数理解析研究所講究録 1378 巻 2004 年 75-86

DIFFERENTIAL OPERATORS OF DIRAC TYPES ON COMPLEX AND QUATERNION MANIFOLDS

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1. INTRODUCTION

We refer to [4] for this section.

Let $\mathbb{R}^2$ be a Euclidean space of dimension 2. We denote by C_2 the associated Clifford algebra. It is well known that C_2 is isomorphic to the field of quaternions $\mathbb{H}$ as an algebra and has $\mathbb{Z}^2$-grading:

$$C_2 \cong C_2^0 \oplus C_2^1.$$

This corresponds to the decomposition $\mathbb{H} \cong \mathbb{C} \oplus \mathbb{C}$. To be more precise, let e_1, e_2 be the standard basis of $\mathbb{R}^2$. We set

$$a + be_2e_1 \cong a + \sqrt{-1}b, \quad ae_1 + be_2 \cong a + \sqrt{-1}b,$$

where a and b are real numbers. Then we obtain the desired identification $C_2^0 \cong \mathbb{C}$ and $C_2^1 \cong \mathbb{C}$. The Euclidean structure with the orientation also induces the complex structure on $\mathbb{R}^2$ which is the same as the complex structure given by $C_2^1 \cong \mathbb{C}$.

Next, we consider the Dirac operator $D : \Gamma(C_2) \to \Gamma(C_2)$, where $\Gamma(C_2)$ is the space of C_2-valued functions. The Dirac operator D is expressed as:

$$Df = \sum_{i=1}^{2} e_i \cdot e_i(f),$$

where $\cdot$ denotes the Clifford multiplication and $e_i(f)$ is the derivative along e_i. If we introduce coordinates on $\mathbb{R}^2$ by $(x, y) \cong xe_1 + ye_2$, then D can be written as:

$$Df = e_1 \cdot \frac{\partial f}{\partial x} + e_2 \cdot \frac{\partial f}{\partial y}.$$

By definition, the Dirac operator respects the $\mathbb{Z}^2$-grading:

$$D : \Gamma(C_2^0) \to \Gamma(C_2^1), \quad D : \Gamma(C_2^1) \to \Gamma(C_2^0).$$

Let $f = u(x,y)1 + v(x,y)e_2e_1$ be a C_2^0-valued function. Then we have

$$Df = u_xe_1 + v_xe_2 + u_ye_2 - v_ye_1 = (u_x - v_y)\, e_1 + (v_x + u_y)\, e_2$$

When we adopt the identification $\mathbb{C}_2^0 \cong \mathbb{C}_2^1 \cong \mathbb{C}$, we obtain

$$Df = \frac{\partial}{\partial \overline{z}}(u + \sqrt{-1}v) = 2\frac{\partial f}{\partial \overline{z}},$$

where

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right).$$

In brief, the Dirac operator is the Cauchy-Riemann operator. Note that the whole setting is preserved by the action of Spin(2).

Let $\mathbb{R}^4$ be a Euclidean space of dimension 4. We denote by C_4 the associated Clifford algebra. It is well known that C_4 is isomorphic to $\mathbb{H}(2)$ the 2×2 matrix algebra over quaternions. If we denote by $e_1,\cdots,e_4$ the standard basis of $\mathbb{R}^4$, then the identification is realised as:

$$e_1 \mapsto \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$
$$e_3 \mapsto \begin{pmatrix}0 & k\\ -k & 0\end{pmatrix}, \quad e_4 \mapsto \begin{pmatrix}0 & -j\\ j & 0\end{pmatrix}$$

Note that $\mathbb{H}$ is not a commutative field. The scalar product on $\mathbb{H}^2$ is provided with multiplication by quaternions on the *right* and the quaternion matrix acts on $\mathbb{H}^2$ from the *left*. Consequently,

$$e_1\begin{pmatrix}1\\ i\end{pmatrix} = \begin{pmatrix}1\\ -i\end{pmatrix}, \quad e_2\begin{pmatrix}1\\ i\end{pmatrix} = \begin{pmatrix}i\\ 1\end{pmatrix} = \begin{pmatrix}1\\ -i\end{pmatrix} i$$
$$e_3\begin{pmatrix}1\\ i\end{pmatrix} = \begin{pmatrix}j\\ -k\end{pmatrix} = \begin{pmatrix}1\\ -i\end{pmatrix} j, \quad e_4\begin{pmatrix}1\\ i\end{pmatrix} = \begin{pmatrix}k\\ j\end{pmatrix} = \begin{pmatrix}1\\ -i\end{pmatrix} k$$

Hence we obtain a $\mathbb{Z}^2$-grading of $\mathbb{H}$ as a module of C_4:

$$\mathbb{H}^2 \cong \mathbb{H}\begin{pmatrix}1\\ i\end{pmatrix} \oplus \mathbb{H}\begin{pmatrix}1\\ -i\end{pmatrix} =: V_0 \oplus V_1.$$

The Dirac operator D is defined in a similar way and respects the $\mathbb{Z}^2$-grading:

$$D : \Gamma(V_0) \to \Gamma(V_1), \quad D : \Gamma(V_1) \to \Gamma(V_0).$$

We introduce coordinates on $\mathbb{R}^4$ by $(x_0, x_1, x_2, x_3) \cong \sum_{i=0}^3 x_i e_{i+1}$ and identify $\mathbb{R}^4$ with $\mathbb{H}$ as $(x_0, x_1, x_2, x_3) \cong x_0 + x_1 i + x_2 j + x_3 k = q$.

Let $f = u_0 + u_1 i + u_2 j + u_3 k$ be a V_0-valued function. (f may be regarded as $\begin{pmatrix}1\\ i\end{pmatrix} f$.) Then we have

$$\begin{aligned} Df &= \sum_{i=0}^{3} e_{i+1}\cdot\left(\frac{\partial f}{\partial x_i}\right) \cong \sum_{i=0}^{3} e_{i+1}\cdot\begin{pmatrix}1\\ i\end{pmatrix}\left(\frac{\partial f}{\partial x_i}\right) \\ &= \begin{pmatrix}1\\ -i\end{pmatrix}\left\{\left(\frac{\partial f}{\partial x_0}\right) + i\left(\frac{\partial f}{\partial x_1}\right) + j\left(\frac{\partial f}{\partial x_2}\right) + k\left(\frac{\partial f}{\partial x_3}\right)\right\} \\ &\cong \frac{\partial}{\partial \overline{q}} f, \end{aligned}$$

where, of course,

$$\frac{\partial}{\partial \overline{q}} = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}.$$

We should also mention that the whole setting is preserved by the action of Spin(4).

2. Generalisation

2.1. **Another interpretation.** Although the Dirac operator can be defined on any dimensional Euclidean space, Dirac operators in the previous section can be re-interpreted from a different viewpoint.

In the 2-dimensional case, this is obvious. Let J denote the complex structure of $\mathbb{R}^2$:

$$Je_1 = e_2, \quad Je_2 = -e_1$$

Consider the complexification $(\mathbb{R}^2)^{\mathbb{C}} \cong \mathbb{C}^2$ of $\mathbb{R}^2$. Then J can be extended as the complex linear transformation of $\mathbb{C}^2$. Since $J^2 = -1$, $\mathbb{C}^2$ is decomposed into the eigenspaces of J:

$$\mathbb{C}^2 = \mathbb{C}_{(1,0)} \oplus \mathbb{C}_{(0,1)},$$

where

$$\mathbb{C}_{(1,0)} = \left\{z \in \mathbb{C}^2 | Jz = \sqrt{-1}z\right\}, \quad \mathbb{C}_{(0,1)} = \left\{z \in \mathbb{C}^2 | Jz = -\sqrt{-1}z\right\},$$

in other words,

$$\frac{1}{2}\left(u - \sqrt{-1}Ju\right) \in \mathbb{C}_{(1,0)}, \quad \frac{1}{2}\left(u + \sqrt{-1}Ju\right) \in \mathbb{C}_{(0,1)},$$

where $u \in \mathbb{R}^2$. We can easily show that $(\mathbb{R}^2, J)$ is isomorphic to $\mathbb{C}_{(1,0)}$ as a complex vector space. The tangent space $T_x\mathbb{R}^2$ at a point $x \in \mathbb{R}^2$ is naturally identified with the vector space $\mathbb{R}^2$.

Let $\mathbb{C}^{2^*}$ be the dual space. According to the decomposition of $\mathbb{C}^2$, we have

$$\mathbb{C}^{2^*} = \mathbb{C}dz \oplus \mathbb{C}d\overline{z}, \quad dz = dx + \sqrt{-1}dy, \ d\overline{z} = dx - \sqrt{-1}dy.$$

Then, for a function f, we have a differential df:

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z},$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right).$$

Taking the complex conjugate of $\mathbb{C}^2$, we define a Hermitian inner product h on $\mathbb{C}^2$ as

$$h(z, w) = g(z, \overline{w})$$

where $g(u, v)$ is the bi-linear extension of the inner product on $\mathbb{R}^2$. Then $\mathbb{C}_{(1,0)} \perp \mathbb{C}_{(0,1)}$. Let $\pi : \mathbb{C}^2 \to \mathbb{C}d\overline{z}$ be the orthogonal projection. We can define a differential operator $\pi \circ d$. For a function f, we write explicitly $\pi \circ df$ down:

$$\pi \circ df = \frac{\partial f}{\partial \overline{z}}d\overline{z} \cong \frac{\partial f}{\partial \overline{z}}$$

and so

$$2\pi \circ df = Df.$$

This consistency is based on the group isomorphism $\mathrm{Spin}(2) \cong \mathrm{U}(1)$. The group $\mathrm{U}(1)$ can be considered as the unit complex numbers:

$$\mathrm{U}(1) = \left\{z \in \mathbb{C} \,\middle|\, |z| = 1\right\}.$$

Next, we concern the 4-dimensional case. We follow S.Salamon's description [9]. When we regard $\mathbb{R}^4$ as $\mathbb{H}$, $\mathbb{H}$ has a natural "quaternion"-Hermitian inner product $h_{\mathbb{H}}$:

$$h_{\mathbb{H}}(p,q) = \overline{q}p, \quad \overline{q} = x_0 - ix_1 - jx_2 - kx_3.$$

The set of unit quaternions has a group structure induced from multiplication of $\mathbb{H}$, which is denoted by Sp(1):

$$\mathrm{Sp}(1) = \left\{ q \in \mathbb{H} \,\middle|\, |q| = 1 \right\}.$$

If we identify $\mathbb{H}$ with $\mathbb{C}^2$ using the identification $i \cong \sqrt{-1}$, then $h_{\mathbb{H}}$ is decomposed into a Hermitian inner product and a complex volume form:

$$\begin{aligned} h_{\mathbb{H}}(p,q) &= \overline{q}p = \overline{(u+jv)}(z+jw) = (\overline{u} - jv)(z+jw) \\ &= (z\overline{u} + w\overline{v}) + j\,(wu - zv) = h\left(\begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right) + \omega\left(\begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right) j. \end{aligned}$$

The group Sp(1) preserves $h_{\mathbb{H}}$ and so, preserves a Hermitian inner product h and a complex volume form ω. This observation yields the isomorphism $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$.

Since unit quaternions act on $\mathbb{H}$ from the both sides and the left action commutes with the right action, $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ acts on $\mathbb{H}$. This action preserves the inner product and the volume form on $\mathbb{R}^4$ and so, we obtain a group homomorphism $\rho : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SO}(4)$. Since $\rho(1,1) = \rho(-1,-1) = \mathrm{Id}$, $\mathrm{Ker}\,\rho \cong \mathbb{Z}^2$. In this way, we have an identification:

$$\mathrm{Sp}(1) \times \mathrm{Sp}(1)/\mathbb{Z}^2 \cong \mathrm{SO}(4), \quad \text{or} \quad \mathrm{Sp}(1) \times \mathrm{Sp}(1) \cong \mathrm{Spin}(4).$$

To define differential operators from the quaternionic viewpoint, we recall the representation theory of $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$. Let $\mathbb{C}^2$ be the standard representation of SU(2). Then the k-th symmmetric tensor product $S^k\mathbb{C}^2$ ($\dim S^k\mathbb{C}^2 = k+1$) is an irreducible representation of SU(2) and each finite dimensional irreducible representation is one of them. In particular, we need an irreducible decomposition of $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \wedge^2\mathbb{C}^2 \oplus S^2\mathbb{C}^2$. Then $\wedge^2\mathbb{C}^2 = \mathbb{C}\omega$ and the classification of irreducible representation yields that

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C} \oplus S^2\mathbb{C}^2.$$

We denote two copies of the standard representaion of Sp(1) by $\mathbb{H}$ and $\mathbb{E}$. The tensor product $\mathbb{H}\otimes_{\mathbb{C}}\mathbb{E}$ (for short, $\mathbb{H}\otimes\mathbb{E}$) is of complex dimension 4. Here we identify $\mathbb{H}$ and $\mathbb{E}$ with $\mathbb{C}^2$ using $i \cong \sqrt{-1}$. Since $ij = -ji$, j is an anti-linear (or conjugate-linear) transformation. We refer to j as the quaternion structure which is preserved by Sp(1). Then $\sigma = j \otimes j$ acts on $\mathbb{H} \otimes \mathbb{E}$ and $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ also preserves σ. By definition, $\sigma^2 = j^2 \otimes j^2 = 1$ and σ is still an anti-linear transformation and so, σ is called the real structure. The invariant subset $(\mathbb{H}\otimes\mathbb{E})^{\mathbb{R}}$ under σ of $\mathbb{H}\otimes\mathbb{E}$ is a real vector space of dimension 4, and $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ acts on $(\mathbb{H} \otimes \mathbb{E})^{\mathbb{R}}$. Hence we recover $\mathbb{R}^4 \cong (\mathbb{H} \otimes \mathbb{E})^{\mathbb{R}}$.

Let f be an $\mathbb{H}$-valued function. Then $\dfrac{\partial f}{\partial x_i} = f_{x_i}$ is also an $\mathbb{H}$-valued function and so,

$$\begin{pmatrix} f_{x_0} \\ f_{x_1} \\ f_{x_2} \\ f_{x_3} \end{pmatrix} \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}^4 \cong \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{E} \cong (\mathbb{C} \oplus S^2\mathbb{H}) \otimes \mathbb{E}$$

We define the orthogonal projections π_i by

$$\pi_1 : (\mathbb{C} \oplus S^2\mathbb{H}) \otimes \mathbb{E} \to \mathbb{C} \otimes \mathbb{E} \cong \mathbb{E}, \quad \pi_2 : (\mathbb{C} \oplus S^2\mathbb{H}) \otimes \mathbb{E} \to S^2\mathbb{H} \otimes \mathbb{E}.$$

We use π_1 to define the differential operator $\pi_1 \circ d : \Gamma(\mathbb{H}) \to \Gamma(\mathbb{E})$.

To compute explicitly, we use the standard basis h_1, h_2 of $\mathbb{H}$ which is a unitary basis and satisfies $h_2 = jh_1$. We also take the standard basis e_1 and e_2 of $\mathbb{E}$. Then the identification $\mathbb{R}^4 \cong \mathbb{H} \otimes \mathbb{E}$ is realised by:

$$\begin{aligned}(x_0, x_1, x_2, x_3) \cong & x_0(h_2 \otimes e_1 - h_1 \otimes e_2) + x_1\sqrt{-1}(h_2 \otimes e_1 + h_1 \otimes e_2) \\ & + x_2(h_1 \otimes e_1 + h_2 \otimes e_2) + x_3\sqrt{-1}(h_1 \otimes e_1 - h_2 \otimes e_2)\end{aligned}$$

Since

$$x_0 + ix_1 + jx_2 + kx_3 = (x_0 + ix_1) + j(x_2 - ix_3),$$

we put

$$z = x_0 + x_1, \quad w = x_2 - ix_3.$$

If $f = uh_1 + vh_2 \in \Gamma(\mathbb{H})$, where u and v are $\mathbb{C}$-valued functions, then

$$\begin{aligned} df =& (u_{x_0}h_1 + v_{x_0}h_2) \otimes (h_2 \otimes e_1 - h_1 \otimes e_2) \\ & + (u_{x_1}h_1 + v_{x_1}h_2) \otimes \sqrt{-1}(h_2 \otimes e_1 + h_1 \otimes e_2) \\ & + (u_{x_2}h_1 + v_{x_2}h_2) \otimes (h_1 \otimes e_1 + h_2 \otimes e_2) \\ & + (u_{x_3}h_1 + v_{x_3}h_2) \otimes \sqrt{-1}(h_1 \otimes e_1 - h_2 \otimes e_2) \\ \xrightarrow[\pi_1]{} & (u_{x_0}e_1 + v_{x_0}e_2) + \sqrt{-1}\,(u_{x_1}e_1 - v_{x_1}e_2) \\ & + (u_{x_2}e_2 - v_{x_2}e_1) - \sqrt{-1}\,(u_{x_3}e_2 + v_{x_3}e_1) \\ =& \left(u_{x_0} + \sqrt{-1}u_{x_1} - v_{x_2} - \sqrt{-1}v_{x_3}\right) e_1 \\ & + \left(v_{x_0} - \sqrt{-1}v_{x_1} + u_{x_2} - \sqrt{-1}u_{x_3}\right) e_2. \end{aligned}$$

We set

$$\partial_z = \frac{\partial}{\partial x_0} - \sqrt{-1}\frac{\partial}{\partial x_1}, \quad \partial_{\bar{z}} = \frac{\partial}{\partial x_0} + \sqrt{-1}\frac{\partial}{\partial x_1},$$
$$\partial_w = \frac{\partial}{\partial x_2} + \sqrt{-1}\frac{\partial}{\partial x_3}, \quad \partial_{\bar{w}} = \frac{\partial}{\partial x_2} - \sqrt{-1}\frac{\partial}{\partial x_3}.$$

Then

$$\begin{aligned}\pi_1 \circ df =& (\partial_{\bar{z}}u - \partial_w v)e_1 + (\partial_z v + \partial_{\bar{w}}u)e_2 \cong (\partial_{\bar{z}}u - \partial_w v) + j(\partial_z v + \partial_{\bar{w}}u) \\ =& \partial_{\bar{z}}u + j\partial_z v + j\,(\partial_{\bar{w}}u + j\partial_w v) = (\partial_{\bar{z}} + j\partial_{\bar{w}})\,(u + jv) = \frac{\partial}{\partial \bar{q}}f.\end{aligned}$$

We obtain

$$\pi_1 \circ df = Df.$$

We also have a differential operator $\mathcal{D} : \Gamma(\mathbb{H}) \to \Gamma(S^2\mathbb{H} \otimes \mathbb{E})$:

$$\mathcal{D} = \pi_2 \circ d,$$

which is called the twistor operator [1].

2.2. **Higher dimensional analogue.** From the viewpoint of complex number field, it is now clear that we have a generalisation of the differential operator $2\pi \circ d = D$ on $\mathbb{R}^2$. We may replace $\mathbb{C} \cong \mathbb{R}^2$ by $\mathbb{C}^n$ or the structure group $\mathrm{U}(1)$ by $\mathrm{U}(n)$. Let $(z_1, \cdots, z_n)$ be the standard coordinates of $\mathbb{C}^n$. Then, for a $\mathbb{C}$-valued function f, we define a system of differential operators:

$$\overline{\partial} f = \left(\frac{\partial}{\partial \overline{z}_1} f, \cdots, \frac{\partial}{\partial \overline{z}_n} f \right).$$

In an invariant way, we regard $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ with a complex structure J. The complex structure J can be extended to a complex linear transformation on the complexified vector space $\mathbb{C}^{2n} \cong \mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$. As before, we obtain the eigenspaces of J:

$$\mathbb{C}_{(1,0)} = \left\{ v \in \mathbb{C}^{2n} \middle| Jv = \sqrt{-1} v \right\}, \quad \mathbb{C}_{(0,1)} = \left\{ v \in \mathbb{C}^{2n} \middle| Jv = -\sqrt{-1} v \right\}$$

and an isomorphism $\mathbb{C}^n \cong \mathbb{C}_{(1,0)}$. Let $\mathbb{C}^{n^*}$ be the dual space of $\mathbb{C}^n$. Since the dual space has also a natural complex structure, we obtain in a similar way

$$\mathbb{C}^{(1,0)} = \left\{ \phi \in \mathbb{C}^{2n^*} \middle| J\phi = \sqrt{-1} \phi \right\}, \quad \mathbb{C}^{(0,1)} = \left\{ \phi \in \mathbb{C}^{2n^*} \middle| J\phi = -\sqrt{-1} \phi \right\}$$

When we use coordinates $z_1, \cdots, z_n$ on $\mathbb{C}^n$, the basis of $\mathbb{C}^{(1,0)}$ (resp. $\mathbb{C}^{(0,1)}$) consists of

$$dz_1, \cdots, dz_n \quad (\text{resp.}\, d\overline{z}_1, \cdots, d\overline{z}_n).$$

Then the differential of f is expressed as:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} dz_i + \sum_{i=1}^{n} \frac{\partial f}{\partial \overline{z}_i} d\overline{z}_i.$$

We can also define the orthogonal projection $\pi : \mathbb{C}^{2n} \to \mathbb{C}^{(0,1)}$. It can be shown that

$$\overline{\partial} f = \pi \circ df$$

Although $2\overline{\partial} = D$ is an elliptic operator in the case $n = 1$, higher dimensional analogue $\overline{\partial}$ is not an elliptic operator when $n \geqq 2$. But we have an elliptic complex. The complex vector space generated by $dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ (resp. $d\overline{z}_{i_1} \wedge \cdots \wedge d\overline{z}_{i_q}$) is denoted by $\bigwedge^{p,0}$ (resp. $\bigwedge^{0,q}$). We can consider a k-form ϕ of bidegree (p, q):

$$\phi = \sum_{\substack{p+q=k \\ 0 \leqq i_1 < \cdots < i_p \leqq n \\ 0 \leqq j_1 < \cdots < j_q \leqq n}} \phi_{i_1, \cdots, i_p, j_1 \cdots, j_q} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q},$$

where $\phi_{i_1,\cdots,i_p,j_1\cdots,j_q}$ is a function on $\mathbb{C}^{2n}$. Since

$$d\phi_{i_1,\cdots,i_p,j_1\cdots,j_q} = \partial_{z_i}\phi_{i_1,\cdots,i_p,j_1\cdots,j_q} dz_i + \partial_{\bar{z}_i}\phi_{i_1,\cdots,i_p,j_1\cdots,j_q} d\bar{z}_j,$$

we have a differential of ϕ:

$$d\phi = \sum\Big(\sum_{i=1}^{n}\partial_{z_i}\phi_{i_1,\cdots,i_p,j_1\cdots,j_q} dz_i \wedge dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{j_q}$$
$$+\sum_{j=1}^{n}\partial_{\bar{z}_j}\phi_{i_1,\cdots,i_p,j_1\cdots,j_q} d\bar{z}_j \wedge dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{j_q}\Big)$$

Hence, we can also define

$$\overline{\partial}\phi = \sum \overline{\partial}\phi_{i_1,\cdots,i_p,j_1\cdots,j_q}\wedge dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{j_q}.$$

Note that

$$\overline{\partial}\phi_{i_1,\cdots,i_p,j_1\cdots,j_q} = \sum_{j=1}^{n}\partial_{\bar{z}_j}\phi_{i_1,\cdots,i_p,j_1\cdots,j_q} d\bar{z}_j,$$

which is already defined. Consequently, we obtain a generalisation of a differential $\overline{\partial}$ for a function to a differential for a k-form which is denoted by the same symbol $\overline{\partial}$. By definition, $\overline{\partial}\circ\overline{\partial} = 0$. We use $\overline{\partial}$ to get an elliptic complex:

$$0 \to \Omega^{p,q} \xrightarrow{\overline{\partial}} \Omega^{p,q+1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{p,n} \to 0,$$

where

$$\Omega^{p,q} = \Gamma(\wedge^{p,q})$$

In the case of the field of quaternions, $\mathbb{H}^n \cong \mathbb{R}^{4n}$ can be taken as a generalisation in an obvious sense [8]. The group $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ is replaced by $\mathrm{Sp}(1)\times\mathrm{Sp}(n)$. We also denote by $\mathbb{E}$ the standard representation of $\mathrm{Sp}(n)$ ($\mathbb{E}\cong\mathbb{C}^{2n}\cong\mathbb{H}^n$). Therefore

$$\mathbb{R}^{4n} \cong (\mathbb{H}\otimes\mathbb{E})^{\mathbb{R}}.$$

Now we define a differential operator $D:\Gamma(\mathbb{H})\to\Gamma(\mathbb{E})$ as

$$D:\Gamma(\mathbb{H}) \xrightarrow{d} \Gamma(\mathbb{H}\otimes\mathbb{H}\otimes\mathbb{E}) \cong \Gamma\left((\mathbb{C}\oplus S^2\mathbb{H})\otimes\mathbb{E}\right) \xrightarrow{\pi_1} \Gamma(\mathbb{E})$$

[9]. We have another differential operator $\mathcal{D}:\Gamma(\mathbb{H})\to\Gamma(S^2\mathbb{H}\otimes\mathbb{E})$ as

$$\mathcal{D}:\Gamma(\mathbb{H}) \xrightarrow{d} \Gamma(\mathbb{H}\otimes\mathbb{H}\otimes\mathbb{E}) \cong \Gamma\left((\mathbb{C}\oplus S^2\mathbb{H})\otimes\mathbb{E}\right) \xrightarrow{\pi_2} \Gamma(S^2\mathbb{H}\otimes\mathbb{E}).$$

Using the Clebsch-Gordan formula, we obtain

$$\mathbb{H}\otimes S^p\mathbb{H} \cong S^{p-1}\mathbb{H}\oplus S^{p+1}\mathbb{H}.$$

Combined with an algebraic homomorphism : $\mathbb{E}\otimes\wedge^q\mathbb{E} \to \mathbb{E}\otimes(\otimes^q\mathbb{E}) \to \wedge^{q+1}\mathbb{E}$, we have two extensions of differential operators D and $\mathcal{D}$:

$$\begin{aligned} D : \Gamma(S^p\mathbb{H}\otimes\wedge^q\mathbb{E}) &\xrightarrow{d}\Gamma\left(S^p\mathbb{H}\otimes\wedge^q\mathbb{E}\otimes\mathbb{H}\otimes\mathbb{E}\right) \\ &\cong\Gamma\left((S^p\mathbb{H}\otimes\mathbb{H})\otimes(\wedge^q\mathbb{E}\otimes\mathbb{E})\right) \xrightarrow{\pi_1} \Gamma\left(S^{p-1}\mathbb{H}\otimes\wedge^{q+1}\mathbb{E}\right) \\ \mathcal{D} : \Gamma(S^p\mathbb{H}\otimes\wedge^q\mathbb{E}) &\xrightarrow{d}\Gamma\left(S^p\mathbb{H}\otimes\wedge^q\mathbb{E}\otimes\mathbb{H}\otimes\mathbb{E}\right) \\ &\cong\Gamma\left((S^p\mathbb{H}\otimes\mathbb{H})\otimes(\wedge^q\mathbb{E}\otimes\mathbb{E})\right) \xrightarrow{\pi_2} \Gamma\left(S^{p+1}\mathbb{H}\otimes\wedge^{q+1}\mathbb{E}\right) \end{aligned}$$

Here D is called a quaternion-Dirac operator and $\mathcal{D}$ is called a twistor operator.

We also obtain an elliptic complex:

$$\begin{aligned} 0 \to C^\infty(\mathbb{R}^{4n}) &\xrightarrow{d}\Gamma(\mathbb{H}\otimes\mathbb{E}) \xrightarrow{\mathcal{D}} \Gamma(S^2\mathbb{H}\otimes\wedge^2\mathbb{E}) \xrightarrow{\mathcal{D}} \cdots \\ &\xrightarrow{\mathcal{D}} \Gamma(S^p\mathbb{H}\otimes\wedge^p\mathbb{E}) \xrightarrow{\mathcal{D}} \cdots \xrightarrow{\mathcal{D}} \Gamma(S^{2n}\mathbb{H}\otimes\wedge^{2n}\mathbb{E}) \to 0. \end{aligned}$$

3. Generalisation II

We have already found *linear* differential operators. Here we consider a *non-linear* problem which relates to our differential operators.

For our purpose, we replace a vector valued function or form by a matirix valued function or form such that

$$A = A_1dx_1 + \cdots + A_ndx_n \quad \text{on } \mathbb{R}^n,$$

where

$$A_i \in \mathbb{C}(r) := \{r\times r \text{ matrices over } \mathbb{C}\}.$$

We introduce a new differentiation ∇:

$$\nabla = d + A, \quad \nabla_i = \partial_i + A_i, \quad \partial_i = \frac{\partial}{\partial x_i}$$

which acts on $\mathbb{C}^r$-valued function. Although $d^2 = 0$ which means that $\partial_i\partial_j = \partial_j\partial_i$, we have $\nabla^2 \neq 0$. In fact,

$$[\nabla_i, \nabla_j] = \nabla_i\nabla_j - \nabla_j\nabla_i = \partial_iA_j - \partial_jA_i + [A_i, A_j]$$

In this way, we obtain a curvature 2-form or a gauge field F which associates to A which is called the connection form or the gauge potential:

$$F = \sum_{i,j} F_{ij}dx_i\wedge dx_j, \quad F_{ij} = \partial_iA_j - \partial_jA_i + [A_i, A_j].$$

Note that F is a non-linear function of a given A. Finally, we can extend the operator ∇ to the covariant exterior differentiation d^∇ which acts on $\mathbb{C}^r$-valued k-forms. For a $\mathbb{C}^r$-valued k-form

$$\phi_{i_1,\cdots,i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k} \in \Gamma(\wedge^k\otimes\mathbb{C}^r) = \Omega^k(\mathbb{C}^r),$$

we define

$$d^{\nabla}\left(\phi_{i_1,\cdots,i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k}\right)=(\nabla\phi_{i_1,\cdots,i_k})\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$
$$=\left(\sum_{i=1}^{n}\nabla_i\phi_{i_1,\cdots,i_k}dx_i\right)\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}\in\Omega^{k+1}(\mathbb{C}^r).$$

Then it is easily shown that

$$d^{\nabla}d^{\nabla}s=Fs\quad\text{for } s\in\Gamma(\mathbb{C}^r)=\Omega^0(\mathbb{C}^r),\quad (d^{\nabla}d^{\nabla}=F)$$

When f is a function, then we have

$$d^{\nabla}d^{\nabla}fs=fd^{\nabla}d^{\nabla}s,$$

and so, F can be regarded as

$$F\in\Omega^2\left(\mathrm{End}(\mathbb{C}^r)\right).$$

3.1. **$\mathbb{C}$-case.** We begin with a vector space $\mathbb{C}^n$ as a *base* space. For a given gauge potential $A=\sum_{i=1}^n\left(A_{z_i}dz_i+A_{\bar z_i}d\bar z_i\right)$ and a $\mathbb{C}^r$ (a *fibre*) -valued (p,q)-form $\phi\in\Omega^{p,q}(\mathbb{C}^r):=\Gamma\left(\wedge^{p,q}\otimes\mathbb{C}^r\right)$, we have

$$d^{\nabla}\phi\in\Omega^{p+1,q}(\mathbb{C}^r)\oplus\Omega^{p,q+1}(\mathbb{C}^r),$$

and so, we can define in an obvious way,

$$\partial^{\nabla}\phi=\pi_1d^{\nabla}\phi\in\Omega^{p+1,q}(\mathbb{C}^r),\quad \overline{\partial}^{\nabla}\phi=\pi_2d^{\nabla}\phi\in\Omega^{p,q+1}(\mathbb{C}^r).$$

Consider a differential equation

$$(*)\qquad \overline{\partial}^{\nabla}s=0,$$

for a $\mathbb{C}^r$-valued function s. If such an s exists, then

$$\overline{\partial}^{\nabla}\overline{\partial}^{\nabla}s=0,$$

which means

$$F^{0,2}s=0.$$

Note that

$$F\in\Omega^2\left(\mathrm{End}(\mathbb{C}^r)\right)\cong\Omega^{2,0}\left(\mathrm{End}(\mathbb{C}^r)\right)\oplus\Omega^{1,1}\left(\mathrm{End}(\mathbb{C}^r)\right)\oplus\Omega^{0,2}\left(\mathrm{End}(\mathbb{C}^r)\right).$$

Indeed, the condition

$$F^{0,2}=0$$

is the integrability condition for the equation $(*)$ [1]. If $F^{0,2}=0$ is satisfied, then we can find locally enough solutions for $(*)$, which provide a basis of $\mathbb{C}^r$ at each point of $\mathbb{C}^n$. In other words, there exist locally defined $s_1,\cdots,s_r$ which satisfy $(*)$ and span the vector space $\mathbb{C}^r_z$ at each point $z\in\mathbb{C}^n$. Hence $\mathbb{C}^r_z$ can be thought as varying holomorphically with z. In this way, we obtain a *holomorphic vector bundle.* Then the *frame* $s_1,\cdots,s_r$ is called the holomorphic *gauge.*

As a consequence, a connection form A satisfying $F^{0,2}=0$ gives a holomorphic vector bundle. Since

$$F^{0,2}_{ij}=\overline{\partial}_iA_{\bar z_j}-\overline{\partial}_jA_{\bar z_i}+\left[A_{\bar z_i},A_{\bar z_j}\right],$$

the equation $F^{0,2} = 0$ is a non-linear equation of the first order and we can find the Cauchy-Riemann operator as the linearisation:

$$\bar{\partial}^{\nabla} B = 0, \quad \text{for} \quad B \in \Omega^{0,1}\left(\mathrm{End}(\mathbb{C}^r)\right).$$

Using again $\bar{\partial}^{\nabla}\bar{\partial}^{\nabla} = F^{0,2} = 0$, we have an elliptic complex:

$$0 \to \Omega^{0,0}\left(\mathrm{End}(\mathbb{C}^r)\right) \xrightarrow{\bar{\partial}^{\nabla}} \Omega^{0,1}\left(\mathrm{End}(\mathbb{C}^r)\right) \xrightarrow{\bar{\partial}^{\nabla}} \cdots \xrightarrow{\bar{\partial}^{\nabla}} \Omega^{0,n}\left(\mathrm{End}(\mathbb{C}^r)\right) \to 0.$$

3.2. **$\mathbb{H}$-case.** Let $\mathbb{R}^{4n} \cong (\mathbb{H} \otimes \mathbb{E})^{\mathbb{R}}$ be the vector space acted by $\mathrm{Sp}(1) \cdot \mathrm{Sp}(n) = \mathrm{Sp}(1) \times \mathrm{Sp}(n)/\mathbb{Z}^2$. Although $\Omega^1 = \Gamma(\mathbb{H} \otimes \mathbb{E})$ is irreducible, $\wedge^2 = \wedge^2(\mathbb{H} \otimes \mathbb{E})$ can be decomposed into irreducible components:

$$\wedge^2 = (S^2\mathbb{H} \otimes \wedge^2 E) \oplus (\wedge^2\mathbb{H} \otimes S^2\mathbb{E}). \tag{3.2.1}$$

We define the orthogonal projection π as:

$$\pi : \wedge^2 \to S^2\mathbb{H} \otimes \wedge^2\mathbb{E}.$$

We consider a connection form A and its curvature 2-form $F \in \Omega^2(\mathrm{End}\mathbb{C}^r)$. Then the equation

$$\pi \circ F = 0 \tag{**}$$

is a non-linear differential equation of the first order for A. By analogy with the $\mathbb{C}$-case, we call a vector bundle with such a connection form ($\pi \circ F = 0$) *quaternion-holomorphic* vector bundle (for example, see [5]).

For brevity, we focus our attention on the case $n = 1$. Then the decomposition (3.2.1) reduces to:

$$\wedge^2 = \wedge_+ \oplus \wedge_-,$$

where a basis of each space is

$$\wedge_+ : \begin{cases} dx_0 \wedge dx_1 + dx_2 \wedge dx_3 \\ dx_0 \wedge dx_2 - dx_1 \wedge dx_3 \\ dx_0 \wedge dx_3 + dx_1 \wedge dx_2, \end{cases} \qquad \wedge_- : \begin{cases} dx_0 \wedge dx_1 - dx_2 \wedge dx_3 \\ dx_0 \wedge dx_2 + dx_1 \wedge dx_3 \\ dx_0 \wedge dx_3 - dx_1 \wedge dx_2. \end{cases}$$

Then the equation (**) is written down as:

$$F^+ = 0$$

which is called the *anti-self-dual* equation [1].

A linearisation of the equation (**) is the composition:

$$\Omega^1\left(\mathrm{End}(\mathbb{C}^r)\right) = \Gamma\left(\mathrm{End}(\mathbb{C}^r) \otimes \mathbb{H} \otimes \mathbb{E}\right) \xrightarrow{d^{\nabla}} \Omega^2\left(\mathrm{End}(\mathbb{C}^r)\right)$$
$$\xrightarrow{\pi} \Gamma\left(\mathrm{End}(\mathbb{C}^r) \otimes S^2\mathbb{H} \otimes \wedge^2\mathbb{E}\right),$$

and so, we obtain the twistor operator $\mathcal{D}^{\nabla} = \pi \circ d^{\nabla}$ coupled to the connection A. We also have

$$\mathcal{D}^{\nabla}\mathcal{D}^{\nabla} = \mathcal{D}^{\nabla}\nabla = \pi \circ d^{\nabla}\nabla = \pi \circ F = 0$$

for a quaternionic holomorphic vector bundle. Consequently, there exists an elliptic complex:

$$0 \to \Gamma(\mathrm{End}(\mathbb{C}^r)) \xrightarrow{d^\nabla} \Gamma(\mathbb{H} \otimes \mathbb{E} \otimes \mathrm{End}(\mathbb{C}^r)) \xrightarrow{\mathcal{D}^\nabla} \Gamma\left(S^2\mathbb{H} \otimes \wedge^2\mathbb{E} \otimes \mathrm{End}(\mathbb{C}^r)\right)$$

$$\xrightarrow{\mathcal{D}^\nabla} \cdots \xrightarrow{\mathcal{D}^\nabla} \Gamma\left(S^{2n}\mathbb{H} \otimes \wedge^{2n}\mathbb{E} \otimes \mathrm{End}(\mathbb{C}^r)\right) \to 0.$$

In particular, in the 4-dimensional case, our elliptic complex is reduced to:

$$0 \to \Gamma(\mathrm{End}(\mathbb{C}^r)) \xrightarrow{d^\nabla} \Omega^1(\mathrm{End}(\mathbb{C}^r)) \xrightarrow{\mathcal{D}^\nabla} \Omega^+(\mathrm{End}(\mathbb{C}^r)) \to 0,$$

which is called the Atiyah-Hitchin-Singer complex [1].

4. Generalisation III

For the first generalisation, after the identification Spin(2) ≅ U(1) and Spin(4) ≅ Sp(1) × Sp(1), the groups U(1) and Sp(1) × Sp(1) are replaced by U(n) and Sp(1) × Sp(n), respectively. We obtained differential operators of Dirac types according to the structure groups.

In the second process of a generalisation, we replace a function by a $\mathbb{C}^r$-valued function or a matrix valued function. As a consequence, we had a vector bundle, a connection form and a curvature form. Then we found non-linear differential equations which relate to the "Dirac equations" via linearisation.

Here we concern a manifold with a structure group U(n) or Sp(1) × Sp(n) and a vector bundle with a connection.

The manifold with a structure group $U(n)$ is called a **Kähler** manifold. The typical example is the complex projective line $\mathbb{C}P^1 \cong S^2$.

The tangent space of a Kähler manifold has a complex structure and can be regarded as a complex vector space with a Hermitian inner product. The parallel transport makes sense, because a Kähler manifold is also a Riemann manifold. Then, the complex structure and the Hermitian metric are preserved by the parallel transport.

Since our construction in the previous sections is purely local in nature, the Cauchy-Riemann operator $\overline{\partial}$ and a holomorphic vector bundle can be defined on a Kähler manifold.

We pay an attention on a complex line bundle L over a compact Kähler manifold M and the resulting elliptic complex:

$$0 \to \Omega^{0,0}(M; L) \xrightarrow{\overline{\partial}_0^\nabla} \Omega^{0,1}(M; L) \xrightarrow{\overline{\partial}_1^\nabla} \cdots \xrightarrow{\overline{\partial}_{n-1}^\nabla} \Omega^{0,n}(M; L) \to 0.$$

Then we can consider the cohomology

$$H^q(M; L) := \mathrm{Ker}\, \overline{\partial}_q^\nabla / \mathrm{Im}\, \overline{\partial}_{q-1}^\nabla$$

of the elliptic complex.

Theorem 4.1. (Kodaira vanishing theorem) (for example, see [2]) *If the holomorphic line bundle L is <u>negative</u> in some sense, then we have*

$$H^q(M; L) = 0 \quad \textit{for} \quad q \leqq n - 1$$

Finally, a manifold with a structure group $\mathrm{Sp}(1)\cdot\mathrm{Sp}(n)$ concerns us. (As seen previously, the group $\mathrm{Sp}(1)\times\mathrm{Sp}(n)$ does not act on $\mathbb{R}^4$ effectively, but $\mathrm{Sp}(1)\cdot\mathrm{Sp}(n)$ really acts effectively.) Such a manifold is called a **quaternion-Kähler** manifold [8]. The typical example of a quaternion-Kähler manifold is the quaternion projective line $\mathbb{H}P^1 \cong S^4$.

Let V be a quaternion-holomorphic vector bundle over a compact quaternion Kähler manifold M. (Note that we do not suppose that V is a line bundle). A related elliptic complex is

$$0 \to \Gamma(M;V) \xrightarrow{d^\nabla} \Gamma(M;\mathbb{H}\otimes\mathbb{E}\otimes V) \xrightarrow{\mathcal{D}^\nabla} \Gamma\left(M;S^2\mathbb{H}\otimes\wedge^2\mathbb{E}\otimes V\right)$$
$$\xrightarrow{\mathcal{D}^\nabla}\cdots\xrightarrow{\mathcal{D}^\nabla}\Gamma(M;S^p\mathbb{H}\otimes\wedge^p\mathbb{E}\otimes V)\xrightarrow{\mathcal{D}^\nabla}\cdots\xrightarrow{\mathcal{D}^\nabla}\Gamma\left(M;S^{2n}\mathbb{H}\otimes\wedge^{2n}\mathbb{E}\otimes V\right)$$
$$\to 0.$$

We again consider the cohomology of the elliptic complex:

$$H^q(M;V) := \mathrm{Ker}\,\mathcal{D}^\nabla/\mathrm{Im}\,\mathcal{D}^\nabla.$$

Theorem 4.2. ([3](4-dimensional case), [6] and [7]) *If a quaternion-Kähler manifold has a positive scalar curvature, then we have*

$$H^q(M;V) = 0 \quad \textit{for} \quad q \geqq n+1.$$

If a quaternion-Kähler manifold has a negative scalar curvature, then we have

$$H^q(M;V) = 0 \quad \textit{for} \quad 1 \leqq q \leqq n+1.$$

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