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Dyadic Subbases and Representations of Topological Spaces

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1 Introduction

In order to define computation over a countably-based Hausdorff space X, we need to represent each $x \in X$ as a sequence of characters. The sequence can be infinite because we can consider a machine which makes stream input/output access on infinite sequences. In this case, the machine continues to work infinitely, and produces longer and longer prefix of the output (i.e. better and better approximation of the output) based on longer and longer prefix of the input (i.e. better and better approximation of the input). This kind machine is nothing but a program which makes stream input and output, and is widely used in actual 'real' programming applications. It is also the foundation of computability analysis, where such a machine is called a Type-2 machine [13].

For such a computation of X, the choice of a representation is very important. A representation of X is a surjective partial function from Σ^ω to X, with Σ^ω the set of infinite sequences of a finite alphabet Σ. For example, binary expansion δ_{bin} is a representation of $\mathcal{I} = [0,1]$ for $\Sigma = \{0,1\}$. However, it is known that the computational notion on real numbers induced by δ_{bin} and Type-2 machines is an odd one in that even the simple function to multiply by 3 is not computable.

There is a class of representations, called admissible representations [13], which connect the continuity notion of X to that of Σ^ω, and which induce natural computational notion on the set X with respect to Type-2 machines. Admissible representations are usually considered as the natural representations, and is the main research topic in computability analysis. One example of an admissible representation is an expansion of $[0,\eta]$ with the golden number $\eta = (1+\sqrt{5})/2$ [4]. That is, $\delta_\eta(p) = x$ if $\Sigma_{n=0}^{\infty}\frac{1}{\eta^{n+1}}p[n] = x$. In this case, when p is a name for x, every sequence obtained by substituting an occurrence of 011 in p with 100 is also a name for x, because $1 = \frac{1}{\eta} + \frac{1}{\eta^2}$. Therefore, this representation is very redundant. It is known that every admissible representation of $\mathcal{I}$ are 'very' redundant. More precisely, when ρ is an admissible representation of $\mathcal{I}$, $\{x \in \mathcal{I} \mid \rho^{-1}(x)$ is an infinite set$\}$ is a fat and dense subset of $\mathcal{I}$ [2].

In this article, we consider different kind of representations, which are less redundant and which induce the same computability notion on X with a machine different from the Type-2 machine. We fix the alphabet to be $\{0,1\}$ and consider a representation ρ such that for each x, $\rho^{-1}(x)$ has the form $A_0A_1A_2\ldots$ for A_i an one point set (i.e. $\{0\}$ or $\{1\}$), or the whole space $\{0,1\}$. That is, our representation has the property that the value of each cell of a name of x is defined independently. For the binary representation, when p is a name of 1/2, $p[n]$ $(n \geq 1)$ has the possibility of both 0 and 1, but $p[n]$ is 0 or 1 depending on whether $p[0] = 1$ or 0, respectively. It is also the case for the expansion by the golden number. On the other hand, with our representation, there are three cases for $p[n]$: determined as 0, determined as 1, or has both possibility. These three cases are determined not depending on other bits of p, but only on x. Therefore, when A_n is $\{0,1\}$ and $p \in \rho^{-1}(x)$ has the n-th bit 0, then this bit does not contribute to the fact that p denotes x, because $p[n = 1]$, which is p with the value of the n-th bit substituted to 1, also denotes x. Therefore, only the locations n with $A_n \neq \{0,1\}$ specify that $\rho(p) = x$.

Such a representation can also be expressed as an injective function φ from X to $\Sigma_\perp^\omega$. $\Sigma_\perp^\omega$ is the set of infinite sequences of $\Sigma_\perp = \{0,1,\perp\}$ also known as Plotkin's T^ω [7]. The symbol $\perp$ means undefinedness, and when $\rho^{-1}(x) = A_0A_1A_2\ldots$, we define $\varphi(x)[n]$ as 0 or 1 when A_n is $\{0\}$ or $\{1\}$, respectively, and $\perp$ when A_n is $\{0,1\}$.

Among such representations, we are particularly interested in the case that the cardinality of$\{k \mid \varphi(x)[k] = \perp\}$ is less than a finite number n for every x. We will write $\Sigma_{\perp,n}^\omega$ $(n = 0,1,\ldots)$ for the subspace of $\Sigma_\perp^\omega$ such that the number of bottoms which appear in a sequence is not more than n. When φ is an injective function from X to $\Sigma_{\perp,n}^\omega$, for the corresponding representation function ρ, the cardinality of the fiber $\rho^{-1}(x)$ is upper-bounded by 2^n for every x. Since a sequence in $\Sigma_\perp^\omega$ may contain some undefined cells, we cannot make ordinary stream input/output on such a sequence. However, we can consider an extended stream access which skips bottom cells and continue the input/output of the rest of the stream. The author defined an IM2-machine (indeterministic multi-head Type-2 machine) which makes multi-head access to input/output bottomed streams and which has indeterministic behavior depending on the head used for input. This machine is defined naturally when the stream is $\Sigma^{\omega}_{\perp,n}$-stream. In this case, a machine has $n+1$ heads.

In order that the computational notion induced by φ and IM2-machines is natural, we require that φ is a topological embedding of X in $\Sigma^{\omega}_{\perp}$. In the next section, we show an example of such an embedding for the unit closed interval $I = [0,1]$, and in Section 3, we explain in which sense an embedding of X in $\Sigma^{\omega}_{\perp}$ induces a natural computational notion, and we consider further conditions on an embedding and define the notion of representing embedding. In Section 4, we reformulate the notion of representing embedding through a subbase structure, and define the notion of a dyadic subbase. Then, in Section 5, consider the case that the induced representation is a total function, and define a full-representing dyadic subbase and overview some properties of such subbases following [12].

2 Gray-code embedding of $\mathcal{I}$

First, we give an example of an embedding of $\mathcal{I}$ in $\Sigma^{\omega}_{\perp,1}$, which is called the Gray-code embedding. *Gray-code embedding* φ_G is a function from $\mathcal{I}$ to $\Sigma^{\omega}_{\perp,1}$ defined as $\varphi_G(x)[n] = P(t^n(x))$ $(n = 0,1,\ldots)$ for $t : \mathcal{I} \to \mathcal{I}$ the tent function

$$t(x) = \begin{cases} 2x & (0 \le x \le 1/2) \\ 2(1-x) & (1/2 < x \le 1) \end{cases}$$

and $P : \mathcal{I} \to \Sigma_{\perp}$ the function

$$P(x) = \begin{cases} 0 & (x < 1/2) \\ \perp & (x = 1/2) \\ 1 & (x > 1/2) \end{cases}.$$

One can see that, when $\perp$ appears in a sequence, the remainder always has the form $1000\ldots$. Therefore, $\perp$ appears at most once in each sequence and thus φ_G is a function to $\Sigma^{\omega}_{\perp,1}$.

Figure 1 shows this embedding. Here, a horizontal line means that the corresponding bit has value 1, and the edges of each line corresponds to the value $\perp$. Thus, for example, $\varphi_G(3/4) = 1\perp1000\ldots$. Therefore, when we consider the corresponding representation $\rho : \Sigma^{\omega} \to \mathcal{I}$, $3/4$ has two names $111000\ldots$ and $101000\ldots$. Note that with the usual binary representation, $3/4$ has two names $110000\ldots$ and $101111\ldots$, which are different at all but one bits.

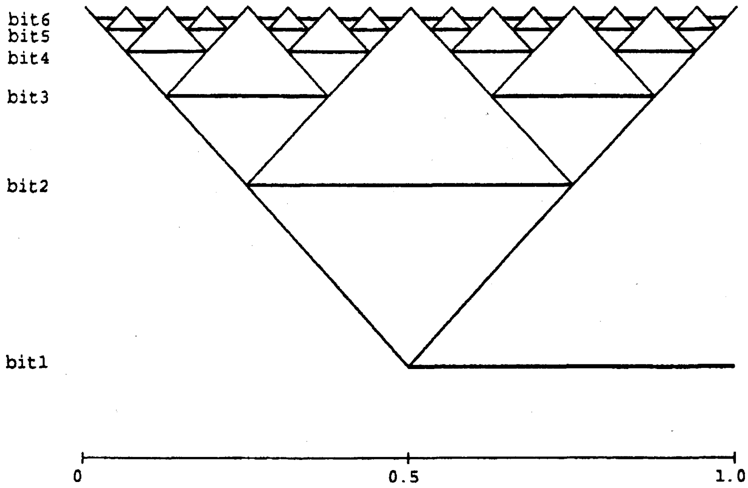

Figure 1: Gray code Embedding of real numbers.

Gray-code embedding is equal to the itinerary of the tent function, which is essential for symbolic dynamical systems [6]. It is also the expansion of $[0, 1]$ with binary reflected Gray-code, which is a binary coding of natural numbers other than the ordinary one [5].

3 Embeddings in $\Sigma_\perp^\omega$

As I said, φ is an embedding of X in $\Sigma_\perp^\omega$, not merely an injective function. We study what does it mean for the induced computation of X, and consider further properties φ must have so that it is a 'good' representation of X. The function φ is determined by the sets $S_n^0 = \{x \mid \varphi(x)[n] = 0\}$ and $S_n^1 = \{x \mid \varphi(x)[n] = 1\}$ $(n = 0, 1, \ldots)$. Therefore, we sometimes consider conditions of these families of sets, instead. We also define $B_n = \{x \mid \varphi(x)[n] = \perp\}$. We consider the topology of $\Sigma_\perp^\omega$ defined by the subbase $\{p \mid p[n] = 0\}$ and $\{p \mid p[n] = 1\}$ $(n = 0, 1, \ldots)$. It is equal to the Scott topology on $\Sigma_\perp^\omega$ considered as a domain, and also equal to the product topology of ${\Sigma_\perp}^\omega$ for the topology on $\Sigma_\perp$ generated by $\{\{0\}, \{1\}\}$. Note that $S_n^0 = \varphi^{-1}(\{p \mid p[n] = 0\})$ and $S_n^1 = \varphi^{-1}(\{p \mid p[n] = 1\})$ $(n = 0, 1, \ldots)$.

Consider the property that φ is an embedding of X in $\Sigma_\perp^\omega$. The continuity of φ is equivalent to saying that S_n^0 and S_n^1 $(n = 0, 1, 2, \ldots)$ are open sets. Therefore, when $\varphi(x)[n]$ is 0 (or 1), for some open neighbourhood $Z \subset X$, $\varphi(y)[n] = 0$ (or 1) for $y \in Z$. Furthermore, since φ is an embedding, the family of sets S_n^0 and S_n^1 $(n = 0, 1, \ldots)$ form a subbase of X. Thus, when

$x \in Z$ for some open set Z, $S_{n_0}^{c_0} \cap \dots S_{n_k}^{c_k} \subset Z$ for some $n_0, \dots, n_k$ and $c_0, \dots, c_k$ such that $\varphi(x)[n_i] = c_i$ $(i = 0, \dots, k)$. . Therefore, only finite number of bits of $\varphi(x)$ determines that $x \in Z$. Consider that there is a tape whose cells are filled with $\perp$ at the beginning, and a machine computing x fills the tape with $\varphi(x)$. More precisely, when $\varphi(x)[i] = 0$ (or 1), the machine obtains this information in some finite time and fills the i-th cell of the tape with 0 (or 1) at some time, and when $\varphi(x)[i] = \perp$, it does not fill the i-th cell eternally, and the value of the cell is lest as $\perp$. Then, at some time of computation, we can observe on the tape enough information to infer that x is in R for each open set R. For the above example, it is just the time the machine fills all the cells with the index $n_0, \dots, n_k$.

The above fact is sometimes called open sets as finitely observable properties [8], and it links observability, which is a computational notion, in a topological term. Note that, though we have explained it with an embedding φ of X in $\Sigma_\perp^\omega$, it is not properties of embeddings in $\Sigma_\perp^\omega$ and the same properties hold for an embedding $\hat{\varphi}$ of X in $\{1\}_\perp^\omega$, which is defined for each countable subbase $b_0, b_1, \dots$ as $\hat{\varphi}(x)[n] = 1$ if $x \in b_n$. Now, we consider additional conditions of the embedding φ which makes use of the fact that the target space is $\{0,1\}_\perp^\omega$, not $\{1\}_\perp^\omega$.

The first one is that B_n is a nowhere-dense closed subset. Since B_n, S_n^0, and S_n^1 are disjoint sets such that $B_n \cup S_n^0 \cup S_n^1 = X$, B_n is always a closed set. If it is not nowhere-dense, it means that there is an open subset R of B_n. Then, from the above observation, when $x \in R$, this fact is determined in a finite time by a machine which computes x and outputs $\varphi(x)$ on a tape. That is, it can write the character $\perp$. This contradicts our interpretation of $\perp$ as a non-terminating computation, which is widely accepted in computer science.

Secondary, we consider here the condition that x is on the boundary of both S_n^0 and S_n^1 when $\varphi(x)[n] = \perp$. The fact that B_n is nowhere-dense and closed means that when $\varphi(x)[n] = \perp$, x must be on the boundary of $S_n^0 \cup S_n^1$. If x is on the boundary of S_n^1 but not on the boundary of S_n^0, then there is an open neighbourhood R of x which is disjoint from S_n^0. This means that we can determine, in a finite time, that $\varphi(x)[n]$ is not 0. Therefore, we can assign 1 to that cell in a finite time, in order to obtain *a* ρ-name of x for ρ the corresponding representation. Thus, it is more natural to erase the name p of x which satisfies $p[n] = 0$, and define $\varphi(x)[n] = 1$. It means to define a new representation defined as $\hat{S_n^a} = int(cl(S_n^a)) \supset S_n^a$ $(a = 0, 1)$. For this representation, $\hat{S_n^0}$ and $\hat{S_n^1}$ are disjoint regular open sets, which are exteriors of each other. In this case, B_n comes to be a nowhere-dense closed subset and this condition subsumes the first one. Thus, we define as follows.

Definition 3.1 *An embedding φ of X in $\Sigma_\perp^\omega$ is representing if S_n^0 and S_n^1 are regular open sets such that S_n^0 is the exterior of S_n^1.*

In [9], it is shown that every separable metric space of dimension n can be embedded in $\Sigma_{\perp,n}^\omega$. With a small modification to this construction, we can show that every separable metric space of dimension n has a representing embedding in $\Sigma_{\perp,n}^\omega$.

4 Dyadic subbase

As we have noted, when φ is an embedding of X in $\Sigma_\perp^\omega$, the family of sets S_n^0 and S_n^1 $(n = 0, 1, \ldots)$ forms a subbase of X. Thus, we define as follows.

Definition 4.1 *Let X be a Hausdorff space. We call a countable subbase $S = (S_0^0, S_0^1, S_1^0, S_1^1 \ldots)$ of X with a pairing and an enumeration of the pairs a dyadic subbase when $S_n^j (n = 0, 1, 2, \ldots, j = 0, 1)$ are regular open and S_n^0 and S_n^1 are exteriors of each other.*

When a dyadic subbase is given, we can define an embedding $\varphi_S : X \to \Sigma_\perp^\omega$ defined as $\varphi_S(x)[n] = 0, 1$ or $\perp$, when $x \in S_n^0$, $x \in S_n^1$, or x is on the boundary of S_n^0 (and also of S_n^1), respectively. Therefore, there is an one-to-one correspondence between a dyadic subbase and a representing embedding. Since φ_S is an embedding, we have a corresponding representation $\rho_S :\subseteq \Sigma^\omega \rightharpoonup X$ defined as $\rho_S(p) = x$ iff $x \in \overline{S_n^{p[n]}}$ for all n such that $p[n] \neq \perp$.

Definition 4.2 *We define $\psi_S, \overline{\psi}_S : \Sigma_\perp^\omega \to \mathcal{P}(X)$ as follows*

$$\begin{aligned} \psi_S(p) &= \bigcap S_n^{p[n]}, \\ \overline{\psi}_S(p) &= \bigcap \overline{S_n^{p[n]}}. \end{aligned}$$

Here, $S_n^\perp$ is defined as X.

These two functions give two interpretations of an infinite sequence p in $\Sigma_\perp^\omega$ as a specification of points in X. ψ_S and $\overline{\psi}_S$ correspond to thinking about each digit a of the n-th cell of p as giving the information that the point is in S_n^a and $\overline{S_n^a}$, respectively. For our study, we place one more condition on dyadic subbases which connects these two interpretations.

Definition 4.3 *A dyadic subbase is proper if $\overline{\psi_S(d)} = \overline{\psi}_S(d)$ for $d \in K(\Sigma_\perp^\omega)$.*

Here, $K(\Sigma_\perp^\omega)$ is the set of finite elements of $\Sigma_\perp^\omega$, that is, the set of elements with finite number of 0 and 1. $\{\psi_S(d) \mid d \in K(\Sigma_\perp^\omega)\}$ forms a base of $\Sigma_\perp^\omega$, corresponding to the subbase.

5 Full-representing dyadic subbases

Suppose that a dyadic subbase S is given. As we have said, we can consider that a program which outputs $\varphi_S(x)$ is computing x. However, if $\overline{\psi}_S(q) = \{x\}$ for some string $q < \varphi_S(x)$, then, we can consider that q is also specifying x. That is, those bits which are 0 or 1 in $\varphi_S(x)$ but $\perp$ in q are providing redundant information, which can be obtained by the sequence q. For example, the following dyadic subbase is redundant in this sense, but the dyadic subbase corresponding to the Gray-code embedding is not.

Example 5.1 (Dedekind subbase) *Fix a numbering q_i of rational numbers in $(0,1)$. Define the dyadic subbase $D = (D_0^0, D_0^1, D_1^0, D_1^1, \ldots)$ as $D_n^0 = [0, q_n)$ and $D_n^1 = (q_n, 1]$. The induced representation $\varphi_D : [0,1] \to \Sigma_\perp^\omega$ is $\varphi_D(x)[n] = 0, \perp,$ and 1 iff $x < q_n$, $x = q_n$, and $x > q_n$, respectively.*

Therefore, we define as follows.

Definition 5.2 *A proper dyadic subbase S is canonically representing iff $\overline{\psi}_S(q) = \{x\}$ implies $q \geq \varphi_S(x)$.*

We define related properties as follows.

Definition 5.3 *A dyadic subbase S is full-representing iff the corresponding representation ρ_S is a total function.*

Definition 5.4 *A dyadic subbase S is independent iff $\psi_S(d) \neq \emptyset$ for all $d \in K(\Sigma_\perp^\omega)$.*

Definition 5.5 *A dyadic subbase is minimal if any proper subset is not a dyadic subbase.*

In [12], the author has proved the following two theorems.

Theorem 5.6 *(1) When S is a proper dyadic subbase, S is full-representing $\Rightarrow$ canonically-representing $\Rightarrow$ independent $\Rightarrow$ minimal.*

(2) When the space X is compact, S is full-representing $\Leftrightarrow$ canonically-representing $\Leftrightarrow$ independent.

Theorem 5.7 *Suppose that S is a full-representing subbase of a space X, the following are equivalent.*
1) X is compact.
2) X is regular.
3) ρ_S is continuous.

Thus, when we only consider regular spaces, the existence of a full-representing subbase ensures that X is compact, and thus the three properties of Theorem 5.6(1) become equivalent. The author has not succeeded in finding a non-compact space with a full-representing subbase. From the above theorem, such a space should be non-regular and the corresponding representing function ρ_S becomes non-continuous.

The author is interested in characterizing compact spaces with independent (i.e. full-representing by Theorem 5.7) subbases. We can easily show that if X has an independent subbase, then X has no isolated points (i.e., X is perfect). Therefore, every countable compact Hausdorff space does not have an independent subbase. However, the characterization of perfect compact Hausdorff spaces with an independent subbase seems rather difficult. As we have seen, The Cantor space Σ^ω and the unit interval $\mathcal{I}$ have such subbases. We can construct such a subbase for I^n, the unit circle S^1, n-dimensional surface S^n, torus T^2, and n-torus nT^2 (orientable closed surface of genus n). Since every orientable closed surface is homeomorphic to one of them, we can conclude that all the orientable closed surfaces have independent subbases. However, it is still open whether non-orientable surfaces like the Möbius ring, Klein bottle, and the projective plane have such subbases.

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