数理解析研究所講究録 1381 巻 2004 年 99-107

Apartness Spaces–an Overview

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Abstract

Within Bishop's constructive mathematics, we briefly introduce apartness spaces and discuss various continuity properties of mappings between these spaces.

The theory of proximity spaces originated apparently in 1908 at the mathematical congress in Rome, when Riesz [24] presented some ideas in his 'theory of enchainment' which have become the basic concepts of the theory. In the early 1950's, the subject was rediscovered by Efremovic [13], [14] when he axiomatically characterized the proximity relation 'A is near B' for subsets of any set X. Recently there has been quite an intense investigation of topological structures in image processing, mostly in connection with the analysis of connectivity and the operation of thinning (see e.g. [3, 12, 19, 21], etc.). An interesting attempt to introduce richer structures than those of topology, and replacing thus 'local' continuity properties by a global notion of nearness, has been done in [20] where the authors contemplated the so called semi–proximity spaces as a theoretical tool in the image processing studies. See also [18, 25].

This paper outlines a few aspects of the theory of apartness spaces. We work entirely within constructive mathematics à la Bishop [1, 2], an informal framework which simultaneously generalises classical, intuitionistic, and recursive mathematics [4], and which has turned out in practice to be just mathematics with intuitionistic logic (see, for example, [23]). Apartness spaces are the constructive analogues of the proximity spaces studied by some classical topologists (see, for instance, [11, 22, 15, 29]), and the theory of apartness spaces is intended to provide a constructively meaningful development of set-theoretic topology[1]. Bridges and Vîţă began with a first-order theory based on two primitive notions of nearness and apartness for a point and a set [30]. They then produced a much smoother theory based on a single primitive notion of point-set apartness [7], abstracted from the context of a metric space (X,ρ) in which a point x is apart from a set S if

$$\exists r>0\,\forall y\in S\ (\rho(x,y)\geq r),$$

and developed it in a series of papers [27, 5, 6, 10, 28, 31, 32, 8, 9].

[1] We should mention here that there is an altogether different approach to constructive topology, known as *pointless topology* [17, 16, 26]. The underlying idea in this approach is to generalise the classical notion of a topological space, characterised by the lattice of its open sets, by working with *frames*.

Let X be a nonempty set. We assume that there is a **set–set apartness** relation $\bowtie$ between pairs of subsets of X, such that the following axioms hold.

B1 $X \bowtie \emptyset$.

B2 $S \bowtie T \Rightarrow S \cap T = \emptyset$.

B3 $R \bowtie (S \cup T) \Leftrightarrow R \bowtie S \wedge R \bowtie T$.

B4 $S \bowtie T \Rightarrow T \bowtie S$.

B5 $\{x\} \bowtie S \Rightarrow \exists T(\{x\} \bowtie T \wedge \forall y(\{y\} \bowtie S \vee y \in T))$.

We then call X an **apartness space**. Defining

$$x \neq y \Leftrightarrow \{x\} \bowtie \{y\}$$

and

$$x \bowtie S \Leftrightarrow \{x\} \bowtie S,$$

we obtain an inequality and a point–set apartness relation associated with the given set–set one.

If we want to restrict our study to a space that has only a point–set apartness $\bowtie$, then we will assume that X is endowed with a non–trivial inequality relation $\neq$, that is, X contains at least two distict points and

$$\begin{aligned} x \neq y &\Rightarrow y \neq x \\ x \neq y &\Rightarrow \neg(x = y). \end{aligned}$$

We also assume that X has an apartness relation $\bowtie$ between points and subsets satisfying the following axioms.

A1 $x \neq y \Rightarrow x \bowtie \{y\}$.

A2 $x \bowtie S \Rightarrow x \notin S$.

A3 $x \bowtie (S \cup T) \Leftrightarrow x \bowtie S \wedge x \bowtie T$.

A4 $x \in -S \subset \sim T \Rightarrow x \bowtie T$.

A5 $x \in -S \Rightarrow \forall y \in X(x \neq y \vee y \in -S)$,

in which we write

$$-S = \{x \in X : x \bowtie S\}.$$

Each subset S of X admits three types of a complement—namely, the **apartness complement** $-S$, the **logical complement** $\neg S = \{x \in X : \forall y \in S \neg(x = y)\}$, and the **complement** $\sim S = \{x \in X : \forall y \in S\,(x \neq y)\}$. We have

$$-S \subset \sim S \subset \neg S.$$

In general, if X is an apartness space, we will think of it not only as a set–set apartness space, but also as a point–set apartness space with the point–set apartness and inequality induced by $\bowtie$. A point–set apartness space that has the property stated in axiom B5 is called **locally decomposable**.
A natural example of a point–set apartness is given by a T_1 topological space (X, τ), where by T_1 we mean the following separation property:

$$x \neq y \Rightarrow \exists U \in \tau \ (x \in U \subset {\sim}\{y\}) .$$

If $x \in X$ and $A \subset X$, we define

$$x \bowtie A \Leftrightarrow \exists U \in \tau \ (x \in U \subset {\sim}A) .$$

This relation satisfies axioms A1–A4. To make X into an apartness space we need to postulate axiom A5. A topological space (X, τ) is a **topological apartness space** if the apartness defined above turns X into a (point–set) apartness space; we then call the apartness structure on X the **topological apartness structure** corresponding to τ.

Conversely, a point–set apartness induces a topology on X as follows. A subset S of an apartness space X is said to be **nearly open** if there exists a family $(A_i)_{i \in I}$ such that $S = \bigcup_{i \in I} -A_i$. It is easy to show that the nearly open sets form a topology—the **apartness topology**—on X for which the apartness complements form a basis.

We say that a topological apartness space is **topologically consistent** if every open subset X is nearly open. In contrast to the classical situation, topological consistency does not automatically hold in constructive mathematics. Local decomposability turns out to be a natural condition which ensures topological consistency. (Note that local decomposability always holds classically). The following proposition gives us a topological equivalent for local decomposability.

Proposition 1 *Let X be a locally decomposable apartness space, and let τ be the corresponding apartness topology. Then for all $x \in X$ and $U \in \tau$,*

$$x \in U \Rightarrow \exists V \in \tau \ (x \in V \wedge \forall y \in X \ (y \in U \vee y \in {\sim}V)) . \tag{1}$$

Conversely, a topological apartness space (X, τ) with this property is locally decomposable.

Proposition 2 *A locally decomposable topological apartness space is topologically consistent.*

On the other hand, if we start off with a point–set apartness space $(X, \bowtie)$ and get the apartness topology $\tau_{\bowtie}$, we can prove that the apartness induced by $\tau_{\bowtie}$ coincides with the original apartness on X.

We now introduce a fundamental example of a (set–set) apartness space. Let X be a set with a nontrivial inequality, and let U, V be subsets of the Cartesian product $X \times X$. A family $\mathcal{U}$ of subsets of the Cartesian product $X \times X$ is called a **uniformity** on X if the following conditions hold.

U1 $\mathcal{U}$ is a **filter:** that is,

- $U \neq \emptyset$ for all $U \in \mathcal{U}$,
- $U \cap V \in \mathcal{U}$ for all $U, V \in \mathcal{U}$, and
- if $U \subset V$ and $U \in \mathcal{U}$, then $V \in \mathcal{U}$.

U2 Every $U \in \mathcal{U}$ contains both the diagonal Δ, and a symmetric member of $\mathcal{U}$.

U3 For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^2 \subset U$

U4 For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that

$$\forall x, y \in X\,((x,y) \in U \vee (x,y) \notin V)\,. \tag{2}$$

Define the **apartness relation** $\bowtie$ between subsets S, T of our uniform space X by setting

$$S \bowtie T \Longleftrightarrow \exists U \in \mathcal{U}\ (S \times T \subset \sim U)\,.$$

The topology $\tau_{\mathcal{U}}$ induced on X by $\mathcal{U}$ has an associated relation **apart** defined by

$$\mathbf{apart}\,(x, S) \Longleftrightarrow \exists V \in \tau_{\mathcal{U}}\ (x \in V \subset \sim S)\,,$$

which coincides with the point–set relation $\bowtie$.

In the classical theory of proximity spaces, every proximity space has uniform structures that are compatible with the denial apartness (the one in which two sets are apart if and only if they are not near each other). Constructively there is no hope of proving constructively that the apartness obtained from the usual point–set apartness on $\mathbf{R}$ has any compatible uniformities.

Therefore the constructive theory of apartness spaces is strictly bigger than that of uniform spaces. Note also that constructively the compact metric subspace $[0,1]$ of $\mathbf{R}$ has an apartness that induces the original topology but is not the same as the original apartness; this contrasts with the classical situation in which any compact topological space has a unique proximity structure compatible with the original compact topology.

It is natural to ask if a point–set apartness relation $\bowtie$ on a set X can be extended to a set–set apartness. The following proposition answers this question.

Proposition 3 *Let $(X, \bowtie)$ be a locally decomposable point–set apartness space, and for $S, T \subset X$ define*

$$S \bowtie T \Leftrightarrow \forall x \in X\,(x \bowtie S \vee x \bowtie T)\,, \tag{3}$$

where, on the right-hand side, $\bowtie$ denotes the original point–set apartness on X. Then the set–set relation $\bowtie$ satisfies axioms B1–B5.

When we turn to consider continuity of functions between apartness spaces things begin to become even more interesting because intuitionistic logic distinguishes between various types of continuity that are classically equivalent.

We first look at continuity properties of mappings between point–set apartness spaces. Keeping in mind that every point–set apartness induces a topology, we say that a mapping $f : X \to Y$ is

▷ **nearly continuous** if

$$\forall A \subset X\,(f(\overline{A}) \subseteq \overline{f(A)});$$

▷ **apart continuous** if

$$\forall x \in X\, \forall A \subset X\ \ (f(x) \bowtie f(A) \Rightarrow x \bowtie A)\,;$$

▷ **topologically continuous** if $f^{-1}(S)$ is nearly open in X for each nearly open $S \subset Y$.

Proposition 4 *A topologically continuous mapping between apartness spaces is nearly continuous.*

Proposition 5 *A topologically continuous mapping between apartness spaces is apart continuous.*

The converse, which holds classically, does not hold constructively. In order to obtain a partial converse to Proposition 5, we introduce the following property of **weak local decomposability** (w.l.d.) for an apartness space X :

$$x \bowtie T \Rightarrow \exists R(x \in -R \wedge (\neg R \subset -T)).$$

Proposition 6 *Let X and Y be apartness spaces, with Y weakly locally decomposable. Then every continuous function $f : X \to Y$ is topologically continuous.*

Note that weak local decomposability is a simple consequence of local decomposability. This observation, taken with Propositions 5, 6, and 4, yields the following.

Corollary 1 *For mappings from an apartness space into a locally decomposable apartness space, continuity and topological continuity are equivalent and imply near continuity.*

It is clear by now that local decomposability—which, remember, holds trivially under classical logic—is an extremely useful constructive property. In its presence, continuity and topological continuity coalesce, a topological apartness space is topologically consistent, and product apartness spaces (not discussed in this paper) have precisely the properties that one would wish for.

A mapping $f : X \to Y$ between set–set apartness spaces is said to be **strongly continuous** if for all subsets S, T of X,

$$f(S) \bowtie f(T) \Rightarrow S \bowtie T.$$

Theorem 1 *A uniformly continuous mapping between uniform spaces is strongly continuous.*

Given a uniform space $(X,\mathcal{U})$, an entourage U of X, and x in X, we denote by $U[x]$ the set $\{y \in X : (x, y) \in U\}$. We say that X is **totally bounded** if for each $U \in \mathcal{U}$ there exists a finitely enumerable subset $\{x_1, \ldots, x_n\}$ of X such that $X = \bigcup_{i=1}^{n} U[x_i]$.

Theorem 2 *Let X and Y be uniform apartness spaces, and f a strongly continuous mapping of X onto Y such that Y is totally bounded. Then f is uniformly continuous.*

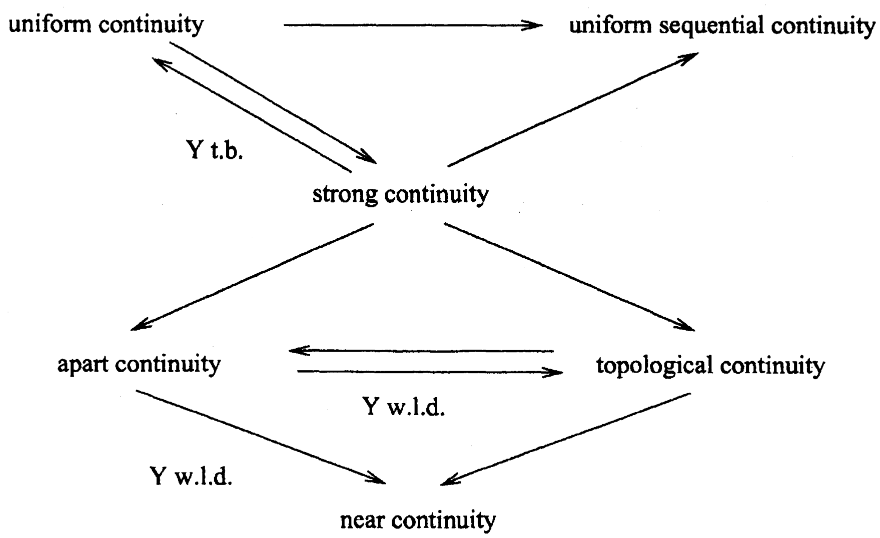
We conclude this paper with a result that shows what can be proved by way of a converse to Theorem 1 when the range of the strongly continuous function is not totally bounded. Two sequences $(x_n)_{n=1}^{\infty}, (x'_n)_{n=1}^{\infty}$ in a uniform space $(X, \mathcal{U})$ are **eventually close** if

$$\forall U \in \mathcal{U} \exists N \forall n \geqslant N \ \left((x_n, x'_n) \in U\right).$$

A mapping f of X into a uniform space Y is **uniformly sequentially continuous** if the sequences $(f(x_n))_{n=1}^{\infty}, (f(x'_n))_{n=1}^{\infty}$ are eventually close in Y whenever $(x_n)_{n=1}^{\infty}, (x'_n)_{n=1}^{\infty}$ are eventually close in X. We have the following converse of Proposition 1.

Theorem 3 *A strongly continuous mapping $f : X \to Y$ between uniform spaces is uniformly sequentially continuous.*

These results can be summarised in the following diagram.



Acknowledgements. The author wishes to thank FoRST New Zealand for supporting her research since 2002 (contract # UOCX0215), Hajime Ishihara for the invitation to visit JAIST and offering her the opportunity to attend the RIMS Symposium in September 2003. The Japanese Society for the Promotion of Science (Grant-in-Aid for Scientific Research (C) # 15500005) is acknowledged with thanks for the financial support offered during the author's stay in Japan in September 2003.

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