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# Construction of elements of a Lie group $G_2$ via spinor group $Spin(8)$

横浜市立大学・数理科学 五明 智 (Satoshi Gomyo)
Department of Mathematical Sciences,
Yokohama City University

## 1 Introduction

Let $\mathfrak{C}$ be the division Cayley algebra over $\boldsymbol{R}$. It is known that the automorphism group $G_2 = \mathrm{Aut}\mathfrak{C}$ is a compact simply connected simple Lie group of type $G_2$ and $\mathfrak{g}_2 = \mathrm{Der}\mathfrak{C}$ is a compact simple Lie algebra of type $G_2$. In this paper, we give a concrete description of elements of the group $G_2$ by using elements of the Cayley algebra.

## 2 Spinor group

Let $T^+(\boldsymbol{R}^n)$ be the even tensor algebra of $\boldsymbol{R}^n$ and $U(\boldsymbol{R}^n)$ the two-sided ideal of $T^+(\boldsymbol{R}^n)$ generated by

$$x \otimes x + (x, x)1 \quad (x \in \boldsymbol{R}^n)$$

where $(\ ,\ )$ is the canonical inner product of $\boldsymbol{R}^n$. Define the even Clifford algebra $C^+(\boldsymbol{R}^n)$ by

$$C^+(\boldsymbol{R}^n) := T^+(\boldsymbol{R}^n)/U(\boldsymbol{R}^n).$$

We denote the multiplication of $\alpha, \beta \in C^+(\boldsymbol{R}^n)$ by $\alpha \cdot \beta$. For $x, y \in \boldsymbol{R}^n$ we have

$$x \cdot y + y \cdot x = -2(x, y). \tag{1}$$

It is known that a spinor group $Spin(n)$ is defined by

$$Spin(n) := \left\{ \alpha = a_1 \cdot a_2 \cdots a_{2l} \in C^+(\boldsymbol{R}^n) \;\middle|\; a_i \in \boldsymbol{R}^n,\ \prod_{i=1}^{2l}(a_i, a_i) = 1 \right\}.$$

The unit element of $Spin(n)$ is $1 = -a \cdot a \quad (a \in \boldsymbol{R}^n,\ (a,a) = 1)$ and an inverse element of

$$\alpha = a_1 \cdot a_2 \cdots a_{2l-1} \cdot a_{2l} \in Spin(n)$$

is

$$\alpha^{-1} = a_{2l} \cdot a_{2l-1} \cdots a_2 \cdot a_1 \in Spin(n).$$

The vector representation $p_1 : Spin(n) \to SO(n)$ is given by

$$p_1(\alpha)x = \alpha \cdot x \cdot \alpha^{-1} \quad (x \in \boldsymbol{R}^n).$$

It is known that $Spin(n)$ is a universal covering group of $SO(n)$ (double covering), and $Spin(n)$ $(n \geq 3)$ is simply connected.

# 3 Cayley algebra and spinor groups

In the division Cayley algebra $\mathfrak{C}$, we denote the multiplication and the canonical conjugation by $xy$ and $\overline{x}$ $(x, y \in \mathfrak{C})$ respectivly. The inner product of $\mathfrak{C}$ is defined by

$$(x,y) := \frac{1}{2}(x\overline{y} + y\overline{x}).$$

We describe here some formulas of the Cayley algebra (use in later). For $x, y, z \in \mathfrak{C}$, we have

$$x(\overline{y}z) + y(\overline{x}z) = 2(x,y)z = (zx)\overline{y} + (zy)\overline{x}, \tag{2}$$

$$(xy)x = x(yx) = xyx, \tag{3}$$

$$(xy)(zx) = x(yz)x, \tag{4}$$

$$(xy, xz) = (x,x)(y,z) = (yx, zx). \tag{5}$$

We identify $\mathfrak{C}$ with $\boldsymbol{R}^8$ and $\mathrm{Im}\mathfrak{C} = \{x \in \mathfrak{C} \mid \overline{x} + x = 0\}$ with $\boldsymbol{R}^7$. Then we see

$$Spin(8) = \left\{\alpha = a_1 \cdot a_2 \cdots a_{2l} \;\middle|\; a_i \in \mathfrak{C},\ \textstyle\prod_{i=1}^{2l}(a_i, a_i) = 1 \right\},$$

$$Spin(7) = \left\{\alpha = a_1 \cdot a_2 \cdots a_{2l} \;\middle|\; a_i \in \mathrm{Im}\mathfrak{C},\ \textstyle\prod_{i=1}^{2l}(a_i, a_i) = 1 \right\}.$$

**Lemma 3.1** *For $\alpha = a_1 \cdot a_2 \cdots a_{2l} \in Spin(8)$ and $x \in \mathfrak{C}$, we have*

$$p_1(\alpha) = a_1(\overline{a}_2(a_3(\cdots a_{2l-1}(\overline{a}_{2l} x \overline{a}_{2l})a_{2l-1} \cdots)a_3)\overline{a}_2)a_1.$$

Proof. It is sufficient to prove the claim for $\alpha = a \cdot b \in Spin(8)$. From (1) we see $b \cdot x \cdot b = (b,b)x - 2(x,b)b$ and

$$\begin{aligned} p_1(\alpha)x &= a \cdot b \cdot x \cdot b \cdot a \\ &= (a,a)(b,b)x - 2(x,a)(b,b)a - 2(x,b)(a,a)b + 4(x,b)(a,b)a. \end{aligned}$$

On the other hand, from (2) we see $\bar{b}x\bar{b} = 2(x,b)\bar{b} - (b,b)\bar{x}$ and

$$a(\bar{b}x\bar{b})a = (a,a)(b,b)x - 2(x,a)(b,b)a - 2(x,b)(a,a)b + 4(x,b)(a,b)a.$$

□

A linear transformation

$$\mathfrak{C} \otimes \mathfrak{C} \to \mathrm{End}(\mathfrak{C}), \ a \otimes b \mapsto -L_a L_{\bar{b}}$$

(where $L_a x = ax$) can be extended to a representation

$$p_2 : T^+(\mathfrak{C}) \to \mathrm{End}(\mathfrak{C}), \ p_2(a \otimes b) \mapsto -L_a L_{\bar{b}}.$$

From (2) we see

$$p_2(a \otimes a + (a,a)1) = -L_a L_{\bar{a}} + (a,a)1 = 0,$$

then we have $p_2(U(\mathfrak{C})) = \{0\}$ and $p_2$ is a representation of $C^+(\mathfrak{C}) := T^+(\mathfrak{C})/U(\mathfrak{C})$. Then we have a representation $p_2$ of $Spin(8)$ on $\mathfrak{C}$

$$p_2(a_1 \cdot a_2 \cdots a_{2l}) = (-1)^q L_{a_1} L_{\bar{a}_2} L_{a_3} \cdots L_{\bar{a}_{2l}}.$$

In a similar way we define a representation $p_3$ of $Spin(8)$ on $\mathfrak{C}$ by

$$p_3(a_1 \cdot a_2 \cdots a_{2q}) = (-1)^q R_{a_1} R_{\bar{a}_2} R_{a_3} \cdots R_{\bar{a}_{2l}}$$

where $R_a x = xa$. Then we have following lemma from (4) and lemma 3.1.

**Lemma 3.2** *For $\alpha \in Spin(8)$ and $x, y \in \mathfrak{C}$, we have*

$$p_1(\alpha)(xy) = (p_2(\alpha)x)(p_3(\alpha)y).$$

From (5), we see

$$p_2(\alpha), p_3(\alpha) \in SO(8) = \{A \in GL(\mathfrak{C}) \mid (Ax, Ay) = (x,y), \quad x, y \in \mathfrak{C}\}.$$

If $p_2(\alpha)1 = 1$, since

$$p_1(\alpha)x = p_1(\alpha)(1x) = (p_2(\alpha)1)(p_3(\alpha)x) = p_3(\alpha)x,$$

we see $p_1(\alpha) = p_3(\alpha)$. Similarly, if $p_3(\alpha)1 = 1$ we see $p_1(\alpha) = p_2(\alpha)$. Hence we have

**Proposition 3.3** *For $\alpha \in Spin(8)$, the following two conditions are equivalent.*

(i) $p_2(\alpha)1 = p_3(\alpha)1 = 1$,

(ii) $p_1(\alpha) = p_2(\alpha) = p_3(\alpha) \in G_2$.

# 4 Main theorem

It is known that the group $G_2$ is a subgroup of $SO(7) = p_1(Spin(7))$.

**Lemma 4.1** *For* $\alpha = a_1 \cdot a_2 \cdots a_{2l} \in Spin(7)$, *2 conditions* $p_2(\alpha)1 = 1$ *and* $p_3(\alpha)1 = 1$ *are equivalent.*

Proof. Since $\overline{a} = -a$ for $a \in \mathrm{Im}\mathfrak{C}$, we see

$$p_2(\alpha) = L_{a_1}L_{a_2}\cdots L_{a_{2l}}, \qquad p_3(\alpha) = R_{a_1}R_{a_2}\cdots R_{a_{2l}}.$$

If $p_2(\alpha)1 = 1$, we see

$$\begin{aligned} 1 = \overline{1} = \overline{p_2(\alpha)1} &= \overline{a_1(a_2(\cdots a_{2l-2}(a_{2l-1}a_{2l})))} \\ &= (((\overline{a}_{2l}\overline{a}_{2l-1})\overline{a}_{2l-2}\cdots)\overline{a}_2)\overline{a}_1 = (((a_{2l}a_{2l-1})a_{2l-2}\cdots)a_2)a_1 \\ &= p_3(\alpha)1. \end{aligned}$$

If $p_3(\alpha)1 = 1$, similarly we have $p_2(\alpha)1 = 1$.

□

From proposition 3.3 and lemma 4.1, we have the following.

**Proposition 4.2** *For* $a_1, a_2, \cdots, a_{2l} \in \mathrm{Im}\mathfrak{C}$, *let us put* $g := L_{a_1}L_{a_2}\cdots L_{a_{2l}}$. *If* $g1 = 1$, $g \in G_2$.

Let us put

$$K := \{g = L_{a_1}L_{a_2}\cdots L_{a_{2l}} \mid a_k \in \mathrm{Im}\mathfrak{C}, g1 = 1\}.$$

Then $K$ is a subgroup of $G_2$. Since $\alpha L_a \alpha^{-1} = L_{\alpha(a)}$, $(\alpha \in G_2)$ the subgroup $K$ is normal. In next section we show $K \neq \{e\}$. Then we have

**Theorem 4.3**

$$G_2 = \{g = L_{a_1}L_{a_2}\cdots L_{a_{2l}} \mid a_k \in \mathrm{Im}\mathfrak{C}, g1 = 1\}.$$

# 5 Example of elements

Let $\{e_0 = 1, e_1, e_2, \cdots, e_7\}$ be a basis of $\mathfrak{C}$ with following conditions.

$$\begin{aligned} &e_0e_k = e_ke_0 = e_k \ (0 \le k \le 7), \quad (e_0 = 1, \text{ the unit element}), \\ &e_k^2 = -e_0 \ (k \neq 0), \quad e_ke_l = -e_le_k \ (1 \le k \neq l \le 7), \\ &e_1 = e_2e_3 = e_4e_5 = e_6e_7, \quad e_2 = e_3e_1 = e_6e_4 = e_5e_7, \\ &e_3 = e_1e_2 = e_4e_7 = e_5e_6, \quad e_4 = e_5e_1 = e_2e_6 = e_7e_3, \\ &e_5 = e_1e_4 = e_7e_2 = e_6e_3, \quad e_6 = e_7e_1 = e_4e_2 = e_3e_5, \\ &e_7 = e_1e_6 = e_2e_5 = e_3e_4. \end{aligned}$$

Define an element $G_{ij}$ $(0 \leq i \neq j \leq 7)$ of $\mathfrak{so}(8) = \{X \in \mathfrak{gl}(\mathfrak{C}) \mid (Xx, y) + (x, Xy) = 0, \quad x, y \in \mathfrak{C}\}$ by

$$G_{ij} e_k = \delta_{jk} e_i - \delta_{ik} e_j.$$

In [F], Freudenthal proved if $e_i e_j = e_k e_l$ (for example $e_2 e_3 = e_4 e_5, e_3 e_1 = e_6 e_4, \cdots$ etc.), $G_{ji} - G_{lk} \in \mathfrak{g}_2$. Let us put

$$a_1 = \cos\frac{\theta}{2} e_i + \sin\frac{\theta}{2} e_j, a_2 = e_i, a_3 = e_k, a_4 = \cos\frac{\theta}{2} e_k + \sin\frac{\theta}{2} e_l,$$
$$h = h_{ijkl}(\theta) = L_{a_1} L_{a_2} L_{a_3} L_{a_4}.$$

By a straightforward calculation, we have

$$h1 = 1 \quad \text{and} \quad he_p = \begin{cases} \cos\theta e_i + \sin\theta e_j & (p = i), \\ -\sin\theta e_i + \cos\theta e_j & (p = j), \\ \cos\theta e_k - \sin\theta e_l & (p = k), \\ \sin\theta e_k + \cos\theta e_l & (p = l), \\ e_p & (\text{others}). \end{cases}$$

This show

$$\frac{d}{d\theta} h_{ijkl}(\theta)\Big|_{\theta=0} = G_{ji} - G_{lk}.$$

## References

[F] Freudenthal, H., Okutaven, Ausnahmegruppen und Okutavengeometrie, Math. Inst. Rijksuniv. te Utrecht, 1951.