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COMMUTING SINGULAR VECTOR FIELDS WITH LINEAR PARTS HAVING JORDAN BLOCKS

カリアリ大学・理学部　トドー　グラムチェフ (Todor Gramchev)[1]
Dipartimento di Matematica,
Università di Cagliari

ABSTRACT. We consider the problem of the simultaneous linearization of commuting singular analytic vector fields in $\mathbb{K}^n$, $\mathbb{K}=\mathbb{C},\mathbb{R}$, with non-semisimple linear parts. We investigate the solvability under compatibility conditions of overdetermined systems of linear homological equations. We also examine the influence of the presence of Jordan blocks for intersections of foliations defined by two commuting real vectors fields in $\mathbb{R}^{2n}$ with odd-dimensional spheres.

Key words: commuting singular vector fields, simultaneous linearization, Jordan blocks, homological equations, Diophantine conditions, transversal intersections

MSC classification: 37C15, 32M25, 37C27.

1. INTRODUCTION

We investigate simultaneous linearization of d of commuting analytic vector fields $X^1,\dots,X^d$ having a common singular point at $0\in\mathbb{K}^n$, $\mathbb{K}=\mathbb{C}$ or $\mathbb{K}=\mathbb{R}$ with

$$X^j=\langle X^j(x),\partial_x\rangle=\sum_{k=1}^{n}X_k^j(x)\partial_{x_j},\qquad j=1,\dots,d, \tag{1.1}$$

and

$$X^j(x)=A^jx+R^j(x),\qquad R^j(x)=O(|x|^2),\ |x|\to 0. \tag{1.2}$$

where $A^j\in M_n(\mathbb{K})$ (the set of all $n\times n$ matrices with entries from $\mathbb{K}$), $R^j\in C^\omega(\Omega:\mathbb{K}^n)$, $\Omega\subset\mathbb{K}^n$ being an open neighbourhood of $0\in\mathbb{K}^n$, $C^\omega(\Omega:\mathbb{K}^n)$ stands for the space of the analytic vector valued functions from Ω to $\mathbb{K}^n$. We emphasize that we do not require semisimpleness of the linear parts A^j, $j=1,\dots,d$.

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A more general and invariant setting is to consider a germ of singular infinitesimal $\mathbb{K}^d$ ($d \geq 2$) actions of class C^B with $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, and $B = \infty$, $B = \omega$ or $B = k$ for some $k > 0$, namely a Lie algebra homomorphism

$$\rho : \mathbb{K}^d \longrightarrow \mathcal{G}^n_B, \tag{1.3}$$

where $\mathcal{G}^n$ denotes a d–dimensional Lie algebra of germs at $0 \in \mathbb{K}^n$ of analytic vector fields vanishing at 0. We denote by $Act(\mathbb{K}^d : \mathbb{K}^n)$ the set of germs of singular infinitesimal analytic $\mathbb{K}^d$ actions in $0 \in \mathbb{K}^n$. It is well known (e.g., cf. [5], [9]) that, by choosing a basis $e_1, \ldots, e_d$ in $\mathbb{K}^n$, the infinitesimal action can be identified with a d–tuple of germs at 0 of commuting vector fields $\rho(e_1), \ldots, \rho(e_d)$. Given $\rho \in Act(\mathbb{K}^d : \mathbb{K}^n)$ and a basis $e_1, \ldots, e_d$ in $\mathbb{K}^d$ we set $X^j = \rho(e_j)$, $j = 1, \ldots, d$. We can define, in view of the commutativity relation, the action

$$\tilde{\rho} : \mathbb{K}^d \times \mathbb{K}^n \longrightarrow \mathbb{K}^n, \tag{1.4}$$

$$\begin{aligned} \tilde{\rho}(s; z) &= X^1_{s_1} \circ \cdots \circ X^d_{s_d}(z) \\ &= X^{\sigma_1}_{s_{\sigma_1}} \circ \cdots X^{\sigma_d}_{s_{\sigma_d}}(z), \quad s = (s_1, \ldots, s_d), \end{aligned} \tag{1.5}$$

for all permutations $\sigma = (\sigma_1, \ldots, \sigma_d)$ of $\{1, \ldots, d\}$, where X^j_t denotes the flow of X^j. We denote by ρ_{lin} the linear action formed by the linear parts of the vector fields defining ρ.

We shall investigate the linearization of ρ, namely, whether there exists an analytic diffeomorphism $x = y + v(y)$,

$$v(y) = \sum_{\alpha \in \mathbb{Z}^n_+(2)} v_\alpha y^\alpha, \qquad v_\alpha \in \mathbb{C}^n, \tag{1.6}$$

such that u conjugates simultaneously $X^1, \ldots, X^d$ into their corresponding linear parts $X^1_{lin}, \ldots, X^d_{lin}$. Here $y^\alpha = y_1^{\alpha_1} \ldots y_n^{\alpha_n}$ for a multi-index $\alpha \in \mathbb{Z}^n_+$, and $\alpha \in \mathbb{Z}^n_+(2)$ means $|\alpha| := \alpha_1 + \ldots + \alpha_n \geq 2$. It is well known that this is means that the unknown function (or forma power series) v satisfies a system of d homological equations

$$L_{A_j} v(x) := \langle A_j x, \partial_x \rangle v - A_j v = R^j(x + v(x)), \qquad j = 1, \ldots, d \tag{1.7}$$

Let $\mathbb{K}^n_2\{x\}$ (respectively, $\mathbb{K}^n_2[x]$) be the set of n vector functions of convergent (respectively, formal) power series of $x \in \mathbb{K}^n$ without constant and linear terms. We will also investigate the solvability of the linear version of the system (1.7)

$$L_{A_j} v(x) = f^j(x), \qquad j = 1, \ldots, d \tag{1.8}$$

where $f = (f^1, \ldots, f^d) \in (\mathbb{K}^n_2\{x\})^d$. Similar equations appear in the study of simultaneous normal forms for commuting holomorphic maps. We note that if $d \geq 2$ the system becomes overdetermined. We recall that in [9], [17], [29] (see also [7], [8], [30], where normal forms in the presence of symmetries have been investigated) the linear parts were supposed to be diagonalizable, while in [35] the existence of analytic first integrals was required. We point out that even for a convergent normal forms of a single analytic vector field or a biholomorphic map in the Siegel domain the results have been usually proved under the assumption of semisimple linear parts cf. [2], [13], [20]. On the other hand, the celebrated results of A. Bruno [3] for convergent normal forms are proved for some cases where the linear part of a singular analytic vector field admits Jordan blocks (the so called (A) condition) plus the arithmetic Bruno condition (ω).

Recently, linearization of single maps and vector fields in a Siegel domain with nontrivial Jordan blocks in the linear part have been investigated (cf. [11], [15], [33], see also [1] where Jordan blocks appear not in the linearization of biholomorphic maps but in an interplay between holomorphic dynamics and singularity theory). We refer to [15], [11], [32] for divergent solutions of a single linear homological equation $(d = 1)$ in the presence of Jordan blocks, implying, in virtue of the general abstract approach in [23], [27], to nonlinerazition results for maps and vector fields (i.e., divergent formal transformations $y + v(y)$).

We mention as another motivation recent results on the solvability in Gevrey classes of first order linear singular equations admitting Jordan blocks (see [16], [21], [22]).

The second goal of our investigations is to generalize results on the intersections of complex flows in the Poincaré and the Siegel domain with odd–dimensional spheres cf. [6], [19], where the linear part is supposed to be diagonalizable). We will outline the new phenomena in the presence of nontrivial Jordan blocks.

One essential ingredient of our approach is to rely on a classical result for the simultaneous reduction of commuting matrices to an upper triangular form (e.g., see [25] where this has been used in the study of the action of commuting hyperbolic diffeomorphisms of the n–dimensional torus $\mathbb{T}^n$). More precisely, we can find a positive integer $m \leq n$ such that $\mathbb{K}^n$ is decomposed into a direct sum of m linear subspaces invariant under all $A^\ell = \nabla X_\ell(0)$ $(\ell = 1, \dots, d)$:

$$(1.9) \qquad \mathbb{K}^n = \mathbb{I}^{s_1} + \cdots + \mathbb{I}^{s_m}, \qquad \dim \mathbb{I}^{s_j} = s_j,\ j = 1, \dots, m, \\ s_1 + \cdots + s_m = n.$$

The minimal polynomial of the Jacobian matrix A^ℓ over $\mathbb{I}^{s_j}$ is a power of an irreducible polynomial $p_{j\ell}(x)$ over $\mathbb{C}$ (respectively, over $\mathbb{R}$). The matrices $A^1, \dots, A^d$ can be simultaneously brought in an upper triangular form, and we write again A^ℓ for the matrices,

$$(1.10)\qquad A^\ell = \begin{pmatrix} A_1^\ell & 0_{s_1\times s_2} & \dots & 0_{s_1\times s_m} \\ 0_{s_2\times s_1} & A_2^\ell & \dots & 0_{s_2\times s_m} \\ \vdots & \vdots & \vdots & \vdots \\ 0_{s_m\times s_1} & 0_{s_m\times s_2} & \dots & A_m^\ell \end{pmatrix}, \quad \ell = 1, \dots, d.$$

If $\mathbb{K} = \mathbb{C}$, the matrix A_j^ℓ is given by

(1.11)

$$A_j^\ell = \begin{pmatrix} \lambda_j^\ell & A_{j,12}^\ell & \dots & A_{j,1s_j}^\ell \\ 0 & \lambda_j^\ell & \dots & A_{j,2s_j}^\ell \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_j^\ell \end{pmatrix}, \ \ell = 1, \dots, d, \ j = 1, \dots, m,$$

with $\lambda_j^\ell, A_{j,\nu\mu}^\ell \in \mathbb{C}$. Next, if $\mathbb{K} = \mathbb{R}$ we have, for every fixed $j \in \{1, \dots, m\}$ two possibilities: firstly the minimal polynomials of all A^ℓ on $\mathbb{I}^{s_j}$ are powers of monomials: $p_{\ell j}(t) = (t - \lambda_j^\ell)^{s_\ell}$, $\lambda_j^\ell \in \mathbb{R}$, $\ell = 1, \dots, d$. Then all A_j^ℓ ($\ell = 1, \dots, d$) are given by (1.11) with $\lambda_j^\ell \in \mathbb{R}$. Secondly, there exists ℓ, $1 \le \ell \le d$ such that the minimal polynomial of A_j^ℓ on $\mathbb{I}^{s_j}$ is a power of irreducible quadratic polynomial with complex conjugate roots $\lambda_j^\ell \pm i\mu_j^\ell$. Then $s_j = 2\tilde{s}_j$ is even and A_j^ℓ is a $\tilde{s}_j \times \tilde{s}_j$ square block matrix

(1.12)

$$A_j^\ell = \begin{pmatrix} R_2(\lambda_j^\ell, \mu_j^\ell) & A_{\ell,j}^{12} & \dots & A_{\ell j}^{1\tilde{s}_j} \\ 0 & R_2(\lambda_j^\ell, \mu_j^\ell) & \dots & A_{\ell j}^{2\tilde{s}_j} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & R_2(\lambda_j^\ell, \mu_j^\ell) \end{pmatrix}, \quad \ell = 1, \dots, d,$$

where

$$(1.13)\qquad R_2(\lambda, \mu) := \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}, \qquad \lambda, \mu \in \mathbb{R},$$

and the matrices

$$(1.14)\quad A_{\ell j}^{rs} = R_2(\lambda_{\ell j}^{rs}, \mu_{\ell j}^{rs}), \quad \lambda_{\ell j}^{rs}, \mu_{\ell j}^{rs} \in \mathbb{R}, \ \ell = 1, \dots, d, 1 \le r < \tilde{s}_j,$$

are 2×2 real matrices provided $\tilde{s}_j \ge 2$.

We define the diagonal part of A_j^ℓ by $A_j^\ell(diag) := \lambda_j^\ell I_{s_j}$ (respectively, $A_j^\ell(diag) := \mathrm{diag}\,\{R_2(\lambda_j^\ell, \mu_j^\ell), \dots, R_2(\lambda^\ell \mathrm{J}, \mu_j^\ell)\}$) provided $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$ and $\lambda_j^\ell \in \mathbb{R}$, $\ell = 1, \dots, d$ (respectively, $\mathbb{K} = \mathbb{R}$, $s_j = 2\tilde{s}_j$ and $\mu_j^\ell \ne 0$ for at least one $\ell \in \{1, \dots, d\}$). The nilpotent parts are defined by $A_j^\ell(nil) := A_j^\ell - A_j^\ell(diag)$. We decompose in a natural way the linear action into the diagonal part ρ_{diag} and the nilpotent part ρ_{nil}.

Throughout the paper we assume that d vectors in $\mathbb{K}^n$ formed by the diagonal parts (respectively, the diagonal elements of the 2×2 real matrices in the real Jordan block form) if $\mathbb{K} = \mathbb{C}^n$ (respectively, $\mathbb{K} = \mathbb{R}$) are linearly independent. Following the decomposition (1.10) (respectively, (1.11)) we define $\tilde{\lambda^j}$ by

$$(1.15) \qquad \tilde{\lambda^k} = (\lambda_1^k, \ldots, \lambda_m^k) \in \mathbb{K}^m, \qquad k = 1, \ldots, d.$$

Clearly

$$(1.16) \qquad \tilde{\lambda^1}, \cdots, \tilde{\lambda^d} \text{ are linearly independent in } \mathbb{K}^m,$$

which implies

$$(1.17) \qquad d \leq m.$$

One can easily see that (1.16) is invariantly defined.

We also define

$$(1.18) \quad \lambda^k = (\underbrace{\lambda_1^k, \ldots, \lambda_1^k}_{n_1 \text{ times}}, \ldots, \underbrace{\lambda_m^k, \ldots, \lambda_m^k}_{n_m \text{ times}}) \in \mathbb{K}^n, \qquad k = 1, \ldots, d.$$

The decomposition (1.9) leads in a natural way to the following notations: give $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ we write $\alpha = (\alpha^1, \ldots, \alpha^m)$, where $\alpha^j \in \mathbb{Z}_+^{s_j}$, $j = 1, \ldots, m$. We set $\tilde{\alpha} = (|\alpha^1|, \ldots, |\alpha^m|) \in \mathbb{Z}_+^m$. Given a positive integer k we define $\mathbb{Z}_+^m(k) = \{\alpha \in \mathbb{Z}_+^m;\ |\alpha| \geq k\}$.

Set

$$(1.19) \quad \tilde{\omega_j}(\tilde{\alpha}) = \sum_{\nu=1}^{d} |\langle \tilde{\lambda}^\nu, \tilde{\alpha} \rangle - \tilde{\lambda}_j^\nu|, \qquad j = 1, \ldots, m, \ \tilde{\alpha} \in \mathbb{Z}_+^m(2)$$

$$(1.20) \quad \tilde{\omega}(\tilde{\alpha}) = \min\{\tilde{\omega_1}(\tilde{\alpha}), \ldots, \tilde{\omega_m}(\tilde{\alpha})\}, \qquad \tilde{\alpha} \in \mathbb{Z}_+^m(2),$$

$$(1.21) \quad \omega_j(\alpha) = \sum_{\nu=1}^{d} |\langle \tilde{\lambda}^\nu, \alpha \rangle - \lambda_j^\nu|, \qquad j = 1, \ldots, n,$$

$$(1.22) \quad \omega(\alpha) = \min\{\omega_1(\alpha), \ldots, \omega_n(\alpha)\}, \qquad \alpha \in \mathbb{Z}_+^n(2),$$

Note that

$$(1.23) \qquad \omega(\alpha) = \tilde{\omega}(\tilde{\alpha}), \qquad , \alpha \in \mathbb{Z}_+^n(2).$$

Definition 1.1. *We say that the $X^1, \ldots, X^d$ are simultaneously non-resonant if*

$$(1.24) \qquad \omega(\alpha) \neq 0, \qquad \forall \alpha \in \mathbb{Z}_+^n(2).$$

If (1.24) holds we say in short that the action ρ is simultaneously non-resonant.

Clearly the simultaneously nonresonant condition is invariant under the change of the basis $A^1, \dots, A^d$.

The first main result of our paper concerns the solvability of the system of linear homological equations (LHE) given by (1.8) in the presence of nontrivial Jordan blocks in some the linear parts A^j. This is done in section 2.

Formal linearization results and an analogue of simultaneous linearization under a simultaneous analogue of the Poincaré domain are presented in section 3.

Finally, we discuss transversal intersections with odd–dimensional spheres of 2 dimensional flows defined by two commuting real matrices in section 4.

2. Overdetermined systems of homological equations

The main goal of these section is to derive an explicit algorithm for compatibility conditions involving the RHS f^j, $j = 1, \dots, d$, in order the system (1.8) to be at least formally solvable.

Theorem 2.1. *Assume that $A^1, \dots, A^d$ are simultaneously nonresonant. Then (1.8) is formally solvable if and only if f satisfies*

$$L_{A_j} f_k = L_{A_k} f_j, \qquad j, k = 1, \dots, d. \tag{2.1}$$

In that case there exists a unique formal solution $S[f] \in \mathbb{C}_2^n[x]$ for every $f \in (\mathbb{C}_2^n[x])^d$. In addition, if $S[f]$ is convergent (i.e., belongs to $\mathbb{C}_2^n\{x\}$) for every $f \in (\mathbb{C}_2^n\{x\})^d$ then the following simultaneous arithmetic condition holds:

$$\inf_{\alpha \in \mathbb{Z}_+^n(2)} \exp(\varepsilon|\alpha|)\omega(\alpha) = \inf_{\alpha \in \mathbb{Z}_+^n(2)} \exp(\varepsilon|\alpha|)\tilde{\omega}(\tilde{\alpha}) > 0 \tag{2.2}$$

for every $\varepsilon > 0$.

Proof. If A_j are semisimple, namely $A_j = \operatorname{diag}\{\lambda_1^j, \dots, \lambda_n^j\}$, per $j = 1, \dots, n$, and the system of LHE is equivalent to

$$((\lambda^j, \alpha) - \lambda_k^j) v_{\alpha;k} = f_{\alpha;k}^j, \qquad j = 1, \dots, d, \ k = 1, \dots, m \tag{2.3}$$

for $\alpha \in \mathbb{Z}_+^n(2)$, where $\lambda^j := (\lambda_1^j 1_{s_1}, \dots, \lambda_m^j 1_{s_m}) \in \mathbb{Z}_+^n$, with 1_p standing for $(1, \dots, 1) \in \mathbb{N}^p$, $p \in \mathbb{N}$ and

$$v_\alpha = \begin{pmatrix} v_{\alpha;1} \\ \vdots \\ v_{\alpha;m} \end{pmatrix} \in \mathbb{C}^n, \quad v_{\alpha;k} = \begin{pmatrix} v_{\alpha;k,1} \\ \vdots \\ v_{\alpha;k,1} \end{pmatrix} \in \mathbb{C}^{s_k} \tag{2.4}$$

We are working in $\mathbb{C}$, if $\mathbb{K} = \mathbb{R}$ and the vector fields are real, as in the case of single nonresonant vector fields, the unique solution of the homological equations will be real valued as well. Note that in view of the definition of $\tilde{\alpha}$ and $\tilde{\lambda}^j$ we have

(2.5) $$\langle \lambda^j, \alpha \rangle = \langle \tilde{\lambda}^j, \tilde{\alpha} \rangle, \qquad j = 1, \dots, d, \alpha \in \mathbb{Z}_+^n(2)$$

Then the compatibility condition (2.1) are written as follows

(2.6)
$$((\lambda^j, \alpha) - \lambda_k^j) f_{\alpha;k}^\ell = ((\lambda^\ell, \alpha) - \lambda_k^\ell) f_{\alpha;k}^j, \quad j, \ell = 1, \dots, d, \ k = 1, \dots, m$$

and we have

(2.7) $$v_{\alpha;k} = \frac{f_{\alpha;k}^j}{(\langle \lambda^j, \alpha \rangle - \lambda_k^j)}$$

for some $j = j(\alpha, k)$ provided

(2.8) $$\langle \lambda^j, \alpha \rangle - \lambda_k^j \neq 0,$$

$k = 1, \dots, m$, $\alpha \in \mathbb{Z}_+^m(2)$. We note that the simultaneous nonresonance condition implies that for every given $k \in \{1, \dots, m\}$, $\alpha \in \mathbb{Z}_+^n(2)$ we can find j satisfying (2.8). Moreover, in view of (2.3) the definition of $v_{\alpha;k}$ is independent from j and the following estimate holds

(2.9) $$|v_{\alpha;k}| \leq \frac{\max_{j=1,\dots,d} |f_{\alpha;k}^j|}{\max_{j=1,\dots,d} |\langle \lambda^j, \alpha \rangle - \lambda_k^j|}$$

and then the proof of (2.2) is straightforward.

In the general case, when nontrivial Jordan blocks appear, we need a decomposition of the lattice $\mathbb{Z}^n$. Let g be expanded into the power series, $g(x) = \sum_{|\alpha| \geq 2} g_\alpha x^\alpha$. Set

(2.10) $$g_k^\beta = \{g_{\alpha;k}\}_{|\tilde{\alpha}=\beta}$$

for $\beta \in Z_+^n(2)$, $k = 1, \dots, m$. We define the linear finite dimensional spaces of polynomials

(2.11) $$S_k^\beta = \{\sum_{|\tilde{\alpha}=\beta} g_{\alpha;k} x^\alpha; g_{\alpha;k} \in \mathbb{C}^{n_k}\}$$

(2.12) $$S^\beta = \{\sum_{|\tilde{\alpha}=\beta} g_\alpha x^\alpha; g_\alpha \in \mathbb{C}^n\}$$

By the simultaneous upper triangular canonical form of $A_1, \dots, A_m$ we get that the system of LHE acts invariantly on S^β and S_k^β for $\beta \in \mathbb{Z}_+^m(2)$, $k = 1, \dots, m$. Next, we have a crucial representation

(2.13) $$\langle A_j, \partial_x \rangle \sum_{\alpha \in Z_+^n(2)} g_{\alpha;k} x^\alpha = \sum_{\alpha \in \mathbb{Z}_+^n(2)} ((\lambda^j, \alpha) - \lambda_k^j) g_{\alpha;k} + M_k^j(\alpha)[g_k^{\tilde{\alpha}}]$$

where

(2.14) $\qquad M_k^j(\alpha)$ is a linear nilpotent operator in $S_k^{\tilde{\alpha}}$,

for $\alpha \in \mathbb{Z}_+^n(2)$, $j = 1, \dots, d$, $k = 1, \dots, m$. In particular.

$$M_k^j(\alpha)[g_k^{\tilde{\alpha}}] = 0 \text{ if } \alpha = \alpha^{;0} \tag{2.15}$$

where $\alpha^{;0} = (\alpha^{1;0}, \dots, \alpha^{m;0})$, $\alpha^{k;0} = (|\alpha^k|, 0, \dots, 0)$. Note that the simultaneous nonresonance condition and (2.14) we get both the explicit recurrent definition of the compatibility conditions on the right–hand sides f_α^j as well as the explicit resolution of the overdetermined systems of LHE. □

Next, we get readily an analogue in the commuting case of a formal simultaneous linearization.

Theorem 2.2. *Let the action ρ satisfy the simultaneous nonresonant nonresonant condition (1.24). Then there exist a formal change of the variables $x = u(y)$, such that it linearizes simultaneously $X^1, \dots, X^d$. Moreover, for every integer $N \geq 2$ we can find a polynomial change of the variables $x = u^N(y)$ such that*

$$u_*^N X^j = \langle A_j y + R^{j;N}(y), \partial_y \rangle, \qquad j = 1, \dots, d, \tag{2.16}$$

where

$$R^{j;N}(y) = O(|y|^{N+1}), \qquad j = 1, \dots, d, \tag{2.17}$$

Here $u_*^N X$ stands for the transformation of the vector field X (defined in the coordinates $x = (x_1, \dots, x_n)$) in the new coordinates $y = (y_1, \dots, y_n)$.

3. Convergent simultaneous Poincaré–Dulac normal forms

We will say that the family of commuting vector fields $X^1, \dots, X^d$ (or equivalently, the action ρ) satisfies the simultaneous Pincaré condition if and only if there exist real numbers c_j $j = 1, \dots, d$ such that

(3.1) $\qquad \widetilde{A^j} := \sum_{j=1}^{d} c_j A^j$ is a vector field in the Poincaré domain.

We have

Theorem 3.1. *Let the action ρ satisfy the simultaneous Poincaré condition and (1.24). Then ρ is linearizable via an analytic transformation.*

Proof. Choose an index j such that $c_j \neq 0$, where $c_1, \dots, c_d$ are the numbers in (3.1). Then we replace X^j by

$$\widetilde{X^j} = \sum_{j=1}^{d} c_j X^j.$$

The vector fields $X^1, \dots X^{j-1}, \widetilde{X^j}, X^{j+1}, \dots, X^d$ are pairwise commuting. In view of the classical Poincaré–Dulac theorem we can find an analytic change of the variables $x = u(y)$ such that $\widetilde{X^j}$ is transformed to

$$u_* \widetilde{X^j} = \widetilde{Y^j} = \langle \widetilde{A^j} + Res^j(y), \partial_y \rangle$$

where $Res^j(y)$ is a polynomial of resonant monomials. By the simultaneous nonresonace condition (1.24) and Theorem 2.2 we obtain that $Res^j(y) \equiv 0$ (possibly after additional polynomial change of the variables, we use the same letter $\widetilde{X^j}$) and

$$Y^k : 0 = u_* X^k = \langle A^k + O(|y|^N), \partial_y \rangle, \qquad k \neq j$$

The commutativity is coordinate invariant property. On the other hand, if N is large enough, the LHE defined by the vector field in (3.1) is nonresonant acting on homogeneous polynomials of degree $\geq N$. In view of the analyticity and the commutation with Y^k, $k \in \{1, \dots, m\}$, $k \neq j$, we obtain that all Y^k, $k \in \{1, \dots, m\}$, $k \neq j$, must be linear as well. We conclude the proof by observing that $Y^j = u_* X^j$ is a linear combination of $\widetilde{Y^j}$ and Y^k, $k \in \{1, \dots, m\}$, $k \neq j$. □

Remark 3.2. *Comparing with the case of a single LHE, cf.* [15], *the presence of Jordan blocks in commuting vector fields requires apparently new approach for dealing with the simultaneous Diophantine type conditions appearing when the simultaneous Poincaré condition is not satisfied. We refer to* [18], *where such problems are investigated by using ideas from the theory of the simultaneous Diophantine approximations (cf.* [12], [26], [28], [34]*). We mention also that the simultaneous arithmetic condition (2.2) for the convergence of the solutions of the linear system (1.8) is less restrictive than the simultaneous Bruno type condition in* [29] *(cf.* [31] *for similar comments about the Bruno condition for the Siegel centralizer problem). In fact, in view of the impossibility in the general case to reduce commuting matrices in the same canonical Jordan block structures (cf.* [14] *for the description of the centralizers of matrices), additional difficulties appear in showing convergent normal forms for* $X^1, \dots, X^d$ *when* $d \geq 2$ *and some of the matrices* $A^1, \dots, A^d$ *admit nontrivial Jordan blocks (see* [18]*).*

4. Transversal intersections for 2-dimensional linear actions in $\mathbb{R}^{2n}$

We are interested in the intersections of real 2-dimensional integral manifolds of two commuting linear singular vector fields in $\mathbb{R}^{2n}$ with the unit sphere S^{2n-1}. If the intersection is transversal, it defines in a natural way integral curves of a smooth tangential nonsingular vector field X on S^{2n-1}. We recall that such problems have been investigated for linear complex flows in $\mathbb{C}^n$ under non degeneracy assumptions excluding the presence of Jordan blocks, cf. [19], where it is shown in particular that if all eigenvalues are distinct, belong to the Poincaré domain and no two of the eigenvalues of A lie on the same line through the origin, then the intersection of the linear flow with $S^{2n-1}(r)$ is transversal, induces a nonsingular smooth tangent vector field X_r which admits exactly n closed orbits. For further generalizations and deep results for intersections of complex flows in the Siegel domain with S^{2n-1} we refer to [5], [6], see also [24]. In particular, for $n = 2$, the problem for transversal intersections is related to the Seifert conjecture, namely the existence of cycles of smooth non-vanishing vector field on S^3.

Our aim is to generalize some of the aforementioned results for a particular classes of $2 - D$ linear actions in $\mathbb{R}^{2n}$ admitting allowing Jordan blocks. We consider 2–D linear action ρ in $\mathbb{R}^{2n}$, defined by two commuting $2n \times 2n$ matrices A and B which satisfy the following condition

$$(4.1) \qquad Ax \text{ and } Bx \text{ are linearly independent for every } x \in \mathbb{R}^{2n} \setminus \{0\}.$$

For the sake of simplicity, we assume that A and B are reduced simultaneously to the same type of real Jordan canonical form. The general case is investigated in [18]. Although this condition is more restrictive than the simultaneous reduction to the upper triangular form, we are in more general situation with respect to the aforementioned papers since we recover as a particular case 1–D complex linear flows in $\mathbb{C}^n = \mathbb{R}^{2n}$ viewed as 2–D real linear action. More precisely, we suppose that

$$(4.2) \qquad A = \begin{pmatrix} A_1 & 0_{2n_1\times 2n_2} & \dots & 0_{2n_1\times 2n_m} \\ 0_{2n_2\times 2n_1} & A_2 & \dots & 0_{2n_2\times 2n_m} \\ \vdots & \vdots & \vdots & \vdots \\ 0_{2n_m\times 2n_1} & 0_{2n_m\times 2n_2} & \dots & A_m \end{pmatrix},$$

$$(4.3) \qquad B = \begin{pmatrix} B_1 & 0_{2n_1\times 2n_2} & \dots & 0_{2n_1\times 2n_m} \\ 0_{2n_2\times 2n_1} & B_2 & \dots & 0_{2n_2\times 2n_m} \\ \vdots & \vdots & \vdots & \vdots \\ 0_{2n_m\times 2n_1} & 0_{2n_m\times 2n_2} & \dots & B_m \end{pmatrix},$$

where

$$\begin{aligned} A_j &= R_2(\alpha_j,\beta_j)I_{n_j}(2) + R(\rho_j,\sigma_j)N_{n_j}(2), \\ B_j &= R_2(\xi_j,\eta_j)I_{n_j}(2) + R(\mu_j,\lambda_j)N_{n_j}(2), \end{aligned}$$

namely,

$$(4.4) \qquad A_j = \begin{pmatrix} R_2(\alpha_j,\beta_j) & R_2(\kappa_j,\lambda_j) & \dots & 0_{2\times 2} \\ 0_{2\times 2} & R_2(\alpha_j,\beta_j) & \dots & 0_{2\times 2} \\ \vdots & \vdots & \vdots & \vdots \\ 0_{2\times 2} & 0_{2\times 2} & \dots & R_2(\alpha_j,\beta_j) \end{pmatrix},$$

$$(4.5) \qquad B_j = \begin{pmatrix} R_2(\xi_j,\eta_j) & R_2(\mu_j,\nu_j) & \dots & 0_{2\times 2} \\ 0_{2\times 2} & R_2(\xi_j,\eta_j) & \dots & R_2(\rho^j,\theta^j) \\ \vdots & \vdots & \vdots & \vdots \\ 0_{2\times 2} & 0_{2\times 2} & \dots & 0_{2\times 2} \end{pmatrix},$$

with $\alpha_j,\beta_j,\xi_j,\eta_j,\mu_j \in \mathbb{R}$ for $j = 1,\dots,m$. We define in an obvious way

$$(4.6) \qquad \begin{aligned} A_{nil} &= A - A_{diag},\ B_{nil} = B - B_{diag}, \\ A_{j,nil} &= A_j - A_{j,diag},\ B_{j,nil} = B - B_{j,diag}, \end{aligned}$$

Set

$$(4.7) \qquad \Gamma(s,t;z) = \exp(sA+tB)z, \qquad s,t\in\mathbb{R}$$

We denote by $\mathcal{F}[\rho] = \mathcal{F}[A,B]$ the foliation by the 2–D integral manifolds of the linear action, defined by

$$\begin{aligned} \mathcal{F}[A,B] &= \bigcup_{z\in\mathbb{R}^{2n}} \mathcal{F}_z[A,B] \\ (4.8) \qquad \mathcal{F}_z[A,B] &= \{\Gamma(s,t;z);\ s,t\in\mathbb{R}\}. \end{aligned}$$

We will be interested in the transversality of the intersections $\mathcal{F}_z[A,B]\bigcap S^{2n-1}$, $z\in S^{2n-1}$. We recall that $\mathcal{F}[A,B]$ interesects transversally S^{2n-1} iff

$$(4.9) \qquad |\langle Az,z\rangle| + |\langle Bz,z\rangle| \neq 0, \qquad z\in S^{2n-1}$$

A linear vector field $\langle Az,\partial_z\rangle$ intersects S^{2n-1} transversally iff S^{2n-1} iff

$$(4.10) \qquad |\langle Az,z\rangle| \neq 0, \qquad z\in S^{2n-1}$$

In view of (4.2), (4.3), (4.4), (4.4) we introduce, as in the introduction, the natural decomposition

$$\text{(4.11)} \qquad \mathbb{R}^{2n} = \mathbb{I}^{2n_1} + \cdots + \mathbb{I}^{2n_m}.$$

with $\dim \mathbb{I}^{2n_j} = 2n_j$, and $\mathbb{I}^{2n_j}$ being invariant for A_j and B_j.

Lemma 4.1. *The condition (4.1) holds if and only if*

$$\text{(4.12)} \qquad \alpha_j\eta_j - \beta_j\xi_j \neq 0, \qquad j = 1, \dots, m,$$

Proof. Let (4.1) be true. If (4.12) does not hold, then $R_2(\alpha_j, \beta_j)$ is a constant times of $R_2(\xi_j, \eta_j)$ for some $j \in \{1, \dots, m\}$. Thus, for every $z \in \mathbb{I}_{eig}^{2n_j}$ we get that Az and Bz are linearly dependent. This contradicts to the assumption (4.1). Suppose now that (4.12) is valid. If there exists $z \neq 0$ such that Az and Bz are linearly dependent, by (4.2), (4.3), (4.4), (4.5) and (4.12) we get that $z^j = 0$ for $j = 1, \dots, m$, which contradicts $z \neq 0$. Hence (4.1) is true. □

We denote by $\pi^j : \mathbb{R}^{2n} \to \mathbb{I}^{2n_j}$ the natural orthogonal projection, $j = 1, \dots, m$, and we set $z^j = \tilde{\pi}^j(z) = (z_1^j, \dots, z_{n_j}^j)$, $z_k^j = (z_{k,1}^j, z_{k,2}^j) \in \mathbb{R}^2$, $k = 1, \dots, n_j$, $j = 1, \dots, m$. Clearly we have

$$\text{(4.13)} \qquad \mathbb{R}^{2n} \ni z = \pi^1(z) + \cdots + \pi^m(z) = (z^1, z^2, \dots z^m)^{tr},$$

which leads to

$$\text{(4.14)} \qquad e^{sA+tB}z = (e^{sA_1+tB_1}z^1, e^{sA_2+tB_2}z^2, \dots, e^{sA_m+tB_m}z^m)^{tr}.$$

Next, in view of (4.4), (4.5), we can write

$$\begin{aligned}
& e^{sA_j+tB_j}z^j = e^{s\alpha_j+t\xi_j}U(s\beta_j + t\eta_j)I_{n_j}(2)e^{sA_{j,nil}+tB_{j,nil}}z^j \\
\text{(4.15)} \quad & e^{sA_{j,nil}+tB_{j,nil}}z^j \\
& = \begin{pmatrix} \sum_{\ell=1}^{n_j} \frac{1}{(\ell-1)!} R_2^{\ell-1}(s\kappa_j + t\mu_j, s\lambda_j + t\nu_j) z_\ell^j \\ \sum_{\ell=2}^{n_j} \frac{1}{(\ell-2)!} R_2^{\ell-1}(s\kappa_j + t\mu_j, s\lambda_j + t\nu_j) z_\ell^j \\ \vdots \\ z_{n_j}^j \end{pmatrix} \\
& = \begin{pmatrix} \sum_{\ell=1}^{n_j} \frac{((s\kappa_j+t\mu_j)^2+(s\lambda_j+t\nu_j)^2)^{(\ell-1)/2}}{(\ell-1)!} U((\ell-1)\phi_j(s,t)) z_\ell^j \\ \sum_{\ell=2}^{n_j} \frac{((s\kappa_j+t\mu_j)^2+(s\lambda_j+t\nu_j)^2)^{(\ell-2)/2}}{(\ell-1)!} U((\ell-2)\phi_j(s,t)) z_\ell^j \\ \vdots \\ z_{n_j}^j \end{pmatrix},
\end{aligned}$$

where

$$\cos\phi_j(s,t) = \frac{s\kappa_j + t\mu_j}{\sqrt{(s\kappa_j + t\mu_j)^2 + (s\lambda_j + t\nu_j)^2}},$$

$$\sin\phi_j(s,t) = \frac{s\lambda_j + t\nu_j}{\sqrt{(s\kappa_j + t\mu_j)^2 + (s\lambda_j + t\nu_j)^2}},$$

for $j = 1, \dots, m$ provided $(s\kappa_j + t\mu_j)^2 + (s\lambda_j + t\nu_j)^2 \neq 0$. We recall that $U(\varphi)$ stands for the 2×2 rotation matrix $R_2(\cos\varphi, \sin\varphi)$.

Before addressing the transversality issue in the presence of Jordan blocks we consider the following example of a linear complex flow

$$L_0 = (w_1 + \varepsilon w_2)\partial_{w_1} + w_2\partial_{w_2}$$

in $\mathbb{C}^2$. It corresponds to the linear complex action defined by the matrix

$$\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}, \qquad \varepsilon \in \mathbb{C}.$$

Straightforward calculations imply that L_0 is transversal to S^3 if and only if $|\varepsilon| < 2$.

We will generalize this fact for the 2–D linear real actions ρ defined by A, B satisfying (4.2), (4.3), (4.4), (4.5).

Theorem 4.2. *Let $\mathcal{F}[A_{diag}, B_{diag}]$ be transversal to S^{2n-1}. Then the following properties hold:*

i) there exists a constant $C_0 > 0$ such that the foliation $\mathcal{F}[A, B]$ intersects S^{2n-1} transversally provided

$$\max_{j=1,\dots m}\{\sqrt{\kappa_j^2 + \lambda_j^2} + \sqrt{\mu_j^2 + \nu_j^2}\} < C_0, \tag{4.16}$$

with the convention $\kappa_j = \lambda_j = \mu_j = \nu_j = 0$ if $n_j = 1$.

ii) suppose in addition that $\langle A_{diag}x, \partial_x\rangle$ intersects transversally, i.e. (4.10) holds. Then $\langle Ax, \partial_x\rangle$ intersects transversally S^{2n-1} iff

$$|\alpha_j| > 2d(n_j)\sqrt{\kappa_j^2 + \lambda_j^2}, \qquad j = 1, \dots, m, \tag{4.17}$$

where

$$d(k) := \max_{x \in S^{k-1}} |x_1x_2| + \dots |x_{k-1}x_k|, \tag{4.18}$$

for $k \geq 2$ with the convention $d(1) = 0$.

iii) we can always find two real constants c_1 and c_2 such that $\langle c_1A_{diag}z + c_2B_{diag}z, \partial_z\rangle$ intersects transversally S^{2n-1}, namely

(4.19)

either $\min_{j=1,\dots,m} c_1\alpha_j + c_2\xi_j > 0$ *or* $\max_{j=1,\dots,m} c_1\alpha_j + c_2\xi_j < 0$.

Proof. By the transversality we have

$$c_0 := \min_{z \in S^{2n-1}} (|\langle A_{diag}z, z\rangle| + |\langle B_{diag}z, z\rangle|) > 0.$$

Next, in view of

$$\langle Mz, z\rangle = \langle M_{diag}z, z\rangle + \langle M_{nil}z, z\rangle, = \sum_{j=1}^{m}(\langle M_{j,diag}z, z\rangle + \langle M_{j,nil}z, z\rangle)$$

for $M = A, B$, we get the estimates

$$|\langle A_{j,nil}z, z\rangle| \leq \{\sqrt{\kappa_j^2 + \lambda_j^2}\}|z|^2, \qquad |\langle B_{nil}z, z\rangle| \leq \max_{j=1,\ldots,m}\{\sqrt{\mu_j^2 + \nu_j^2}\}|z|^2,$$

we get

$$\begin{aligned}(4.20)\quad &\min_{z\in S^{2n-1}} |\langle Az, z\rangle| + |\langle Bz, z\rangle| \\ &\geq C_0 - \max_{j=1,\ldots,m}\big(\max_{z^j\in S^{2n_j-1}} |\langle A_{j,nil}z^j, z^j\rangle| + \max_{z^j\in S^{2n_j-1}} |\langle B_{j,nil}z^j, z^j\rangle|\big).\end{aligned}$$

We can write

$$\begin{pmatrix} \kappa_j & -\lambda_j \\ \lambda_j & \kappa_j \end{pmatrix} = \sqrt{\kappa_j^2 + \lambda_j^2}U(\varphi_j)$$

(respectively,

$$\begin{pmatrix} \mu_j & -\nu_j \\ \nu_j & \mu_j \end{pmatrix} = \sqrt{\mu_j^2 + \nu_j^2}U(\psi_j)$$

with

$$\cos(\varphi_j) = \frac{\kappa_j}{\sqrt{\kappa_j^2 + \lambda_j^2}}, \ \sin(\varphi_j) = \frac{\lambda_j}{\sqrt{\kappa_j^2 + \lambda_j^2}}$$

provided $\sqrt{\kappa_j^2 + \lambda_j^2} \neq 0$ (respectively,

$$\cos(\psi_j) = \frac{\mu_j}{\sqrt{\mu_j^2 + \nu_j^2}}, \ \sin(\psi_j) = \frac{\nu_j}{\sqrt{\mu_j^2 + \nu_j^2}}$$

provided $\sqrt{\mu_j^2 + \nu_j^2} \neq 0$). Thus

$$(4.21)\qquad \langle A_{j,nil}z, z\rangle = \sqrt{\kappa_j^2 + \lambda_j^2}\sum_{\ell=1}^{n_j-1}\langle z_\ell^j, U(\varphi_j)z_{\ell+1}^j\rangle$$

$$(4.22)\qquad \langle B_{j,nil}z, z\rangle = \sqrt{\mu_j^2 + \nu_j^2}\sum_{\ell=1}^{n_j-1}\langle z_\ell^j, U(\psi_j)z_{\ell+1}^j\rangle$$

with the convention $\langle A_{j,nil}z, z\rangle = \langle A_{j,nil}z, z\rangle = 0$ if $n_j = 1$. The definition of $d(k)$ and (4.20), (4.21), (4.22) lead to

$$|\langle Az, z\rangle| + |\langle Bz, z\rangle| \geq c_0 - \max_{j=1,\ldots,m}(\{(\sqrt{\kappa_j^2 + \lambda_j^2} + \sqrt{\mu_j^2 + \nu_j^2})d(n_j)\}$$

for $z \in S^{2n-1}$, which yields (4.10) by choosing

$$C_0 = c_0 \left(\max_{j=1,\dots,m} (\{(\sqrt{\kappa_j^2+\lambda_j^2} + \sqrt{\mu_j^2+\nu_j^2}) d(n_j))\} \right)^{-1}.$$

Next, we deal with ii). We assume without loss of generality that $\alpha_j > 0$, $j = 1, \dots, m$. Then we have

$$\begin{aligned} \langle Az, z\rangle &= \sum_{j=1}^{m} \langle A_j z^j, z^j\rangle \\ &= \sum_{j=1}^{m} (\alpha_j |z^j|^2 + \sqrt{\kappa_j^2+\lambda_j^2} \sum_{\ell=1}^{m-1} \langle z_\ell^j, U(\varphi_j) z_{\ell+1}^j\rangle) \end{aligned}$$

and

$$\max_{z^j \in S^{2n_j-1}} \langle A_j z^j, z^j\rangle = \alpha_j + d(n_j)\sqrt{\kappa_j^2+\lambda_j^2} \tag{4.23}$$

$$\min_{z^j \in S^{2n_j-1}} \langle A_j z^j, z^j\rangle = \alpha_j - d(n_j)\sqrt{\kappa_j^2+\lambda_j^2} \tag{4.24}$$

for $j = 1, \dots, m$. The proof of (4.17) is complete.

In order to proof iii), we need the following lemma on quadratic forms

Lemma 4.3. *Let $a = A[x]$ and $B[x]$ be two quadratic form defined as follows*

$$A[x] = |x'|^2 - |x''|^2, \qquad B[x] = \sum_{s=1}^{p} a_j z_j^2 - \sum_{k=p+1}^{n} b_j x_j^2$$

where $x' = (x_1, \dots, x_p)$, $x'' = (x_{p+1}, \dots, x_n)$, $1 < p < n$, $a_j, b_k \in \mathbb{R}$, $j = 1, \dots, p$, $k = p+1, \dots, n$. If

$$\{x \in \mathbb{R}^n \setminus 0;\ A[x] = B[x] = 0\} = \emptyset \tag{4.25}$$

we can find $c_1, c_2 \in \mathbb{R}$ such that $c_1 A[x] + c_2 B[x]$ is positive.

Proof of the lemma. In view of (4.25), we may assume without loss of generality, multiplying with -1 if necessary, that $|x'|^2 - |x''|^2 \geq 0$, $x = (x', x'') \neq 0$ implies $B[x] > 0$. Therefore, we have $a_0 := \min_{s=1,\dots,p} a_s > 0$. Set $b_0 = \min_{k=p+1,\dots,n} b_k$. Letting $|x'| = |x''|$ we get that

$$\min_{|x'|=|x''|=r>0} B[x] = (a_0 - b_0) r^2 > 0,$$

which yields $a_0 > b_0$. We can choose and fix $0 < \varepsilon \ll 1$ to satisfy $(1-\varepsilon)a_0 > b_0$. Then

$$-(1-\varepsilon)a_0 A[x] + B[x] > 0, \qquad x \in \mathbb{R}^n \setminus 0,$$

since $a_j - (1-\varepsilon)a_0 > 0$ and $(1-\varepsilon)a_0 > b_0 \geq b_k$ for all $j = 1, \dots, p$, $k = p+1, \dots, n$, which concludes the proof of the lemma, and hence the proof of iii). □

Remark 4.4. *The sharpness result for the transversality of a single vector (4.17) is true for the 2–D foliation defined by a complex flow. However, in general we may have transversality in the presence of Jordan blocks without restrictions on the nilpotent part, as the following example shows*

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 & a & 0 \\ 1 & 0 & 0 & a \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

for all $a \neq 0$. More complete analysis of such problems is carried out in [18].

In view of (4.19) we will assume without loss of generality that if $\mathcal{F}[A,B]$ intersects transversally S^{2n-1} then

$$\alpha_j > 0 \quad j = 1, \dots, m, \tag{4.26}$$

We observe that we have always at least m cycles on the intersection of S^{2n-1} with a 2–D linear foliation $\mathcal{F}[A,B]$, defined by commuting A and B. The intersections $\mathcal{F}[A,B] \bigcap S^{2n-1}$ are defined implicitly by the equation

$$F(s,t;z) := \|e^{sA+tB}z\|^2 - 1 = 0 \tag{4.27}$$

for $z \in S^{2n-1}$.

Proposition 4.5. *The intersection of the 2–D linear foliation $\mathcal{F}[A,B]$ with S^{2n-1} admits m nontrivial closed curves $\ell_1, \dots, \ell_m$ which are defined by*

$$\begin{aligned} \ell_k : z = Z_k^{per}(t) &= (0, \dots, 0, z_k^{per}(t), 0, \dots)^{tr}, \\ z_k^{per}(t) &= (U((-\beta_k\xi_k/\alpha_k + \eta_k)t)z_1^k, 0, \dots, 0)^{tr}, \end{aligned} \tag{4.28}$$

where $z_1^k \in \mathbb{R}^2$, $\|z_1^k\| = 1$, $k = 1, \dots, m$.

Proof: We choose $z^k_{eig} = (z^k_1, 0, \dots, 0) \in \mathbb{I}^{2n_k}_{eig} \bigcap S^{2n-1}$ and set $z^{k,eig} = (0, \dots, z^k_{eig}, 0, \dots, 0)^{tr}$. Then

$$(4.29)\quad \Gamma(s,t;z^{k,eig}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \exp(\alpha_k s + \xi_k t) U(\beta_k s + \eta_k t) I_2(n_k) z^k_{eig} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$(4.30)\quad \exp(\alpha_k s + \xi_k t) U(\beta_k s + \eta_k t) I_2(n_k) z^k_{eig} = \begin{pmatrix} \exp(\alpha_k s + \xi_k t) U(\beta_k s + \eta_k t) z^k_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Since

$$(4.31)\quad \begin{aligned} \|\Gamma(s,t;z^{k,eig})\|^2 &= \exp(2\alpha_k s + 2\xi_k t) \|U(\beta_k s + \eta_k t) z^k_1\|^2 \\ &= \exp(2\alpha_k s + 2\xi_k t) r^2 \end{aligned}$$

we obtain that $F(s,t;z^{k,eig}) = 0$ if and only if $\alpha_k s + \xi_k t = 0$. □

We introduce a new notion of resonances related to the nilpotent part which will play an essential role in the sharp estimates for the number of periodic orbits of nonsingular tangential vector fields on S^{2n-1} of $\mathcal{F}[A,B]$ provided the SPC condition holds, which imply after a suitable change the generating matrices A and B, that $\alpha_k > 0$ for all $k = 1, \dots, m$.

Definition 4.6. *We shall say that that the nilpotent part of the action $[A,B]$ is nonresonant if for every $k \in \{1, \dots, m\}$, such that $n_k > 1$*

$$(4.32)\quad (\alpha_k \lambda_k - \xi_k \kappa_k)^2 + (\alpha_k \nu_k - \xi_k \mu_k)^2 \neq 0$$

or equivalently, the vectors (α_k, ξ_k), (κ_k, λ_k) and (μ_k, ν_k) do not lie on a line passing through the origin.

Remark 4.7. *The definition above is invariant with respect to the choice of the matrices A and B. Moreover, one verifies that all nilpotent parts of linear 1–dimensional complex linear actions are nonresonant (passing to 2–D real whenever $n_k > 1$). Therefore such phenomena appear only for 2–D real actions. Of course, the definition is void if both*

A and B are semisimple. Detailed investigations of the resonances of the nilpotent parts are done in [18].

Next, we investigate the integral curves of a nonsingular tangent vector field X in S^{2n-1} obtained by by the transversal intersection $\mathcal{F}[A,B]$.

Theorem 4.8. *Let A, B satisfy (4.19), (4.26) and satisfy the smallness condition for its nilpotent part which implies that $\mathcal{F}[A,B]$ intersects $S^{2n-1}(r)$. Then if (4.32) holds and if*

$$\alpha_k\xi_j - \alpha_j\xi_k \neq 0 \quad \text{for all } j,k = 1,\dots,m,\ k \neq j, \tag{4.33}$$

then X admits exactly m periodic orbits defined by (4.28). Let now $z \notin \mathbb{I}^{2n_j}_{eig}$. Then the curve $\ell[z]$ defined by $F(s,t;z) =$ could be parameterized by the implicit function theorem by $s = \theta(t) = \theta(t,z)$ and

$$\ell[z] : Z(t) = \Gamma(\theta(t), t; z), \qquad t \in \mathbb{R}, \tag{4.34}$$

is not periodic and satisfies the following properties

$$\lim_{t\to+\infty} dist(Z(t), O[\ell[\overline{k}(z)]]) = 0 \tag{4.35}$$

$$\lim_{t\to-\infty} dist(Z(t), O[\ell[\underline{k}(z)]]) = 0 \tag{4.36}$$

where $\overline{k}(z)$ (respectively $\underline{k}(z)$) stands for the largest (respectively) smallest integer $k \in \{1,\dots,m\}$ such that $z^k \neq 0$. Here $O[\ell_k] = \{Z^k_{per}(t) : t \in \mathbb{R}\}$ stands for the orbit of the periodic curve ℓ_k.

Next, if at least one of the two conditions (4.32) and (4.33) is not satisfied, then X has infinitely many periodic orbits.

Finally, $\mathcal{F}[A,B] \bigcap S^{2n-1}(r)$ defines a Hopf foliation, i.e., every orbit of X is periodic, provided the vectors (α_j, β_j), $j = 1,\dots,m$, lie on a half–line containing the origin, i.e.,

$$\frac{\xi_1}{\alpha_1} = \dots = \frac{\xi_m}{\alpha_m} =: \tau; \tag{4.37}$$

for every $k \in \{1,\dots,m\}$, such that $n_k > 1$, we have

$$(\alpha_k\mu_k - \xi_k\kappa_k)^2 + (\alpha_k\nu_k - \xi_k\lambda_k)^2 = 0 \tag{4.38}$$

and $\beta_k\tau - \eta_k$, $k = 1,\dots,m$, are rationally dependent, i.e., there exist integers $p_1,\dots,p_m$ and a positive real number ω such that

$$\frac{\beta_1\tau - \eta_1}{p_1} = \dots = \frac{\beta_m\tau - \eta_m}{p_m} = \omega. \tag{4.39}$$

Proof. Set $\tau_k = \xi_k/\alpha_k$, $k = 1, \dots, m$. Without loss of generality we may assume that

$$(4.40) \qquad \tau_1 \le \tau_2 \le \dots \le \tau_m.$$

First we observe that the definition of τ_k and 4.40) imply that (4.33) is equivalent to

$$(4.41) \qquad \tau_1 < \dots < \tau_m \quad \text{if } m > 1,$$

Next, if (4.41) is true, we define

$$(4.42) \qquad \delta_0 := \min_{j=1,\dots,m-1} (\tau_{j+1} - \tau_j) > 0 \quad \text{if } m > 1,$$

By $\alpha_k > 0$ and the smallness of the nilpotent parts we may assume without loss of generality that $F_s(s,t;z) > 0$. Therefore we determine uniquely, by the implicit function theorem applied to (4.45), a real analytic function $s = \theta(t) = \theta(t;z)$, $t \in \mathbb{R}$ such that

$$(4.43) \qquad F(\theta(t),t;z) := \|e^{\theta(t)A+tB}z\|^2 - r^2 = 0, \qquad t \in \mathbb{R}.$$

Set

$$r_k(t) = \theta_k(t) + \tau_k t, \qquad k = 1, \dots, m.$$

In view of the definitions of $\overline{k} = \overline{k}(z)$ and $\underline{k} = \overline{k}(z)$ and (4.14), (4.15), we can write

$$\begin{aligned} \|e^{\theta(t)A+tB}z\|^2 &= \sum_{k=\underline{k}}^{\overline{k}} \|e^{\theta(t)A_k+tB_k}z^k\|^2 \\ &= \sum_{k=\underline{k}}^{\overline{k}} e^{2\alpha_k\theta(t)+2\xi_k t}\|e^{\theta(t)A_{k,nil}+tB_{k,nil}}z^k\|^2 \\ (4.44) \qquad &= \sum_{k=\underline{k}}^{\overline{k}} e^{2\alpha_k(\theta(t)+\tau_k t}\|N_k[t;z^k]\|^2 \end{aligned}$$

where $N_k[t;z^k]$ is defined by

(4.45)

$$\begin{pmatrix} U(\theta(t)\beta_k + t\eta_k)\sum_{\ell=1}^{n^k)} \frac{((\theta(t)\kappa_k+t\mu_k)^2+(\theta(t)\lambda_k+t\nu_k)^2)^{(\ell-1)/2}}{(\ell-1)!} U((\ell-1)\phi_k(\theta(t),t))z_\ell^k \\ U(\theta(t)\beta_k + t\eta_k)\sum_{\ell=2}^{n_k} \frac{((\theta(t)\kappa_k+t\mu_k)^2+(\theta(t)\lambda_k+t\nu_k)^2)^{(\ell-2)/2}}{(\ell-1)!} U((\ell-2)\phi_k(\theta(t),t))z_\ell^k \\ \vdots \\ U(\theta(t)\beta_k + t\eta_k)z_{n_k}^k \end{pmatrix}$$

Next, for given $z^k \in \mathbb{F}^{2n_k}$, $z^k \ne 0$, we define by $n_k^+(z)$ the largest integer ℓ, $1 \le \ell \le n_k$, such that $z_\ell^k \ne 0$. In particular, if $n_k > 1$ and

$n_k^+(z) < n_k$, we get that $N_k[t; z^k]$ is given by

(4.46)

$$\begin{pmatrix} U(\theta(t)\beta_k + t\eta_k) \sum_{\ell=1}^{n_k^+(z))} \frac{((\theta(t)\kappa_k+t\mu_k)^2+(\theta(t)\lambda_k+t\nu_k)^2)^{(\ell-1)/2}}{(\ell-1)!} U((\ell-1)\phi_k(\theta(t),t)) z_\ell^k \\ U(\theta(t)\beta_k + t\eta_k) \sum_{\ell=2}^{n_k^+(z)} \frac{((\theta(t)\kappa_k+t\mu_k)^2+(\theta(t)\lambda_k+t\nu_k)^2)^{(\ell-2)/2}}{(\ell-1)!} U((\ell-2)\phi_k(\theta(t),t)) z_\ell^k \\ \vdots \\ U(\theta(t)\beta_k + t\eta_k) z^k_{n_k^+(z)} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

We observe by the definition of $n_k^+(z)$ that there exists a positive constant $C_k = C_k(z^k)$ such that the following estimates are true

$$\begin{aligned} \|N_k[t; z^k]\| &\geq C_k^{-1} \left((\theta(t)\kappa_k + t\mu_k)^2 + (\theta(t)\lambda_k + t\nu_k)^2\right)^{(n_k^+(z)-1)/2} \\ \|N_k[t; z^k]\| &\leq C_k \left((\theta(t)\kappa_k + t\mu_k)^2 + (\theta(t)\lambda_k + t\nu_k)^2\right)^{(n_k^+(z)-1)/2} \end{aligned} \tag{4.47}$$

provided $(\theta(t)\kappa_k + t\mu_k)^2 + (\theta(t)\lambda_k + t\nu_k)^2 \geq 1$, $k = 1, \dots, m$.

Now we write two crucial decompositions associated to the choice of $\overline{k}$ and $\underline{k}$, by using the the definition of δ_0 and (4.44)

$$\|e^{\theta(t)A+tB}z\|^2 = e^{2\alpha_{\overline{k}} r_{\overline{k}}(t)} \left(\|N_{\overline{k}}[t; z^k]\|^2 + E_{\overline{k}}^+(t; z)\right) \tag{4.48}$$

$$\|e^{\theta(t)A+tB}z\|^2 = e^{2\alpha_{\underline{k}} r_{\underline{k}}(t)} \left(\|N_{\underline{k}}[t; z^k]\|^2 + E_{\underline{k}}^-(t; z)\right) \tag{4.49}$$

with $E_{\overline{k}}^+(t; z)$, $E_{\underline{k}}^-(t; z)$ satisfying

$$E_{\overline{k}}^+(t; z) \leq c^{-1} \exp(-c(\delta_0|t| + r_{\overline{k}}(t))), \qquad t \geq 1 \tag{4.50}$$

$$E_{\underline{k}}^-(t; z) \leq c^{-1} \exp(-c(\delta_0|t| + r_{\underline{k}}(t))), \qquad t \leq -1 \tag{4.51}$$

The estimates (4.51), (4.50), combined with (4.48), (4.49), imply that

$$\lim_{t\to+\infty} \frac{r_{\overline{k}}(t)}{t} = 0 \tag{4.52}$$

$$\lim_{t\to-\infty} \frac{r_{\underline{k}}(t)}{t} = 0 \tag{4.53}$$

Indeed, if, for example, (4.52) is not true, by (4.48) and the estimates (4.47) we contradict $\|e^{\theta(t)A+tB}z\|^2 = 1$ for all $t \in \mathbb{R}$ for a sequence $t_q \to +\infty$ for $q \to \infty$, with similar arguments for $t \to -\infty$ if (4.49) fails.

Now, in view of (4.48), (4.50), (4.52) (respectively, (4.49), (4.51), (4.53)), we obtain that

$$(4.54)\qquad \lim_{t\to+\infty} \pi^k(\Gamma(\theta(t),t;z)) = 0 \text{ for } k \neq \overline{k};$$

(respectively,

$$(4.55)\qquad \lim_{t\to-\infty} \pi^k(\Gamma(\theta(t),t;z)) = 0 \text{ for } k \neq \underline{k}).$$

In particular, if $n^{+}_{\overline{k}}(z) = 1$ (respectively, $n^{+}_{\underline{k}}(z) = 1$), (4.54) (respectively, (4.55)) yields (4.35) (respectively, (4.35)).

Consider the case $n^{+}_{\overline{k}}(z) > 1$. Now the nonresonance condition on the nilpotent parts will play a fundamental role for proving (4.35). We get, after replacing

$$\theta(t) = r_{\overline{k}}(t) - \xi_{\overline{k}}t/\alpha_{\overline{k}} = r_{\overline{k}}(t) - \tau_{\overline{k}}t$$

the following estimate

$$(4.56)\qquad \begin{aligned} &\sqrt{(\theta(t)\kappa_{\overline{k}} + t\mu_{\overline{k}})^2 + (\theta(t)\lambda_{\overline{k}} + t\nu_{\overline{k}})^2} \\ &= \frac{t}{\alpha_{\overline{k}}}(\omega_{\overline{k}} + o(1) \quad t \to +\infty \end{aligned}$$

where

$$\omega_{\overline{k}} = \sqrt{(-\xi_{\overline{k}}\kappa_{\overline{k}} + \alpha_{\overline{k}}\mu_{\overline{k}})^2 + (-\xi_{\overline{k}}\lambda_{\overline{k}} + \alpha_{\overline{k}}\nu_{\overline{k}})^2}.$$

By (4.32) we get $\omega_k \neq 0$ provided $n_k > 1$, $k = 1, \ldots, m$.

The definition of $Z^k(t)$, combined with (4.47), (4.56) and the fact that $\lim_{t\to+\infty} \|Z_k(t)\| = r$, imply that

$$(4.57)\qquad 0 < \inf_{t\geq 1} \|Z^k_1(t)\| < \sup_{t\geq 1} \|Z^k_1(t)\| < +\infty$$

and

$$(4.58)\qquad \lim t \to +\infty \|Z^k_{p+1}(t)\| = 0, \qquad p = 1, \ldots, n^{+}_{\overline{k}} - 1,$$

which lead to (4.35). We show in a similar way (4.35).

The Hopf foliation part follows immediately if one observes that under the hypothesis (4.38) $\tau_1 = \ldots = \tau_m = \tau$ we get $-\tau_1 A_{nil} + B_{nil} = 0$, which yields

$$\|\Gamma(-\tau t, t; z)\| = \|z\|$$

for every $t \in \mathbb{R}$, $z \in \mathbb{R}^{2n}$. The periodicity of $\Gamma(-\tau t, t; z)$ follows from the fact that (4.39) implies that $\cos((-\tau\beta_k + \eta_k)t)$, $\sin((-\tau\beta_k + \eta_k)t)$, $k = 1, \ldots, m$, are $2\pi/p\omega$ periodic, where p stands for the minimal common divisor of $p_1, \ldots, p_m$. □

Remark 4.9. *Let $\mathbb{R}^{2n} = \mathbb{C}^n$ via a constant complex structure J, $J^2 = -I_{2n}$, and let $\mathcal{F}[A, B]$ coincide with the linear complex foliation defined by $\mathcal{F}[C]$. Then (4.38) and (4.39) are equivalent to the condition: C is semisimple and all its eigenvalues lie on a half–line containing the origin. The following example show that the Hopf foliation may be defined by 2–D linear action with at least one of the matrices A and B admitting Jordan blocks. Consider the linear $\mathbb{R}^2$ action in $\mathbb{R}^4$ defined by*

$$A = \begin{pmatrix} 1 & 0 & \varepsilon & 0 \\ 0 & 1 & 0 & \varepsilon \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.59}$$

$$B = \begin{pmatrix} \xi & -\eta & \rho & -\sigma \\ \eta & \xi & \sigma & \rho \\ 0 & 0 & \xi & -\eta \\ 0 & 0 & \eta & \xi \end{pmatrix}, \tag{4.60}$$

where $\varepsilon, \xi, \eta, \rho, \sigma \in \mathbb{R}$. Then $\mathcal{F}[A, B] \bigcap S^3(r)$ defines Hopf bifurcation if and only if $\rho = -\xi$, $\sigma = 0$. For more detailed analysis cf. [18]

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