(x)$ converges to 0 as $s \to \infty$, we obtain

$$H^{-1} \circ Q \circ H(x) = x + \omega.$$

Therefore Reduction Theorem is proved.

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