数理解析研究所講究録 1377 巻 2004 年 76-80

MONODROMIES OF SCHWARZIAN DIFFERENTIAL EQUATIONS OF A CERTAIN TYPE

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ABSTRACT. We consider a second-order linear homogenous ordinary differential equations in connection with the Teichmüller space of a four-times punctured sphere. Interests will be focused on the mysterious relation between the shape of the Teichmüller space and the Fibonacci sequence. The monodromy homomorphism induced by the differential equations plays a decisive role in our framework. This note is based on the author's paper [7] and the joint paper [3] with Y. Komori.

1. AN ORDINARY DIFFERENTIAL EQUATION

Let Y be a four-times punctured sphere $\widehat{\mathbb{C}} \setminus \{0, 1, \infty, \lambda\}$ with hyperbolic metric $\rho_Y(z)|dz|$ of Gaussian curvature -4. It is known that the space $B_2(Y)$ of holomorphic quadratic differentials $\psi(z)dz^2$ with finite norm

$$\|\psi\|_Y = \sup_{z \in Y} \rho_Y(z)^2 |\psi(z)|$$

is one dimensional vector space over $\mathbb{C}$. We may, for instance, take

$$\psi_0(z)dz^2 = \frac{dz^2}{z(z-1)(z-\lambda)}$$

as a basis of $B_2(Y)$. Let $\pi : \mathbb{D} \to Y$ be a holomorphic universal covering map of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto Y and let Γ be the covering transformation group. Then Γ is a Fuchsian group of the first kind acting on $\mathbb{D}$. It can be seen that the Schwarzian derivative $S_{\pi^{-1}}$ of a local inverse of π is independent of the choice of branch and even of the particular choice of π. Hence it defines a (single-valued) analytic function ν_Y on Y. Here the Schwarzian derivative S_f of f is defined by

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

Date: 20 March, 2004.
2000 *Mathematics Subject Classification.* Primary 30F60; Secondary 30F40, 34A20.

The function ν_Y is sometimes called the *uniformizing connection* of Y. Then it is known that ν_Y has the form

$$\nu_Y(z) = \frac{1}{2z^2(z-1)^2} + \frac{1}{2(z-\lambda)^2} + \frac{c(\lambda)}{z(z-1)(z-\lambda)},$$

where $c(\lambda)$ is a constant determined by λ (see, for instance, [1, Ch. X, p. 492]). This constant is known as an accessory parameter. It is generally difficult to compute $c(\lambda)$. A method of numerical computation of $c(\lambda)$ is given in [3] and the method will be indicated in Remark 2.

We now consider the second-order homogeneous linear ordinary differential equation

$$2y'' + (\nu_Y + t\psi_0)y = 0, \tag{1.1}$$

where t is a complex parameter. We fix a base point z_0 in Y and let y_0 and y_1 are fundamental (local) solutions of (1.1) around z_0, namely, they are determined by the initial conditions $y_0(z_0) = 1, y_0'(z_0) = 0, y_1(z_0) = 0$ and $y_1'(z_0) = 1$. It is easy to see that the Wronskian $y_0y_1' - y_0'y_1$ is identically 1. For a curve γ with initial and terminal points at z_0, the solutions y_0 and y_1 can be analytically continued along γ to, say, $\tilde{y}_0$ and $\tilde{y}_1$, respectively. These are written as linear combinations of y_0 and y_1, say,

$$\tilde{y}_1 = ay_1 + by_0$$
$$\tilde{y}_0 = cy_1 + dy_0,$$

where a, b, c, d are complex constants. Note that $ad - bc = 1$ holds. By the monodromy theorem, these constants depend only on the homotopy class $[\gamma]$ of γ. In this way, we obtain a homomorphism $\chi_t : \pi_1(Y, z_0) \to \mathrm{SL}(2, \mathbb{C})$ satisfying

$$\chi_t \ : \ [\gamma] \quad \mapsto \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The homomorphism is called the *monodromy homomorphism* or *holonomy representation* of $\pi_1(Y, z_0)$ with respect to $t\psi_0$. We write $G_t = \chi_t(\pi_1(Y, z_0))$.

In particular, when $t = 0$, the quotient $f = y_1/y_0$ of fundamental solutions to (1.1) has Schwarzian derivative $S_f = \nu_Y$, and thus, $f = L \circ \pi^{-1}$ locally, where L is a Möbius transformation. Therefore, the image $\chi_0(\pi_1(Y, z_0))$ is Möbius conjugate with the uniformizing Fuchsian group Γ of Y.

Remark 1. Up to Möbius conjugate, the group G_0 may be regarded as a lift of $\Gamma \subset \mathrm{PSU}(1,1)$ to $\mathrm{SU}(1,1)$. This lift is determined by the condition $\operatorname{tr}\chi_0(\gamma) = -2$ for every simple closed curve rounding about a puncture of Y.

Let $\tilde{T}(Y)$ be the set of those elements in $B_2(Y)$ of the form $t\psi_0$ for which G_t is a quasiconformal deformation of G_0. The connected component of $\tilde{T}(Y)$ which contains the origin is called (the Bers embedding of) the *Teichmüller space* of Y and will be denoted by $T(Y)$. (Rigorously, our $T(Y)$ is the Teichmüller space of the mirror image Y^* of Y. To avoid confusion, we adopt a different definition here from the standard one.)

It is known that $T(Y)$ is a Jordan domain with $\{\psi \in B_2(Y) : \|\psi\|_Y < 2\} \subset T(Y) \subset \{\psi \in B_2(Y) : \|\psi\|_Y < 6\}$ (see [3]). The quantities

$$\iota(T(Y)) = \inf\{\|\psi\|_Y : \psi \in \partial T(Y)\} \quad \text{and} \quad o(T(Y)) = \sup\{\|\psi\|_Y : \psi \in \partial T(Y)\}$$

are called the *inner radius* and the *outer radius* of $T(Y)$, respectively. Therefore, we have $2 \leq \iota(T(Y)) \leq o(T(Y)) \leq 6$. (In fact, these inequalities are all strict.)

2. Pinching deformation along a simple closed curve

Let τ be a complex number with $\operatorname{Im}\tau > 0$. Then $T = \mathbb{C}/\langle 1, \tau\rangle$ becomes a complex torus, where $\langle 1, \tau\rangle$ denotes the lattice group generated by 1 and τ over $\mathbb{Z}$. We denote by $[z]$ the equivalence class represented by $z \in \mathbb{C}$ with respect to the action of the lattice. Let $\wp$ be the Weierstrass $\wp$-function with period lattice $\langle 1, \tau\rangle$ and set $e_1 = \wp(1/2), e_2 = \wp(\tau/2), e_3 = \wp((1+\tau)/2)$. As is well known, the quantity

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2}$$

is an elliptic modular function. We now choose τ so that $\lambda(\tau) = \lambda$. Then the mapping $p : T \to \widehat{\mathbb{C}}$ defined by $p([z]) = (\wp(z) - e_2)/(e_1 - e_2)$ is a two-sheeted branched covering map of T onto $\widehat{\mathbb{C}}$ with branch point of order 2 at $[0], [1/2], [\tau/2], [(1+\tau)/2]$. Therefore, the four-times punctured torus $Z = T \setminus \{[0], [1/2], [\tau/2], [(1+\tau)/2]\}$ is a two-sheeted unbranched covering space over Y.

A simple closed curve is called peripheral if it is homotopic to either a point or a puncture. Let $\gamma : [0,1] \to Y$ be a non-peripheral simple closed curve in Y and $\tilde{\gamma}$ be a lift of γ via the composition of covering maps $\mathbb{C} \setminus \langle 1/2, \tau/2\rangle \to Z \to Y$. The difference $\tilde{\gamma}(1) - \tilde{\gamma}(0)$ of the terminal point and the initial point of $\tilde{\gamma}$ has the form $m + n\tau$ for relatively prime integers m and n. The ratio $r = n/m \in \widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ is called the slope of γ. It is known that the ratio determines the homotopy class of γ and that any number in $\widehat{\mathbb{Q}}$ is realized as the ratio of a non-peripheral simple closed curve in Y.

Let $[\gamma_r]$ be an element of $\pi_1(Y, z_0)$ with slope $r \in \widehat{\mathbb{Q}}$ and define the function σ_r to be $\sigma_r(t) = \operatorname{tr}^2 \chi_t([\gamma_r])$. Note that σ_r is an entire function and $\sigma_r(0) > 4$.

Remark 2. The origin is a special point where the entire functions σ_r take real values simultaneously. In turn, this property can be used to compute the accessory parameter $c(\lambda)$. See [3] for details.

We next recall fundamental facts about Farey triangles (cf. [6]). For a more detailed explanation, see [3]. The reader also finds a good account for Farey sequences in [2] as well as an interesting historical remark.

For three points a, b, c in $\widehat{\mathbb{R}}$, we denote by $\Delta(a,b,c)$ the hyperbolic triangle formed by three hyperbolic geodesics in the upper half plane $\mathbb{H}$ connecting two of the three points a, b and c. Let $\Delta = \Delta(0,1,\infty)$. Then $\mathbb{H}$ is tessellated by Δ and its conjugates by the modular group $\mathrm{PSL}(2,\mathbb{Z})$. Note that the stabilizer of Δ in $\mathrm{PSL}(2,\mathbb{Z})$ consists of three elements and permutes the vertices of Δ cyclically. Each triangle which is conjugate with Δ by the action of $\mathrm{PSL}(2,\mathbb{Z})$ is called a *Farey triangle.* The initial Farey triangle Δ and its reflection $\Delta' = \Delta(0,-1,\infty)$ in the imaginary axis form a fundamental domain of the modular group $\Gamma_2 = \{\pm C \in \mathrm{PSL}(2,\mathbb{Z}) : C \equiv I \mod 2\}$ of level 2. We will say that both Δ and Δ' are of level 0. A Farey triangle which shares a side with that of level 0 will be called of level 1 unless it is of level 0. Similarly, a Farey triangle which shares a side with that of level n will be called of level $n+1$ unless it is of level $\leq n$. It is important to note that the corresponding graph to the above tessellation is a tree, namely, there is no closed circuit.

It is well known that the orbit of 0 under the action of $\mathrm{PSL}(2,\mathbb{Z})$ coincides with $\widehat{\mathbb{Q}}$. We denote by $\tilde{\mathcal{F}}(n)$ the set of rationals which appear as vertices of Farey triangles of level $\leq n$. Set $\mathcal{F}(n) = \tilde{\mathcal{F}}(n) \setminus \tilde{\mathcal{F}}(n-1)$ for $n = 0, 1, \dots$. For instance, $\mathcal{F}(0) = \{-1, 0, 1, \infty\}$, $\mathcal{F}(1) = \{-2, -1/2, 1/2, 2\}$, $\mathcal{F}(2) = \{-3, -3/2, -2/3, -1/3, 1/3, 2/3, 3/2, 3\}$ and so on. We note that $\#\mathcal{F}(n) = 2^{n+1}$ for $n \geq 1$. An element r in $\mathcal{F}(n)$ will be called of level n and designated by $\mathrm{level}(r) = n$. Note that if p_1/q_1 and p_2/q_2 are of level $\leq n$ and if $p_1q_2 - q_1p_2 = \pm 1$, then $(p_1+p_2)/(q_1+q_2)$ is of level $\leq n+1$.

Note that the rationals $\pm r_n$ and $\pm 1/r_n$ belong to $\mathcal{F}(n)$, where $r_n = b_{n+1}/b_n$ and b_n is the n-th Fibonacci number, namely, b_n is determined by $b_0 = 1, b_1 = 1, b_n = b_{n-1} + b_{n-2}$ $(n = 2, 3, 4, \dots)$.

It is shown in [3] that the connected component P_r of the inverse image $\sigma_r^{-1}([4, \sigma_r(0)])$ containing 0 is a closed analytic Jordan arc for each rational r. The arc P_r is called the *pleating ray* with slope r. The the other end point of P_r, denoted by $\beta(r)$, lies on the boundary of the Teichmüller space $T(Y)$ and corresponds to a cusp group. The deformation along P_r can be regarded as a pinching deformation of the Riemann surface Y along the hyperbolic geodesic with slope r.

When $\lambda = 1/2$, the following observation was made in [7] by the aid of Mathematica for, at least, small n's.

Conjecture 1. *When $\lambda = 1/2$, then the following relation holds for each n :*

$$\max_{r \in \tilde{\mathcal{F}}(n)} \|\beta(r)\| = \|\beta(r_n)\|.$$

Note also that $\|\beta(r_n)\| = \|\beta(-r_n)\| = \|\beta(1/r_n)\| = \|\beta(-1/r_n)\|$ by symmetry in the case $\lambda = 1/2$. It is well known that r_n converges to the golden ratio $(\sqrt{5}-1)/2$. Since cusps are dense in the boundary [5], if the above conjecture is valid, then the next conjecture is valid, too.

Conjecture 2 ([7]). *If $\lambda = 1/2$, then*

$$o(T(Y)) = \|\beta((\sqrt{5}-1)/2)\|$$

holds, where $\beta((\sqrt{5}-1)/2)$ is the end point of the pleating ray with irrational slope $(\sqrt{5}-1)/2$.

Based on the last conjecture, we numerically obtained $o(T(Y)) \approx 2.9386$ when $\lambda = 1/2$. It is naturally expected that, for a general λ, a similar statement would hold for some irrational number r whose continued fraction expansion has the same tail as that of the golden ratio.

At present, we have no idea to prove the above conjectures in a rigorous way. It is also interesting to see the asymptotic behaviour of the monodromy homomorphism χ_t in connection with the configuration of exotic projective structures (see [4]). We end this note with the specialized problem.

Problem 3. *Investigate the asymptotic behaviour of the entire function $\sigma_r(t)$ as $t \to \infty$ for an extended rational number $r \in \widehat{\mathbb{Q}}$.*

REFERENCES

1. A. R. Forsyth, *Theory of Differential Equations Part III, Vol. IV,*, Cambridge University Press, 1902.
2. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, fifth ed., The Clarendon Press, Oxford University Press, New York, 1979.
3. Y. Komori and T. Sugawa, *Bers embedding of the Teichmüller space of a once-punctured torus*, to appear in Conform. Geom. Dyn.
4. Y. Komori, T. Sugawa, M. Wada, and Y. Yamashita, *Drawing Bers embeddings of the Teichmüller space of once-punctured tori*, preprint.
5. C. McMullen, *Cusps are dense*, Ann. of Math. (2) **133** (1991), 217–247.
6. Y. Minsky, *The classification of punctured-torus groups*, Ann. of Math. (2) **149** (1999), 559–626.
7. T. Sugawa, *Estimates of hyperbolic metric with applications to Teichmüller spaces*, Kyungpook Math. J. **42** (2002), 51–60.

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