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Learning about Conic Sections with Geometric Algebra and Cinderella

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Abstract

Over time an astonishing and sometimes confusing variety of descriptions of conic sections has been developed. This article will give a brief overview over some interesting descriptions, showing formulations in the three geometric algebras of Euclidean spaces, projective spaces, and the conformal model of Euclidean space. Systematic illustrations with Cinderella created Java applets are included. I think a combined geometric algebra & illustration approach can motivate students to explorative learning.

1 Introduction

Conic sections are the familiar plane objects of points, x-shaped pairs of lines, circles, ellipses, parabolas and hyperbolas. These curves have an enormous practical significance. They describe rotation trajectories, the orbits of planets, the trajectories of comets and soccer balls, commercial satellites, the ideal form of well focused antennas (nowadays popular for satellite TV), etc. They are so important that every student in the engineering sciences has to study them as part of his first year curriculum.

In this article we first briefly touch upon the history of geometric algebra [1], to motivate the description of conic sections with geometric algebras. The descriptions in the geometric algebras of two-dimensioal and three-dimensional Euclidean space of section 2 are mainly taken from [2]. The presentation of each description consists of the relevant formulas accompanied by an illustrative set of figures. The figures were created with the interactive geometry software Cinderella[3]. Cinderella allows purely interactive construction and animations with export functions to Java applets [4, 5, 6] and postscript format graphics.

One of the finest descriptions of conic sections was given by B. Pascal.[7] Grassmann later gives a general formula for it in terms of Grassmann algebra.[8] We translate this into both projective[9] and conformal[10, 11, 12] geometric algebra. For a subset of conic sections, the conformal model [10, 11, 12, 13, 14] provides an even more elegant "linear" description.

1.1 A new branch of mathematics

In 1844, just about 200 years after Pascal discovered his theorem, the German mathematics school teacher Hermann Grassmann (1809-1877) invented his "Extension Theory" [15], which he republished in 1862[8]. He saw his "new branch of mathematics" indeed to "... form the keystone of the entire structure of mathematics."[8]

Also the popular German mathematician Albrecht Beutelspacher considers the extension theory to comprise a number of "theoretical milestones" and "gems." Amongst the latter he counts:

> Without using coordinates, he could represent the equation of a conic section through five points (A, B, C, D, E) in general position in a plane.[16]

Later we will state Grassmann's representation of conic sections in precise formal terms. But before doing that we shall follow up the historical development of this new "keystone of mathematics."

1.2 Geometric algebra

In 1878, one year after Grassmann's death, William K. Clifford (1845-1879) published his "Applications of Grassmann's extensive algebra."[17], in which he successfully unified Grassmann's extensive algebra[8, 15, 18] with Hamilton's quaternion [19] description of rotations. This was the birth of (Clifford) *Geometric Algebra*[1], (which needs to be thoroughly distinguished from algebraic geometry.)

During the last 50 years or so geometric algebra has become quite popular as a rather universal tool for mathematics and its applications [20], including engineering.[21, 22] But the development of applications seems not finished yet. Projective geometry is by now well integrated in geometric algebra.[9] Especially for applications in computer vision and robotics it proves very versatile to adopt a higher dimensional geometric algebra model, the socalled *conformal* geometric algebra.[10, 11, 12, 13, 14]

1.3 Conformal model of Euclidean space

The conformal model of Euclidean space simply interprets the point of origin and spatial infinity as two extra linear dimensions of space, whose vectors have the peculiar property that they square to zero.[10] This can be seen as borrowing from the description of the propagation of light in space and time. Light propagates at the invariant vacuum speed of light and is therefore relativistic. The propagation of light in four-dimensional space-time also happens along vectors which square to zero. For a point light source, all these vectors form together the light cone.

Defining an even higher dimensional "light cone", the socalled *horosphere* in our five dimensional space of origin, 3-space and infinity, we get the socalled conformal model of Euclidean space. In this conformal model, every point on the horosphere is in one-to-one correspondence with every point in Euclidean space. This idea can be implemented with a host of geometric and computational benefits for areas like: computer vision, computer

[1] Clifford wrote: "The chief classification of *geometric algebras* is into those of odd and even dimensions ..."[17] (italics added).

graphics, robotics, etc.[11, 21, 22] The idea of the horosphere is not at all new, it was already defined by F.A. Wachter (1792-1817), an assistant of Gauss.[23]

Based on the conformal model, a number of computer programs have been developed for various applications, using object oriented programming languages, such as C++ and Java.[11, 12, 24, 25] The description of points, pairs of points, lines, planes, circles and spheres is of great elegance, just using one, two, three or four points. (In the case of lines and planes one of these points will be at infinity.) But a yet unsolved question is, whether we can find in the conformal model a similarly elegant description for conic sections, only using the five general points A, B, C, D, E (comp. Fig. 8) which Grassmann used.

The answer will be worked out in this paper. We will find, that in the conformal model, the implementation of Grassmann's formula for the conic sections given by five general points in the plane is indeed possible. But so far the resulting description will continue to be "quadratic" in each point and not "linear". This is in contrast to the "linear" descriptions of e.g. circles (and lines) in the conformal model.

2 Euclidean description of plane conic sections

2.1 Cone and plane

Plane conic sections are simply the curves of intersection of a cone and a plane. This can be beautifully visualized with colorful, interactive Cinderella[3] created Java applets. Suitable exported applets allow to freely manipulate the position and orientation of the plane in space (comp. Fig. 1).[4, 5, 6] Cinderella's *Spherical view* (a central ball projection to the surface of a ball) allows even to visualize what happens at infinity. In this view it is seen that parabolas close at infinity, but hyperbolas remain divergent.

2.2 The semi-latus rectum formula

A wellknown formula for the unified analytic description of ellipses, parabolas and hyperbolas is the *semi-latus rectum formula.* The radial distance $\mid \mathbf{r} \mid$ of a point on a conic curve from a focus point in the direction of $\hat{\mathbf{r}} = \mathbf{r}/\mid \mathbf{r} \mid$ is given by

$$\mid \mathbf{r} \mid = \frac{l}{1+\boldsymbol{\varepsilon} * \hat{\mathbf{r}}}, \tag{1}$$

where the semi-latus rectum is defined as the length of the excentricity vector (perpendicular to the directrix, attached to the focus), times the length of the distance of the focus from the directrix

$$l = \mid \boldsymbol{\varepsilon} \mid\mid \mathbf{d} \mid . \tag{2}$$

The asterisk product in eq. (1) means the scalar product of vectors. Depending on the scalar magnitude of the excentricity we obtain for

- $\mid \boldsymbol{\varepsilon} \mid < 1$ an ellipse,
- $\mid \boldsymbol{\varepsilon} \mid = 1$ a parabola,
- $\mid \boldsymbol{\varepsilon} \mid > 1$ a hyperbola.

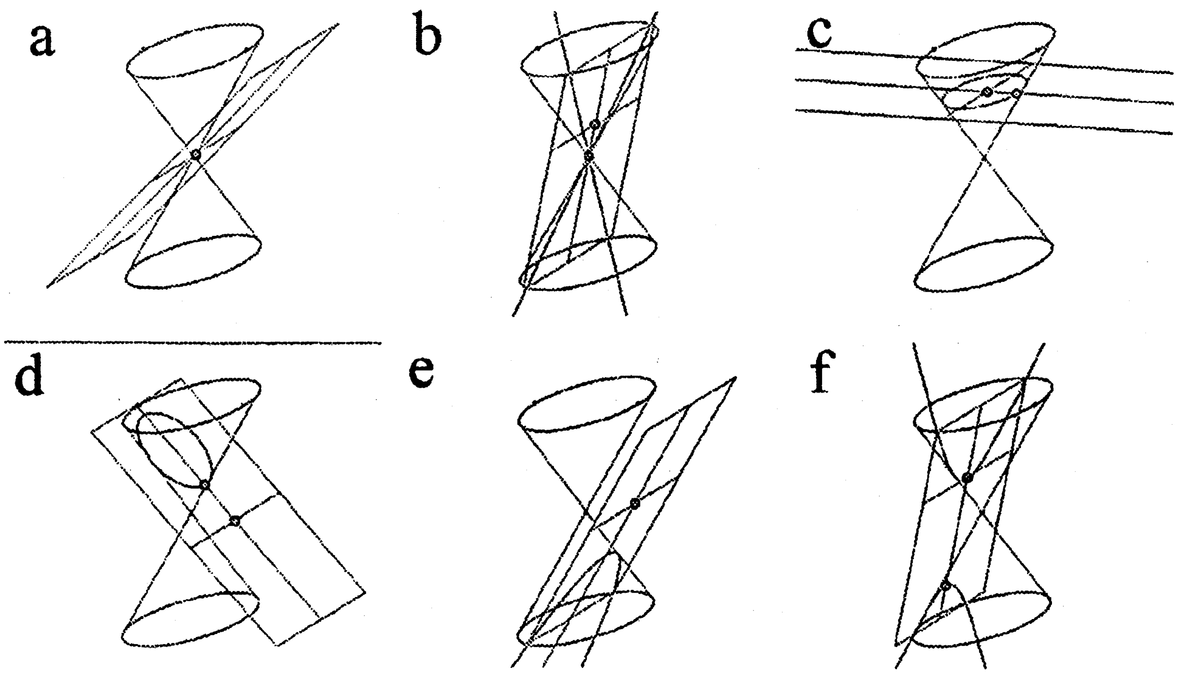

Figure 1: Conic (inter)sections: a) Point, b) pair of intersecting straight lines, c) circle, d) ellipse, e) parabola, f) hyperbola.

The semi-latus rectum formula can again be colorfully visualized with a Cinderella created applet (compare Fig. 2).[4, 5, 6] It is possible to interactively change the directrix, the focal distance, the excentricity, and vary the radial direction by moving a point on the directrix. The semi-latus rectum l appears as the distance between the focus, and the intersection point of the conic section with a line parallel to the directrix through the focus. This happens precisely when the scalar product in eq. (1) vanishes, i.e. when $\hat{\mathbf{r}}$ is parallel to the directrix.

2.3 Polar angle description of ellipse

The *polar angle parameter description of an ellipse* is perhaps the most common description of the ellipse studied already in highschool. Usually two mutually orthogonal vectors, the semi-major axis vector **a** and the semi-minor axis vector **b** with

$$\mathbf{a} * \mathbf{b} = 0 \Longleftrightarrow \mathbf{a} \perp \mathbf{b} \tag{3}$$

are linearly combined with trigonometric coefficients to give the distance of a point on the ellipse in the direction specified by the polar angle φ

$$\mathbf{r} = \mathbf{a}\cos\varphi + \mathbf{b}\sin\varphi. \tag{4}$$

Cinderella created Java applets[4, 6] both allow to see an *animation* with the polar angle φ as animation parameter, and an *interactive* version where the two semi-axis, and the polar angle φ can be changed at will (compare Fig. 3).

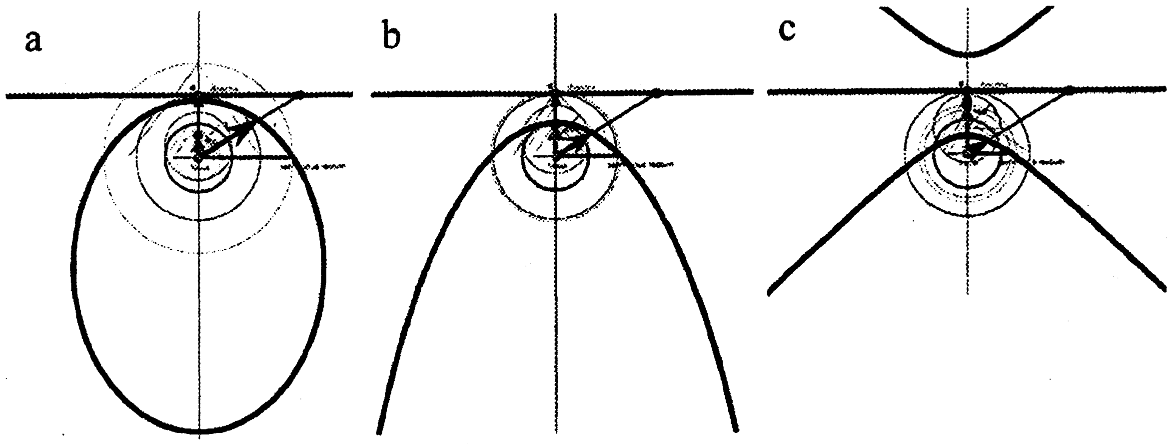

Figure 2: Semi-latus rectum formula: a) Ellipse, b) parabola, c) hyperbola.

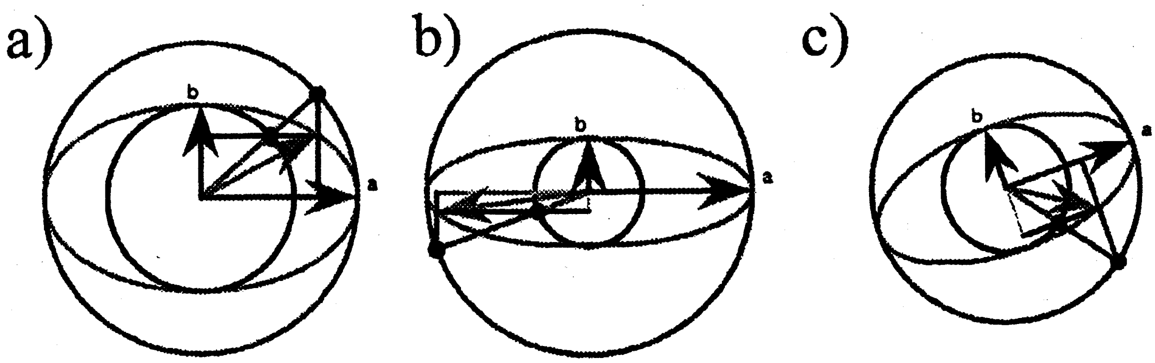

Figure 3: Polar angle parameter description an ellipse: a) polar angle in first quadrant, b) polar angle in third quadrant with change of | **b** |, c) changing the semi-axis.

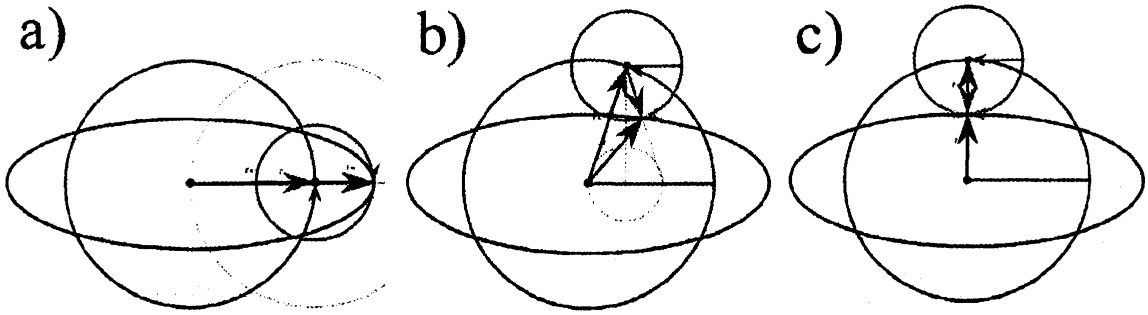

Figure 4: Coplanar circular description of ellipse: a) $\varphi = 0$, b) $0 < \varphi < \pi/2$, c) $\varphi = \pi/2$.

2.4 Coplanar circle description of ellipse

The description of an ellipse by means of a linear combination of two circular motions in one plane (coplanar) is very instructive. Especially engineering students learn in this way an easy-to-apply method for generating elliptical motions from circular motions:

$$\mathbf{r} = \mathbf{r}_+ + \mathbf{r}_-, \tag{5}$$

with the first circular motion in the unit bivector **i**-plane of the geometric algebra of the embedding vector space

$$\mathbf{r}_+ = \mathbf{r}_{+0} \exp(\mathbf{i}\varphi) \tag{6}$$

and the second circular motion with opposite sense of rotation in the same **i**-plane

$$\mathbf{r}_- = \mathbf{r}_{-0} \exp(-\mathbf{i}\varphi). \tag{7}$$

That for fixed $\mathbf{r}_{+0}$ and $\mathbf{r}_{-0}$ the trajectory of $\mathbf{r}$ describes indeed an ellipse can be intuitively illustrated with Cinderella created applets[4, 6], both interactively (with free interactive choices of $\mathbf{r}_{+0}$, $\mathbf{r}_{-0}$ and φ) and animated (compare Fig. 4).

2.5 Non-coplanar circle description of ellipse

It is further possible to describe an ellipse as a linear combination of two circular motions in two planes of different orientation (non-coplanar). The two circular motions are supposed to have equal amplitude, frequency and phase. The two circle planes are characterized in the geometric algebra of Euclidean three space by their respective unit bivectors $\mathbf{i}_1$ and $\mathbf{i}_2$. That the planes are not coplanar means, that the intersection (or *meet*) will be a linear one-dimensional subspace represented by the vector

$$\mathbf{a} = \alpha\, \mathbf{i}_1 \vee \mathbf{i}_2 = \alpha\, \mathbf{i}_1 \llcorner (i\mathbf{i}_1), \tag{8}$$

where the symbol " $\llcorner$ " represents the *right contraction*[26] and i the grade three pseudoscalar of the geometric algebra of three-dimensional Euclidean space. The real scalar α allows to change the length and orientation of $\mathbf{a}$. The symbol $\mathbf{a}$ in (8) for the vector along

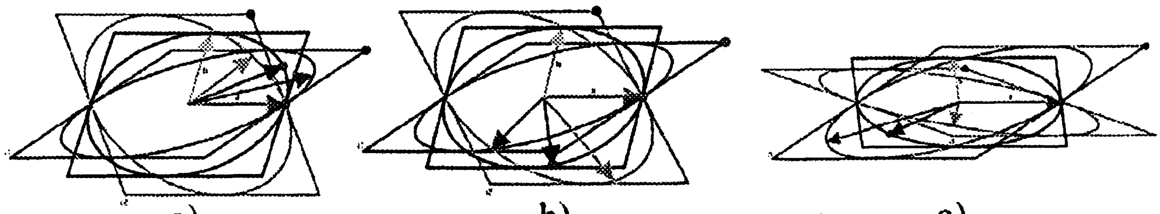

Figure 5: Non-coplanar circle description of ellipse: a) $0 < \varphi < \pi/2$, b) $3\pi/2 < \varphi < 2\pi$, c) changed orientation of planes.

the intersection of the two planes indicates rightly, that it will serve as the semi-major axis vector of the ellipse to be generated. The semi-minor axis vector will be

$$\mathbf{b} = \frac{1}{2}\mathbf{a}(\mathbf{i}_1 + \mathbf{i}_2). \tag{9}$$

The formula for the ellipse to be generated is

$$\mathbf{r} = \frac{1}{2}\mathbf{a}\{\exp(\mathbf{i}_1\varphi) + \exp(\mathbf{i}_2\varphi)\}, \;\; 0 \leq \varphi < 2\pi. \tag{10}$$

The ellipse generated according to (10) can be illustrated by interactive or animated Cinderella created applets.[4, 6] The intereactive construction allows to change the length of $\mathbf{a}$ and the individual orientations of the planes. The dependence of the semi-minor axis (9) on the two plane bivectors $\mathbf{i}_1$ and $\mathbf{i}_2$ is thus well visualized (compare Fig. 5).

2.6 Conic sections as second order curves

Cinderella's *Locus mode* is very suitable for visualizing the fact that conic sections are equivalent to second order curves according to the formula

$$\mathbf{r}(\lambda) = \frac{\mathbf{a}_0 + \mathbf{a}_1\lambda + \mathbf{a}_2\lambda^2}{\alpha + \lambda^2}, \tag{11}$$

with the vectors $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$. The values of the real scalar α decide wether the resulting quadratic curve is for

- $\alpha > 0$ an ellipse,
- $\alpha = 0$ a parabola or
- $\alpha < 0$ a hyperbola.

The real scalar λ parametrizes the curves. Interactive Cinderella created applets[4, 6] allow to *freely vary* the vectors $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ and the scalars α and λ. One kind of animation allows to show how the curves a swept out by the vector $\mathbf{r}$ of equ. (11) with λ as the animation parameter. (compare Fig. 6)

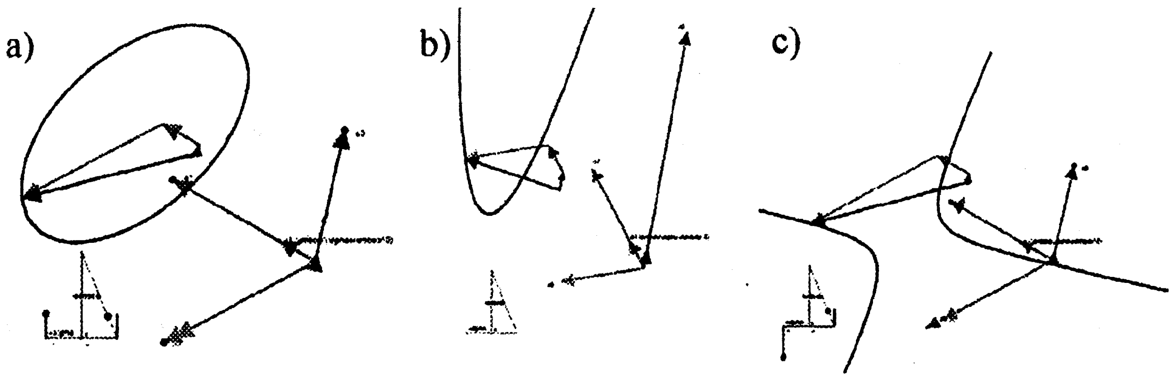

Figure 6: Conic sections are second order curves: a) $\alpha > 0$ ellipse, b) $\alpha = 0$ parabola, c) $\alpha < 0$ hyperbola. Apart from the complete curves, the graphs show the vectors $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ and a linear combination of the three vectors with scalar coefficients $\frac{1}{\alpha+\lambda^2}\mathbf{a}_0$, $\frac{\lambda}{\alpha+\lambda^2}\mathbf{a}_1$, $\frac{\lambda^2}{\alpha+\lambda^2}\mathbf{a}_2$ for a certain value of λ.

3 Projective description of plane conic sections

3.1 Pascal's mystic hexagon

Blaise Pascal (1623-1662, Fig. 7) researched the foundations of hydrodynamics, stating that the pressure is the same at all points in a fluid. This is the basis for hydraulic lifts.[27] But Pascal is also famous for his works in mathematics, both in theory and application. He developed and sold e.g. a calculator machine. In his religious writings he famously stated[28]:

> If God does not exist, one will lose nothing by believing in him, while if he does exist, one will lose everything by not believing.

Our present point of interest is Pascal's work on conic sections. At the age of 16 he found what is now called "Pascal's mystic hexagon" or less glamorous "Pascal's theorem":

> If a hexagon $(ABCDEX)$ is inscribed in a conic section, then the three points $(S_1, S_2$ and $S_3)$ where opposite side (lines) meet are collinear.[7]

The theorem is illustrated[7, 4, 6] in Fig. 8. The theorem is equally true for all plane conic sections previously mentioned.

Formally speaking Pascal's theorem belongs to the field of "higher geometry," "geometry of position," "descriptive geometry," or in modern terms to "projective geometry." The six basic axioms of projective geometry are easy to understand[29]:

- If A and B are distinct points on a plane, there is at least one line containing both A and B.
- If A and B are distinct points on a plane, there is not more than one line containing both A and B.

Figure 7: Blaise Pascal (1623-1662)[28]

Figure 8: Pascal's mystic hexagon: a) ellipse[7], b) parabola, c) hyperbola.

Point	= Intersection of lines
S_1	XA and CD
S_2	AB and DE
S_3	BC and EX

Table 1: Construction of S_1, S_2, and S_3

- Any two lines in a plane have at least one point of the plane (which may be the point at infinity) in common.
- There is at least one line on a plane.
- Every line contains at least three points of the plane.
- All the points of the plane do not belong to the same line.

3.2 Conic sections from five points

Pascal's construction of Fig. 8 can be interpreted in two important ways, an analytic and a constructive way. The analytic interpretation was given in the introduction.

The constructive interpretation means using the theorem for the construction of a conic section from five general points on a plane. Assume five points A, B, C, D, E to be given. Construct the four lines AB, BC, CD, DE and the point of intersection S_2 of the lines AB and DE. Next draw any line g through the point S_2 and construct the intersection points S_1 and S_3 of the line g with CD and BC, respectively. After that draw the lines S_1A and S_3E. According to Pascal's theorem the point X of intersection of the lines S_1A and S_3E is also a point on the conic section. By conducting this construction for every angle of the line g through the point S_2, X will sweep out the whole conic section. This can be interactively realized with a Cinderella created applet.

A consequence is, that any point X in the plane will be part of the conic section iff it can be reached by changing the angle of line g through point S_2. Therefore to decide whether a point X is on the conic section or not, we only need to check, whether S_1, S_2, and S_3 are collinear (on g) or not. The positions of S_1, and S_3 in this examination will critically depend on the position of X (compare Table 1).

3.3 Grassmann's formula

Grassmann used precisely this method for obtaining his "equation of a conic section that goes through the five points A, B, C, D, E, no three of which lie on the same straight line" [8].

For this purpose he stated: "By planimetric multiplication I mean relative multiplication in the plane as a domain of third order, ..." [8] This hint to the plane as a domain of third order is very important, because it shows that Grassmann actually expands the plane projectively by adding an extra dimension, commonly interpreted as the origin.

Grassmann further obtains the expression AB of a line from the outer product of two points A and B on the line. Grassmann omits the product symbol. Therefore the expression AB comes to mean both the product of two projective points A and B that results in an algebraic representation of a line and the common symbolic representation AB of a line through two points A and B.

Let us go into further geometric and algebraic details. The three-dimensional basis of the projective space of a plane is given in terms of three orthonormal vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_0\}$. The first two vectors span the familiar non-projective Euclidean plane. The third vector $\mathbf{e}_0$, is the additional third dimension for lifting the origin to $\mathbf{e}_0$. We represent a point in the Euclidean plane by a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$:

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2, \tag{12}$$

where a_1 and a_2 are simply the two-dimensional Cartesian coordinates. The projective representation[13] of the point A is obtained by adding $\mathbf{e}_0$

$$A = \mathbf{a} + \mathbf{e}_0. \tag{13}$$

Projective points are homogeneous, i.e. λA represents the same point. The Euclidean equivalent of a projective point A is obtained by

$$\mathbf{a} = \frac{A - A * \mathbf{e}_0\, \mathbf{e}_0}{A * \mathbf{e}_0}, \tag{14}$$

I now deliberately introduce the product symbol "$\wedge$" for the exterior product in order to ease he distinction of the symbolic representation of a line AB and Grassmann's algebraic representation $A \wedge B$. The exterior product is antisymmetric:

$$A \wedge B = (\mathbf{a} + \mathbf{e}_0) \wedge (\mathbf{b} + \mathbf{e}_0) = \mathbf{a} \wedge \mathbf{b} + (\mathbf{a} - \mathbf{b}) \wedge \mathbf{e}_0. \tag{15}$$

$\mathbf{a} \wedge \mathbf{b}$ is the moment bivector of a line and $(\mathbf{b} - \mathbf{a})$ is its direction vector.[2] The two entities suffice to construct the line.[11] Grassmann's planimetric product of two lines AB and DE can be realized in the *geometric algebra* of the projective three-dimensional space spanned by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_0\}$ by

$$S_2 = (A \wedge B) \llcorner [I_3(D \wedge E)], \tag{16}$$

where the symbol "$\llcorner$" represents the *right contraction*[26] and $I_3 = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_0 = \mathbf{i}\mathbf{e}_0$ is the volume 3-vector of the projective space ($\mathbf{i} = \mathbf{e}_1 \wedge \mathbf{e}_2$ is the unit bivector of the Euclidean plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$). $[I_3(D \wedge E)]$ results in the dual complement vector perpendicular to the projective line bivector $D \wedge E$. Finally the right contraction with the line bivector $A \wedge B$ results in the element S_2 in the line $A \wedge B$, which is perpendicular to $[I_3(D \wedge E)]$ in $A \wedge B$, and therefore also contained in $D \wedge E$. Inserting (13) and simplifying the expressions algebraically, we get for the intersection

$$S_2 = \lambda_2\mathbf{s}_2 + \lambda_2\mathbf{e}_0 = (\mathbf{a} - \mathbf{b})[\mathbf{i}(\mathbf{d} \wedge \mathbf{e})] - (\mathbf{d} - \mathbf{e})[\mathbf{i}(\mathbf{a} \wedge \mathbf{b})] + \mathbf{i}[(\mathbf{d} - \mathbf{e}) \wedge (\mathbf{a} - \mathbf{b})]\mathbf{e}_0 \tag{17}$$

In projective geometry points are identical up to scalar factors. We therefore divide by

$$\lambda_2 = \mathbf{i}[(\mathbf{d} - \mathbf{e}) \wedge (\mathbf{a} - \mathbf{b})] = (d_1 - e_1)(a_2 - b_2) - (d_2 - e_2)(a_1 - b_1) \tag{18}$$

to get according to eq. (14) the plane Euclidean vector

$$\mathbf{s}_2 = \frac{1}{\lambda_2}\{(\mathbf{a}-\mathbf{b})[\mathbf{i}(\mathbf{d}\wedge\mathbf{e})] - (\mathbf{d}-\mathbf{e})[\mathbf{i}(\mathbf{a}\wedge\mathbf{b})]\} \tag{19}$$

Inserting coordinates (13) we explicitly get

$$\mathbf{s}_2 = \frac{1}{\lambda_2}\{(d_1e_2 - d_2e_1)(\mathbf{a}-\mathbf{b}) - (a_1b_2 - a_2b_1)(\mathbf{d}-\mathbf{e})\} \tag{20}$$

In the very same way Grassmann calculates S_1 and S_3 by planimetric products as

$$S_1 = (X\wedge A)\llcorner[I_3(C\wedge D)],\ \ S_3 = (B\wedge C)\llcorner[I_3(E\wedge X)], \tag{21}$$

So we can finally express the collinearity of S_1, S_2 and S_3 by

$$S_1\wedge S_2\wedge S_3 = 0, \tag{22}$$

i.e.

$$\{(X\wedge A)\llcorner[I_3(C\wedge D)]\}\wedge\{(A\wedge B)\llcorner[I_3(D\wedge E)]\}\wedge\{(B\wedge C)\llcorner[I_3(E\wedge X)]\} = 0. \tag{23}$$

This is Grassmann's formula for the conic sections through five general points (A, B, C, D, E) in a plane expressed in the geometric algebra of the projective space $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_0\}$. Every point X, that fulfills equation (23) will be on the conic section. The equation is quadratic in X and in all of the five points A, B, C, D, E. With the help of the anticommutator "$\times$"

$$B_1\times B_2 = \frac{1}{2}(B_1B_2 - B_2B_1) \tag{24}$$

we can rewrite (23) as

$$\{[(X\wedge A)\times(C\wedge D)]\times[(A\wedge B)\times(D\wedge E)]\}\wedge\{I_3[(B\wedge C)\times(E\wedge X)]\} = 0. \tag{25}$$

4 Conformal geometric algebra description of plane conic sections

4.1 Grassmann's formula for the conformal model

The five-dimensional conformal model [10, 11, 12, 13, 14] adds to the three-dimensional Euclidean space two dimensions: one for representing the origin and one for representing infinity. This is done by introducing two null-vectors, which square to zero and are perpendicular to the vectors of Euclidean space:

$$\{\bar{\mathbf{n}}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{n}\}, \tag{26}$$

where $\bar{\mathbf{n}}$ and $\mathbf{n}$ represent the origin and infinity, respectively. The conformal representation of a point A is obtained by adding two contributions

$$A = \mathbf{a} + \frac{1}{2}a^2\mathbf{n} + \bar{\mathbf{n}}, \tag{27}$$

with $a^2 = \mathbf{aa}$. A straight Euclidean line AB in the conformal model is given by the outer product of wo points on the line with infinity forming the trivector

$$A \wedge B \wedge \mathbf{n} = \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{n} + (\mathbf{b} - \mathbf{a})N = \mathbf{m}_1\mathbf{n} + \mathbf{d}_1 N, \tag{28}$$

where the unit bivector $N = \mathbf{n} \wedge \bar{\mathbf{n}}$ represents the additional two dimensional (Minkowski) space. A point X is on the line AB iff

$$X \wedge A \wedge B \wedge \mathbf{n} = 0. \tag{29}$$

Similarly the line DE is given by the trivector

$$D \wedge E \wedge \mathbf{n} = \mathbf{d} \wedge \mathbf{e} \wedge \mathbf{n} + (\mathbf{e} - \mathbf{d})N = \mathbf{m}_2\mathbf{n} + \mathbf{d}_2 N, \tag{30}$$

where $\mathbf{m}_1$ and $\mathbf{m}_2$ are the Euclidean moment bivectors of the lines AB and DE, and the vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are their direction vectors, respectively. The intersection S_2^c of two lines AB and DE is obtained in a fashion very similar to (16)

$$S_2^c \wedge \mathbf{n} = (A \wedge B \wedge \mathbf{n}) \llcorner [I_4(D \wedge E \wedge \mathbf{n})], \quad S_2^{c2} = 0, \tag{31}$$

where $I_4 = \mathbf{i}N$, with $\mathbf{i}$ being the bivector that represents the plane shared by A, B, D and E.

We now perform a detailed calculation of the right side of the first equation of (31) in order to show that the special form of the bivector on the left side is indeed justified. (A reader less familiar with geometric algebra may skip this calculation.)

$$\begin{aligned}
&(A \wedge B \wedge \mathbf{n}) \llcorner [I_4(D \wedge E \wedge \mathbf{n})] = \langle(\mathbf{m}_1\mathbf{n} + \mathbf{d}_1 N)\mathbf{i}N(\mathbf{m}_2\mathbf{n} + \mathbf{d}_2 N)\rangle_2 && (32)\\
&= \langle \mathbf{m}_1\mathbf{n}\mathbf{i}N\mathbf{m}_2\mathbf{n} + \mathbf{m}_1\mathbf{n}\mathbf{i}N\mathbf{d}_2 N + \mathbf{d}_1 N\mathbf{i}N\mathbf{m}_2\mathbf{n} + \mathbf{d}_1 N\mathbf{i}N\mathbf{d}_2 N\rangle_2 && (33)\\
&= \langle \mathbf{i}\mathbf{m}_1\mathbf{n}N\mathbf{n}\mathbf{m}_2 + \mathbf{i}\mathbf{m}_1\mathbf{n}\mathbf{d}_2 N + \mathbf{d}_1\mathbf{i}\mathbf{m}_2\mathbf{n} + \mathbf{d}_1\mathbf{i}\mathbf{d}_2 N\rangle_2 && (34)\\
&= \langle 0 - \mathbf{i}\mathbf{m}_1\mathbf{d}_2\mathbf{n} + \mathbf{i}\mathbf{m}_2\mathbf{d}_1\mathbf{n} + \mathbf{i}\mathbf{d}_1\mathbf{d}_2\bar{\mathbf{n}} \wedge \mathbf{n}\rangle_2 && (35)\\
&= \langle \underbrace{-\mathbf{i}\mathbf{m}_1\mathbf{d}_2 + \mathbf{i}\mathbf{m}_2\mathbf{d}_1}_{-\lambda_2 \mathbf{s}_2} + \underbrace{\langle \mathbf{i}\mathbf{d}_1\mathbf{d}_2\rangle_0}_{-\lambda_2}\bar{\mathbf{n}}\rangle_1 \wedge \mathbf{n}. && (36)
\end{aligned}$$

Angular brackets with an integer grade index $\langle\ \rangle_k, k \geq 0$ refer to the operation of grade k selection. The equality in (32) is a simple application of the definition of the contraction of a vector $I_4(D \wedge E \wedge \mathbf{n})$ (= the *dual* of the trivector $(D \wedge E \wedge \mathbf{n})$) from the right onto a trivector to the left.[26] The equality between lines (33) and (34) uses the following identities: $\mathbf{m}_1\mathbf{n}\mathbf{i} = \mathbf{i}\mathbf{m}_1\mathbf{n}$, $\mathbf{m}_2\mathbf{n} = \mathbf{n}\mathbf{m}_2$, $N\mathbf{i} = \mathbf{i}N = I_4$, and $NN = 1$. The equality between the lines (34) and (35) uses the following identities: $\mathbf{n}N = \mathbf{n}$, $\mathbf{nn} = 0$, and hence $\mathbf{n}N\mathbf{n} = 0$. It further uses $\mathbf{n}\mathbf{d}_2 = -\mathbf{d}_2\mathbf{n}$, $\mathbf{d}_1\mathbf{i} = -\mathbf{i}\mathbf{d}_1$, $\mathbf{d}_1\mathbf{m}_2 = -\mathbf{m}_2\mathbf{d}_1$, and that $N = \mathbf{n} \wedge \bar{\mathbf{n}} = -\bar{\mathbf{n}} \wedge \mathbf{n}$ is already a bivector. It remains to be observed that in line (36) the entities $\mathbf{i}\mathbf{m}_1$, $\mathbf{i}\mathbf{m}_2$ and $\langle \mathbf{i}\mathbf{d}_1\mathbf{d}_2\rangle_0 = -\lambda_2$ are all scalars, whereas $\mathbf{d}_1$, $\mathbf{d}_2$ and $\bar{\mathbf{n}}$ are all vectors.

The explicit calculation of (31) yields therefore

$$S_2^c = -\lambda_2(\mathbf{s}_2 + \frac{1}{2}s_2^2\mathbf{n} + \bar{\mathbf{n}}), \tag{37}$$

with the same Euclidean vector $\mathbf{s}_2$ as in equations (19) and (20). Note that $\mathbf{n} \wedge \mathbf{n} = 0$, but the term $(s_2^2/2)\mathbf{n}$ is inserted to fulfill $S_2^{c2} = 0$, the second part of (31). Similar to (31) we obtain

$$S_1^c \wedge \mathbf{n} = (X \wedge A \wedge \mathbf{n}) \llcorner [I_4(C \wedge D \wedge \mathbf{n})], \;\; S_1^{c2} = 0, \tag{38}$$

and

$$S_3^c \wedge \mathbf{n} = (B \wedge C \wedge \mathbf{n}) \llcorner [I_4(E \wedge X \wedge \mathbf{n})], \;\; S_3^{c2} = 0. \tag{39}$$

Using the three conformal points of intersection S_1^c, S_2^c and S_3^c we can finally give the equation for the conic sections through five general points on a plane in the conformal model as

$$S_1^c \wedge S_2^c \wedge S_3^c \wedge \mathbf{n} = 0. \tag{40}$$

This is the conformal equivalent of Grassmann's formula for conic sections, which in turn has been seen to be based on Pascal's theorem. Every conformal point $X = x + \frac{1}{2}x^2\bar{\mathbf{n}} + \mathbf{n}$ that fulfills (40) is on the plane conic section through A, B, C, D, E. By construction, equation (40) is again quadratic in X and in all of the five points A, B, C, D, E.

4.2 Cirlces

This quadratic property of (40) is in contrast to the simpler representation of circles[11] by three conformal points A_1, A_2 and A_3

$$A_1 \wedge A_2 \wedge A_3. \tag{41}$$

According to (28) straight lines are simply circles through infinity $\mathbf{n}$. All points X on the circle through A_1, A_2 and A_3 are simply obtained from

$$X \wedge A_1 \wedge A_2 \wedge A_3 = 0. \tag{42}$$

This equation is linear in X and in the three general defining points A_1, A_2 and A_3. The explicit form of (41) becomes

$$\begin{aligned} &\frac{1}{2}(a_1^2\mathbf{a}_2 \wedge \mathbf{a}_3 + a_2^2\mathbf{a}_3 \wedge \mathbf{a}_1 + a_3^2\mathbf{a}_1 \wedge \mathbf{a}_2)\mathbf{n} \\ &\qquad +(\mathbf{a}_3 \wedge \mathbf{a}_1 + \mathbf{a}_2 \wedge \mathbf{a}_3 + \mathbf{a}_1 \wedge \mathbf{a}_2)\bar{\mathbf{n}} \\ &+\frac{1}{2}([a_2^2 - a_3^2]\mathbf{a}_1 + [a_3^2 - a_1^2]\mathbf{a}_2 + [a_1^2 - a_2^2]\mathbf{a}_3)N. \end{aligned} \tag{43}$$

The expected first term $\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3$ will be zero, because we assume a circle in two Euclidean dimensions[2] and not three, i.e. the origin $\bar{\mathbf{n}}$ will always be in the circle plane. Separating off the circle center vector $\mathbf{c}$ and the radius r $(x_1^2 = x_2^2 = x_3^2 = 1)$

$$\mathbf{a}_1 = \mathbf{c} + r\mathbf{x}_1, \;\; \mathbf{a}_2 = \mathbf{c} + r\mathbf{x}_2, \;\; \mathbf{a}_3 = \mathbf{c} + r\mathbf{x}_3, \tag{44}$$

we finally get for (43)

$$\mathbf{c}(\mathbf{c} \wedge I_c)\mathbf{n} + \frac{1}{2}(r^2 - c^2)I_c\mathbf{n} + I_c\bar{\mathbf{n}} - \mathbf{c}I_cN, \tag{45}$$

[2] A more general treatment of circles in the conformal model in three dimensions is included in [12].

where we set the bivector of the circle plain to

$$I_c = (\mathbf{a}_3 - \mathbf{a}_2) \wedge (\mathbf{a}_1 - \mathbf{a}_2). \tag{46}$$

Because we assume, that we are just dealing with the plane two dimensional case the circle center $\mathbf{c}$ must also be in the I_c-plane (i.e. $c \wedge I_c = 0$) and hence

$$A_1 \wedge A_2 \wedge A_3 = [\frac{1}{2}(r^2 - c^2)\mathbf{n} + \bar{\mathbf{n}} - \mathbf{c}N]I_c. \tag{47}$$

We therefore see how (41) includes component by component the circle plane I_c, the center $\mathbf{c}$ and the radius r. Equation (42) is a condition for all points X on the circle (41). By inserting $X = \mathbf{x} + \frac{1}{2}x^2\mathbf{n} + \bar{\mathbf{n}}$ into (42) we get after some algebra

$$(\mathbf{x} - \mathbf{c})^2 = r^2. \tag{48}$$

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> The heavens declare the glory of God; the skies proclaim the work of his hands. Day after day they pour forth speech; night after night they display knowledge. There is no speech or language where their voice is not heard. Their voice goes out into all the earth, their words to the ends of the world.[30]

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