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Fourier transforms of invariant functions on finite reductive Lie algebras

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Abstract: Let G be a connected reductive group defined over $\mathbb{F}_q$ with Lie algebra $\mathcal{G}$. We give two definitions of a Deligne-Lusztig induction for the $\overline{\mathbb{Q}}_\ell$-valued functions on $\mathcal{G}(\mathbb{F}_q)$ which are invariant under the adjoint action of $G(\mathbb{F}_q)$ on $\mathcal{G}(\mathbb{F}_q)$. The first definition is based on the two-variable Green functions defined in group theoritical terms (using ℓ-adic cohomology) and then transfered to the Lie algebra by means of a G-equivariant bijection $G_{uni} \to \mathcal{G}_{nil}$. The second one involves the Lie algebra version of Lusztig's character sheaves theory. We formulate a conjecture about a commutation formula between Deligne-Lusztig induction and Fourier transforms. Using those two definitions of Deligne-Lusztig induction, we establish this conjecture in almost all cases. The importance of such a conjecture comes from the fact that it reduces [Let03b] the computation of the trigonometric sums [Spr76] on $\mathcal{G}(\mathbb{F}_q)$ to the computation of some fourth roots of unity coming from Fourier transforms [Lus87] and the values of the generalized Green functions defined by Lusztig.

Introduction

Let G be a connected reductive group over an algebraic closure $\mathbb{F}$ of the finite field $\mathbb{F}_q$ with q elements and let p be the characteristic of $\mathbb{F}$. Assume that G is defined over $\mathbb{F}_q$ with associated Frobenius endomorphism F. Then the Lie algebra $\mathcal{G}$ of G and the adjoint action of G on $\mathcal{G}$ are also defined over $\mathbb{F}_q$. We still denote by F the corresponding Frobenius endomorphism on $\mathcal{G}$. We then denote by G^F (resp. $\mathcal{G}^F$) the set of the elements of G (resp. $\mathcal{G}$) which are fixed by F. Let ℓ be a prime $\neq p$ and let $\overline{\mathbb{Q}}_\ell$ be an algebraic closure of the field $\mathbb{Q}_\ell$ of ℓ-adic numbers. We denote by $\mathcal{C}(\mathcal{G}^F)$ the $\overline{\mathbb{Q}}_\ell$-vector space of $\overline{\mathbb{Q}}_\ell$-valued functions on $\mathcal{G}^F$ which are invariant under the adjoint of G^F on $\mathcal{G}^F$. Assume that p and q are large enough so that there exists a G-invariant biblinear form $\mu : \mathcal{G} \times \mathcal{G} \to \mathbb{F}$ defined over $\mathbb{F}_q$, and let $\Psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$ be a non-trivial additive character of $\mathbb{F}_q$. Then the Fourier transform $\mathcal{F}^{\mathcal{G}} : \mathcal{C}(\mathcal{G}^F) \to \mathcal{C}(\mathcal{G}^F)$ with respect to the pair (μ, Ψ) is defined by the following formula

$$\mathcal{F}^{\mathcal{G}}(f)(x) = |\mathcal{G}^F|^{-\frac{1}{2}} \sum_{y \in \mathcal{G}^F} \Psi\big(\mu(x,y)\big) f(y)$$

where $f \in \mathcal{C}(\mathcal{G}^F)$ and $x \in \mathcal{G}^F$. The functions of the form $\mathcal{F}^{\mathcal{G}}(\xi_{\mathcal{O}})$, where $\xi_{\mathcal{O}}$ is the characteristic function of a G^F-orbit $\mathcal{O}$ of $\mathcal{G}^F$, form a basis of $\mathcal{C}(\mathcal{G}^F)$ and are called *trigonometric sums.* They were first introduced by Springer [Spr71] [Spr76] in connection with the $\overline{\mathbb{Q}}_\ell$-character theory of finite groups of Lie type: it was shown by Kazhdan [Kaz77], using the results of [Spr76], that the values of the Green functions of finite groups of Lie type can be expressed (via the exponential map) in terms of the values of trigonometric sums of the form $\mathcal{F}(\xi_{\mathcal{O}})$ with $\mathcal{O}$ semi-simple regular.

The first motivation of this work is to study trigonometric sums using the techniques developped principally by Lusztig to study the irreducible $\overline{\mathbb{Q}}_\ell$-characters of finite groups of Lie type. In particular this suggests the existence of a "twisted" induction for Lie algebras which would fit to the study of trigonometric sums, that is, which would commute with Fourier transforms. Gus Lehrer has proved [Leh96] that Harish-Chandra induction commutes with Fourier transforms, suggesting thus to define the required twisted induction as a generalization of Harish-Chandra induction. A natural reflex would be to adapt the definition of Deligne-Lusztig induction [DL76] to the Lie algebra case, however the definition is not directly adaptable since there is no "action" of the Lie algebra on the cohomology of Deligne-Lusztig varieties. The definition of Deligne-Lusztig we give here uses the "character formula" where the "two-variable Green functions" are defined in group theoritical terms and then transferred to the Lie algebra via a G-equivariant homeomorphism from the nilpotent variety $\mathcal{G}_{nil}$ onto the unipotent variety G_{uni}. Our definition of Deligne-Lusztig induction is thus available if such a map $\mathcal{G}_{nil} \to G_{uni}$ is well-defined which is the case if p is good for G [Spr69]. Let $\mathcal{L}$ be the Lie algebra of an F-stable Levi subgroup L of G and let $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}} : \mathcal{C}(\mathcal{L}^F) \to \mathcal{C}(\mathcal{G}^F)$ denote the Deligne-Lusztig induction; the author conjectured the following commutation formula

$$(*)\quad \mathcal{R}^{\mathcal{G}}_{\mathcal{L}} \circ \mathcal{F}^{\mathcal{L}} = \epsilon_G \epsilon_L \mathcal{F}^{\mathcal{G}} \circ \mathcal{R}^{\mathcal{G}}_{\mathcal{L}}$$

where $\mathcal{F}^{\mathcal{L}}$ is the Fourier transforms with respect to $(\mu|_{\mathcal{L}\times\mathcal{L}}, \Psi)$ and $\epsilon_G = (-1)^{\mathbb{F}_q-rank(G)}$. If L is a Levi subgroup of an F-stable parabolic subgroup of G, then the fomula (*) is a result of G. Lehrer [Leh96] since in that case $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}}$ is the Harish-Chandra induction. Using the Lie algebra version of Lusztig character sheaves theory, we have another definition of Deligne-Lusztig induction which does not involve any map $\mathcal{G}_{nil} \to G_{uni}$ (proving thus the independence of our definition of Deligne-Lusztig induction from the choice of such a map). Using these two definitions of Deligne-Lusztig induction, the above commutation formula is proved in many cases (including the cases where the root system G does not have components of type D_n or where L is a maximal torus). Now using the commutation formula (*), we can reduce the computation of trigonometric sums on $\mathcal{G}^F$ to the computation of some constants coming from Fourier transforms [Lus87] (called Lusztig's constants) and the computation of the generalized Green functions defined by Lusztig [Lus85] (a preliminary version of these results is available from [Let03b]). The Lusztig constants have been computed by Digne-Lehrer-Michel [DLM97] in the case of groups of type A_n, by Waldspurger [Wal01] in the case of groups of type C_n and in the case of the special orthogonal groups $SO_n(\mathbb{F})$, and by Kawanaka [Kaw86] in the exceptional cases E_8, F_4 and G_2. Moreover Lusztig has given an algorithm which reduces the computation of the values of generalized Green functions to the computation of some roots of unity whose values are known in many cases (Shoji has recently computed these roots of unity in type A_n).

This paper is essentially a résumé of [Let03b]. In section 1, we study some properties of algebraic groups and their Lie algebras related to the characteristic p in order to have an explicit range of values of p for which the Lie algebra version of Lusztig character sheaves theory applies. In sections 2 and 3, we give the two definitions of Deligne-Lusztig induction mentionned above. In sections 4, we explain how the conjecture (*) reduces to verify a property on the Luzstig constants [Lus87] attached to the "cuspidal pairs" of the simple groups of classical type. In section 5, we give a formula for the Lusztig constants attached to the "cuspidal pairs" of simple groups, generalizing a preliminary formula given in [DLM97] for the "regular" case. However our formula is not explicit enough to verify the required property on Lusztig's constants. So we have to use the results of [DLM97], [Wal01]; we then see that only the case of the spin groups of type D_n remains. Finally we state our results concerning (*).

Notation 0.1. Let H be a linear algebraic group over $\mathbb{F}$. If $x \in H$, we denote by x_s the semi-simple part of x and by x_u the unipotent part of x. We denote by H^o the neutral component of H and by Z_H the center of H. If $x \in H$, the centralizer of x in H is denoted by $C_H(x)$; it will be more convenient to denote the neutral component of $C_H(x)$ by $C_H^o(x)$ rather than by $C_H(x)^o$. Let $\mathcal{H} = \mathrm{Lie}(H)$ be the Lie algebra of H, for $x \in \mathcal{H}$, we denote by x_s the semi-simple part of x and by x_n the nilpotent part of x. We denote by $[,]$ the Lie product on $\mathcal{H}$ and by $z(\mathcal{H}) := \{x \in \mathcal{H} | \forall y \in \mathcal{H}, [x,y] = 0\}$. We have an inclusion $\mathrm{Lie}(Z_H) \subseteq z(\mathcal{H})$. If $f : H \rightarrow X$ is a morphism of algebraic varieties over $\mathbb{F}$, we denote by df its differential at 1.The adjoint action of $H \rightarrow \mathrm{GL}(\mathcal{H})$ is denoted by $\mathrm{Ad} = \mathrm{Ad}_H$ and we put $\mathrm{ad} = \mathrm{ad}_{\mathcal{H}} = d(\mathrm{Ad}_H)$; recall that $\mathrm{ad}(x)(y) = [x,y]$. Let K be a subgroup of H, by "H-orbit of $\mathcal{H}$", we shall mean "$\mathrm{Ad}(K)$-orbit of $\mathcal{H}$" and if $x \in \mathcal{H}$, we denote bu $\mathcal{O}_x^K$ the K-orbit of x. If $x \in \mathcal{H}$, then we denote by $C_H(x)$ the centralizer of x in H i.e. $C_H(x) = \{h \in H | \mathrm{Ad}(h)x = x\}$ and by $C_{\mathcal{H}}(x) := \{y \in \mathcal{H} | [x,y] = 0\}$. If $x \in \mathcal{H}$ is semi-simple, we have $\mathrm{Lie}\big(C_H(x)\big) = C_{\mathcal{H}}(x)$ [Bor, 9.1].

Notation 0.2. Let now G be a connected reductive algebraic group over $\mathbb{F}$ with Lie algebra $\mathcal{G}$. We assume that G is defined over $\mathbb{F}_q$, with q a power of a prime p, and we denote by F the corresponding Frobenius endomorphisms on G and on $\mathcal{G}$. If P is a parabolic subgroup of G, we will denote by U_P the unipotent radical of P and by $\mathcal{U}_P$ the Lie algebra of U_P. If $P = LU_P$, with corresponding Lie algebra decomposition $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$, is a Levi decomposition in G, we denote by $\pi_P : P \rightarrow L$ and $\pi_{\mathcal{P}} : \mathcal{P} \rightarrow \mathcal{L}$ the corresponding canonical projections. The letter T will denote a maximal torus of G and its Lie algebra will be denoted by $\mathcal{T}$. The dimension of T is called the *rank* of G and is denoted by $rk(G)$. As usual, we denote by $X(T)$ the group of algebraic group homomorphisms $T \rightarrow \mathbb{F}^\times$ and by $\Phi = \Phi(T) \subset X(T)$ the root system of G with respect to T. The $\mathbb{Z}$-sublattice of $X(T)$ generated by Φ is denoted by $Q(\Phi)$ and the $\mathbb{Z}$-lattice of weights is denoted by $P(\Phi)$. The group G is said to be semi-simple if $Q(\Phi)$ is of finite index in $X(T)$ (which condition is equivalent to $Q(\Phi) \subseteq X(T) \subseteq P(\Phi)$) and G is said to be simple if it is semi-simple and if Φ is irreducible. The group G is then said to be adjoint if $X(T) = Q(\Phi)$ and simply connected if $X(T) = P(\Phi)$. Recall that an F-stable torus $H \subset G$ of rank n is said to be split if there exists an isomorphism $H \xrightarrow{\sim} (\mathbb{F}^\times)^n$ defined over $\mathbb{F}_q$. The $\mathbb{F}_q$ rank of an F-stable maximal torus of G is defined to be the rank of its maximum split torus. An F-stable maximal torus of G is said to be G-split if it is maximally split in G. The $\mathbb{F}_q$-rank of G is the $\mathbb{F}_q$-rank of its G-split maximal tori. An F-stable Levi subgroup L of G is G-split if it has a G-split maximal torus; this is equivalent of saying that L is the Levi subgroup of an F-stable parabolic subgroup of G.

1 About reductive groups and their Lie algebras

The following results are well-known, however their proof are not always easily available in the literature. For complete proof of the following results which are not refered, see [Let03b]. The following result gives a necessary and sufficient condition on p for $\mathrm{Lie}(Z_G) \subseteq z(\mathcal{G})$ to be an equality:

Proposition 1.1. *The following assertions are equivalent:*
(i) the prime p does not divide $|\big(X(T)/Q(\Phi)\big)_{tor}|$.
(ii) $\mathrm{Lie}(Z_G) = z(\mathcal{G})$.

This result has the following easy consequence:

Corollary 1.2. *Assume G semi-simple and let $G = G_1 \ldots G_r$ be the decomposition of G as a product of simple algebraic groups G_i. If p does not divide $|X(T)/Q(T)|$, then $\mathcal{G} = \bigoplus_i Lie(G_i)$.*

By a G-invariant bilinear form μ on $\mathcal{G}$, we shall mean a symmetric bilinear form $\mu : \mathcal{G} \times \mathcal{G} \to \mathbb{F}$ such that for any $g \in G$, x, $y \in \mathcal{G}$, we have $\mu(\mathrm{Ad}(g)x, \mathrm{Ad}(g)y) = \mu(x, y)$. A well-known example of such a form is the Killing form defined on $\mathcal{G} \times \mathcal{G}$ by $(x, y) \mapsto \mathrm{Trace}(\mathrm{ad}(x) \circ \mathrm{ad}(y))$. As far as I know, no necessary and sufficient condition on p for the existence of non-degenerate G-invariant bilinear forms on $\mathcal{G}$ has been given in the literature. Here we give such a condition on p when G is simple of type A_n or when G is simply connected of type either B_n, C_n or D_n.

Recall that a prime is said to be good for G if it does not divide the coefficient of the highest root of Φ. If a good prime for G does not divide $|P(\Phi)/Q(\Phi)|$, it is said to be very good for G. Recall that if Φ does not have irreducible components of type A_n, then the very good primes for G are the good ones.

From [SS70, I, 5.3], it is known that if G is simple and if p is very good for G, or $G = GL_n(\mathbb{F})$, then there exists a non-degenerate G-invariant bilinear form on $\mathcal{G}$. Using a Lie algebra isomorphism $\mathcal{G} \simeq \mathrm{Lie}(Z_G) \oplus \big(\mathcal{G}/\mathrm{Lie}(Z_G)\big)$, it follows from 1.2 applied to G/Z_G^o, that the above result can be extended to the case of reductive groups, that is if p is very good for G reductive, there exists a non-degenerate G-invariant bilinear form on $\mathcal{G}$. We have the following proposition:

Proposition 1.3. *Assume G simple and let (*) be the proposition "there exists a non-degenerate G-invariant bilinear form on $\mathcal{G}$".*

(i) If G is of type A_n, then (*) holds if and only if p is very good for G or p divides both $|X(T)/Q(\Phi)|$ and $|P(\Phi)/X(T)|$.

(ii) If G is simply connected of type either B_n, C_n or D_n, then (*) holds if and only if p is good for G.

Note that the restriction to $z(\mathcal{G})$ of a non-degenerate G-invariant bilinear form on $\mathcal{G}$ might be degenerate, this happens for instance if we take the form $(x, y) \mapsto \mathrm{Trace}(xy)$ on $\mathcal{G}$ with $G = GL_n(\mathbb{F})$ and $p|n$. However if p is very good for G, this situation does not happen, more precisely we have:

Proposition 1.4. *Assume that p is very good for G and let μ be a non-degenerate G-invariant bilinear form on $z(\mathcal{G}) \oplus \big(\mathcal{G}/z(\mathcal{G})\big) \simeq \mathcal{G}$. Then the subspace $z(\mathcal{G})$ is the orthogonal complement of $\mathcal{G}/z(\mathcal{G})$ in $\mathcal{G}$ with respect to μ. In particular, the restrictions of μ to $z(\mathcal{G})$ and to $\mathcal{G}/z(\mathcal{G})$ remain non-degenerate.*

Lemma 1.5. *[Leh96, proof of 4.3] Let μ be a non-degenerate G-invariant bilinear form on $\mathcal{G}$. The restriction of μ to any Levi subalgebra is still non-degenerate.*

Now let L be a Levi subgroup of G with Lie algebra $\mathcal{L}$. Note that if $x \in \mathcal{G}$ satisfies $C_G^o(x) = L$, then $x \in z(\mathcal{L})$. Define

$$z(\mathcal{L})_{reg} := \{x \in \mathcal{G} | \, C_G^o(x) = L\}.$$

Proposition 1.6. *(i) If p is good for G, then for any semi-simple element $x \in \mathcal{G}$, the group $C_G^o(x)$ is a Levi subgroup of G.*

(ii) If p is good for G and if p does not divide $|\big(X(T)/Q(\Phi)\big)_{tor}|$, then for any Levi subgroup L of G, the set $z(\mathcal{L})_{reg}$ is not empty.

The assertion 1.6(i) comes from the fact that for $x \in \mathrm{Lie}(T)$, the set $\{\alpha \in \Phi \mid d\alpha(x) = 0\}$ is a $\mathbb{Q}$-closed root subsystem of Φ [Slo80, 3.14]. The assertion 1.6(ii) is proved using 1.6(i) and 1.1.

2 Twisted induction: a first definition

For a full detailed version of this section, see [Let03a].

Assumption 2.1. *In this section, we assume that p is good for G so that there exists a G-equivariant homeomorphism $\overline{\phi} : \mathcal{G}_{nil} \to G_{uni}$ defined over $\mathbb{F}_q$, where G acts by the adjoint action on the nilpotent variety $\mathcal{G}_{nil}$ and by conjugation on the unipotent variety G_{uni}.*

Lemma 2.2. *[Bon02, Lemma 3.2] For any Levi decomposition $P = LU_P$ in G with corresponding Lie algebra decomposition $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$, we have:*
(i) $\overline{\phi}(\mathcal{L}_{nil}) = L_{uni}$,
(ii) for any $x \in \mathcal{L}_{nil}$, $\overline{\phi}(x + \mathcal{U}_P) = \overline{\phi}(x)U_P$.

For a variety X over $\mathbb{F}$, we denote by $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ the i-th group of ℓ-adic cohomology with compact support as in [Del77].

Let L be an F-stable Levi subgroup of G, let $P = LU_P$ be a Levi decomposition of a (possibly non F-stable) parabolic subgroup P of G and let $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$ be the corresponding Lie algebra decomposition. We denote by $\mathcal{L}_G$ the Lang map $G \to G, x \mapsto x^{-1}F(x)$. The variety $\mathcal{L}_G^{-1}(U_P)$ is endowed with an action of G^F on the left and with an action of L^F on the right. These actions induce actions on the cohomology and so make $H^i_c(\mathcal{L}_G^{-1}(U_P), \overline{\mathbb{Q}}_\ell)$ into a G^F-module-L^F. The virtual $\overline{\mathbb{Q}}_\ell$-vector space $H^*_c(\mathcal{L}_G^{-1}(U_P)) := \sum_i (-1)^i H^i_c(\mathcal{L}_G^{-1}(U_P), \overline{\mathbb{Q}}_\ell)$ is thus a G^F-module-L^F.

The *two-variable Green function* $\mathcal{Q}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}} : \mathcal{G}^F_{nil} \times \mathcal{L}^F_{nil} \to \mathbb{Z}$ is defined by

$$\mathcal{Q}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(u, v) = |L^F|^{-1}\mathrm{Trace}\left((\overline{\phi}(u), \overline{\phi}(v)^{-1}) \mid H^*_c(\mathcal{L}_G^{-1}(U_P))\right).$$

We "extend" this function to a function $S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}} : \mathcal{G}^F \times \mathcal{L}^F \to \overline{\mathbb{Q}}_\ell$ as follows: for $(x, y) \in \mathcal{G}^F \times \mathcal{L}^F$, define

$$S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x, y) = \sum_{h \in G^F \mid \mathrm{Ad}(h)y_s = x_s} |C^\circ_L(y_s)^F||C^\circ_G(y_s)^F|^{-1} \mathcal{Q}^{C_{\mathcal{G}}(y_s)}_{C_{\mathcal{L}}(y_s)}(\mathrm{Ad}(h^{-1})x_n, y_n).$$

Remark 2.3. (i) If $(u, v) \in \mathcal{G}^F_{nil} \times \mathcal{L}^F_{nil}$, then $S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(u, v) = |L^F|\mathcal{Q}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(u, v)$.

(ii) The function $S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ is the Lie algebra analogue of the function $G^F \times L^F \to \overline{\mathbb{Q}}_\ell$ given by $(g, l) \mapsto \mathrm{Trace}((g, l) | H^*_c(\mathcal{L}_G^{-1}(U_P)))$ as it can be seen from [DM91, 12.3].

Definition 2.4. *The Deligne-Lusztig induction $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}} : \mathcal{C}(\mathcal{L}^F) \to \mathcal{C}(\mathcal{G}^F)$ is defined by:*

$$\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f)(x) = |L^F|^{-1} \sum_{y \in \mathcal{L}^F} S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x, y) f(y) \quad \textit{for } f \in \mathcal{C}(\mathcal{L}^F) \textit{ and } x \in \mathcal{G}^F.$$

Deligne-Lusztig induction satisfies the following elementary properties analogous to the group case:

Proposition 2.5. *(i) If P is F-stable, then $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ coincide with Harish-Chandra induction, that is*

$$\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f)(x) = |P^F|^{-1} \sum_{g\in G^F | Ad(g)x\in\mathcal{P}^F} f\big(\pi_{\mathcal{P}}(Ad(g)x)\big).$$

(ii) Deligne-Lusztig induction is transitive, and satisfies the Mackey formula.

(iii) $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ does not depend on P, and commutes with the duality map.

3 Twisted induction: a second definition

Starting from [Lus87] and by adapting Lusztig's ideas to the Lie algebra case, we have a Lie algebra version of Lusztig's character sheaves theory under the condition "p is acceptable" (see below) leading to the definition of a twisted induction which is better adapted to the study of Fourier transforms. This section is a dense résumé of [Let03b, Chapter 3].

In the following assumption, by a cuspidal pair of G, we shall mean a cuspidal pair $(S, \mathcal{E})$ of G in the sense of [Lus84, 2.4] such that S contains a unipotent conjugacy class of G.

Assumption 3.1. *In this section, we assume that p is* acceptable *for G i.e. that p satisfies the following conditions:*

(i) p is good for G.

(ii) p does not divide $|(X(T)/Q(\Phi))_{tor}|$.

(iii) There exists a non-degenerate G-invariant bilinear form μ on $\mathcal{G}$.

(iv) p is very good for any Levi subgroup of G supporting a cuspidal pair.

(v) There exists a G-equivariant isomorphism $\overline{\phi} : \mathcal{G}_{nil} \to G_{uni}$.

The following result can be easily deduced from the results of section 1 and the classification of the cuspidal data of G [Lus84]:

Lemma 3.2. *(i) If p is acceptable for G, then it is acceptable for any Levi subgroup of G.*

(ii) If p is very good for G, then it is acceptable for G.

(iii) All primes are acceptable for $G = GL_n(\mathbb{F})$.

(iv) If G is simple, the very good primes are the acceptable ones for G.

3.1 Admissible complexes (or character-sheaves) on $\mathcal{G}$

Notation 3.3. Let X be a variety over $\mathbb{F}$. We denote by $Sh(X)$ the abelian category of $\overline{\mathbb{Q}}_\ell$-sheaves on X and we denote by $\overline{\mathbb{Q}}_\ell$ the constant sheaf on X. We denote by $\mathcal{D}^b_c(X)$ the bounded "derived category" of $\overline{\mathbb{Q}}_\ell$-(constructible) sheaves as in [BBD82, 2.2.18]. By a *complex* on X we shall mean an object of $\mathcal{D}^b_c(X)$. For $K \in \mathcal{D}^b_c(X)$, the i-th cohomology sheaf of K is denoted by $\mathcal{H}^iK$. If $f : X \to Y$ is a morphism of varieties, we have the usual functors $f_* : Sh(X) \to Sh(Y)$ (direct image), $f_! : Sh(X) \to Sh(Y)$ (direct image with compact support), $f^* : Sh(Y) \to Sh(X)$ (inverse image) and the functors $Rf_* : \mathcal{D}^b_c(X) \to \mathcal{D}^b_c(Y)$, $Rf_! : \mathcal{D}^b_c(X) \to \mathcal{D}^b_c(Y)$ and $Rf^* : \mathcal{D}^b_c(Y) \to \mathcal{D}^b_c(X)$ as in [Gro73, Exposé XVII]. The functors Rf_*, $Rf_!$, Rf^* commute with the shift operations $[m]$ (if $K \in \mathcal{D}^b_c(X)$, the m-th shift of K is denoted by $K[m]$; for any integer i, we have $\mathcal{H}^i(K[m]) = \mathcal{H}^{i+m}K$). If there is no ambiguity we will denote by f_*, $f_!$ and f^* the functors Rf_*, $Rf_!$ and Rf^*. We denote by $\mathcal{M}(X)$ the full subcategory of $\mathcal{D}^b_c(X)$ consisting of perverse sheaves on X. Recall that $\mathcal{M}(X)$ is abelian. Note that if X is smooth of pure dimension, then for any $\xi \in ls(X)$, the complex $\xi[\dim X]$ is a perverse sheaf on X. For a locally closed smooth irreducible subvariety Y of X together with a local system ξ on Y, we denote by $\mathrm{IC}(\overline{Y},\xi) \in \mathcal{D}_c^b(\overline{Y})$ the corresponding intersection cohomology complex defined by Goresky-MacPherson and Deligne [BBD82]. Then the complex $\mathrm{IC}(\overline{Y},\xi)[\dim Y]$ is a perverse sheaf on $\overline{Y}$; moreover it is simple if ξ is irreducible. Recall that any simple perverse sheaf on X is of the form $j_!(\mathrm{IC}(\overline{Y},\xi)[\dim Y])$ with $j : \overline{Y} \hookrightarrow X$ for some (Y,ξ) as above with ξ irreducible.

Notation 3.4. Let H denote a **connected** linear algebraic group over $\mathbb{F}$ acting algebraically on X. Let $\mathcal{S}h_H(X)$ (resp. $\mathcal{M}_H(X)$) be the category of H-equivariant sheaves (resp. H equivariant perverse sheaves) on X. They are respectively full subcategories of $\mathcal{S}h(X)$ and $\mathcal{M}(X)$. If $\pi : H \times X \to X$ is the second projection and $\rho : H \times X \to X$ is the action of H on X, then the H-equivariant sheaves, resp. the H-equivariant perverse sheaves, on X can be identified with $\{\zeta \in \mathcal{S}h(X) \mid \pi^*(\zeta) \simeq \rho^*(\zeta)\}$, resp. $\{K \in \mathcal{M}(X) \mid \pi^*(K) \simeq \rho^*(K)\}$. We denote by $ls_H(X)$ the full subcategory of $ls(X)$ consisting of H-equivariant local systems on X.

Notation 3.5. Assume that X is defined over $\mathbb{F}_q$ with Frobenius endomorphism $F : X \to X$. A complex (or sheaf) K on X is said to be F-stable if $F^*(K)$ is isomorphic to K. An F-equivariant complex (resp. sheaf) on X is a pair (K,ϕ) with $K \in \mathcal{D}_c^b(X)$ (resp. $K \in \mathcal{S}h(X)$) and $\phi : F^*(K) \xrightarrow{\sim} K$ an isomorphism. The morphisms of F-equivariant complexes (or sheaves) are the obvious ones. If (K,ϕ) is an F-equivariant complex on X, we define the characteristic function $\mathbf{X}_{K,\phi} : X^F \to \overline{\mathbb{Q}}_\ell$ of (K,ϕ) by $\mathbf{X}_{K,\phi}(x) = \sum_i (-1)^i Trace(\phi_x^i, \mathcal{H}_x^i K)$ where ϕ_x^i is the automorphism of $\mathcal{H}_x^i K$ induced by ϕ. If $(\mathcal{E},\phi)$ is an F-equivariant sheaf on X, the characteristic function $X_{\mathcal{E},\phi} : X^F \to \overline{\mathbb{Q}}_\ell$ of $(\mathcal{E},\phi)$ is then defined by $X_{\mathcal{E},\phi}(x) = Trace(\phi_x, \mathcal{E}_x)$. If (K,ϕ) and (K',ϕ') are two isomorphic F-equivariant complexes (or sheaves), then their characteristic functions are equal. Let (K,ϕ) and (K',ϕ') be two F-equivariant simple perverse sheaves (or two irreducible local systems) on X such that $K \simeq K'$, then $\phi = c\phi'$ for some $c \in \overline{\mathbb{Q}}_\ell^\times$. If moreover if $c = 1$, then $(K,\phi) \simeq (K',\phi')$. Now let H and ρ be as in 3.4. If H and ρ are both defined over $\mathbb{F}_q$, then the characteristic function of any F-equivariant H-equivariant perverse sheaf (or sheaf) on X is an H^F-invariant function on X^F.

Notation 3.6. If Σ is a G-stable (for the adjoint action) locally closed, smooth, irreducible subset of $\mathcal{G}$ and if $\mathcal{E}$ is a G-equivariant local system on Σ, then we will denote by $K(\Sigma,\mathcal{E})$ the G-equivariant perverse sheaf $j_!(\mathrm{IC}(\overline{\Sigma},\mathcal{E})[\dim \Sigma])$ where $j : \overline{\Sigma} \hookrightarrow \mathcal{G}$.

3.7. We define the parabolic induction of equivariant perverse sheaves as in [Lus87]: let P be a parabolic subgroup of G and LU_P be a Levi decomposition of P. Let $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$ be the corresponding Lie algebra decomposition. Recall that $\pi_{\mathcal{P}} : \mathcal{P} \to \mathcal{L}$ denotes the canonical projection. Define $V_1 = \{(X,h) \in \mathcal{G} \times G \mid \mathrm{Ad}(h^{-1})X \in \mathcal{P}\}$ and $V_2 = \{(X,hP) \in \mathcal{G} \times (G/P) \mid \mathrm{Ad}(h^{-1})X \in \mathcal{P}\}$. Then we have the following diagram

$$\mathcal{L} \xleftarrow{\pi} V_1 \xrightarrow{\pi'} V_2 \xrightarrow{\pi''} \mathcal{G}$$

where $\pi''(X,hP) = X$, $\pi'(X,h) = (X,hP)$, $\pi(X,h) = \pi_{\mathcal{P}}(\mathrm{Ad}(h^{-1})X)$. Let K be an object in $\mathcal{M}_L(\mathcal{L})$. The morphism π is smooth with connected fibers of dimension $m = \dim G + \dim U_P$ and is P-equivariant with respect to the action of P on V_1 and on $\mathcal{L}$ given respectively by $x.(X,h) = (X,hx^{-1})$ and $x.X = \mathrm{Ad}(\pi_P(x))X$. Hence $\pi^*K[m]$ is a P-equivariant perverse sheaf on V_1 and since π' is a locally trivial principal P-bundle there exists a unique perverse sheaf $\tilde{K}$ on V_2 such that $\pi^*K[m] = (\pi')^*\tilde{K}[\dim P]$. Now we define the induced complex $\mathrm{ind}_{\mathcal{P}}^{\mathcal{G}}K$ of K by $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}K = (\pi'')_!\tilde{K} \in \mathcal{D}_c^b(\mathcal{G})$. This process defines a functor $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}$ from the category $\mathcal{M}_L(\mathcal{L})$ of L-equivariant perverse sheaves on $\mathcal{L}$ to $\mathcal{D}_c^b(\mathcal{G})$. If $K \in \mathcal{M}_L(\mathcal{L})$ is such that $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K \in \mathcal{M}(\mathcal{G})$ then $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K$ is automatically a G-equivariant perverse sheaf on $\mathcal{G}$; indeed the morphisms π, π' and π'' are all G-equivariant if we let G acts on V_1 and V_2 by Ad on the first coordinate and by left translation on the second coordinate, and on $\mathcal{L}$ trivially. Note that if P, L and K are all F-stable and if $\phi : F^*K \to K$ is an isomorphism, then ϕ induces a canonical isomorphism $\psi : F^*(\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K) \xrightarrow{\sim} \mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K$ such that $\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(X_{K,\phi}) = \mathbf{X}_{\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}} K,\psi}$ where $\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}$ is the Harish-Chandra induction (see 2.5(i)).

3.8. Let $(P, L, \Sigma, \mathcal{E})$ be a tuple where P is a parabolic subgroup of G, L is a Levi subgroup of P, $\Sigma = \mathcal{Z} + C$ with C a nilpotent orbit of $\mathcal{L}$ and $\mathcal{Z}$ a closed irreducible smooth subvariety of $z(\mathcal{L})$, and where $\mathcal{E}$ is an L-equivariant irreducible local system on Σ. Let $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$ be the Lie algebra decomposition corresponding to the decomposition $P = LU_P$. Then the complex $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(K(\Sigma, \mathcal{E}))$ is a G-equivariant perverse sheaf on $\mathcal{G}$. If moreover the local system $\mathcal{E}$ is of the form $\zeta \boxtimes \xi$ with $\xi \in ls_L(C)$ and $\zeta \in ls(\mathcal{Z})$ such that $\zeta[\dim \mathcal{Z}]$ is of geometrical origin in the sense of [BBD82, 6.2.4], then the perverse sheaf $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(K(\Sigma, \mathcal{E}))$ is semi-simple.

3.9. Let $(P, L, \Sigma, \mathcal{E})$ be as in 3.8 and assume moreover that $\mathcal{Z}_{reg} := \mathcal{Z} \cap z(\mathcal{L})_{reg} \neq \emptyset$. In this situation, we can regard the perverse sheaf $\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(K(\Sigma, \mathcal{E}))$ as an intersection cohomology complex on $\mathcal{G}$ as follows. Let $\Sigma_{reg} := \mathcal{Z}_{reg} + C$ and put $Y = \bigcup_{g\in G} \mathrm{Ad}(g)(\Sigma_{reg})$. The subset Y is then locally closed in $\mathcal{G}$, irreducible and smooth of dimension $\dim G - \dim L + \dim \Sigma$. We now construct following [Lus84] a G-equivariant semi-simple local system on Y: we have a diagram $\Sigma \xleftarrow{\alpha} Y_1 \xrightarrow{\alpha'} Y_2 \xrightarrow{\alpha''} Y$ where $Y_1 := \{(X, g) \in \mathcal{G} \times G | \mathrm{Ad}(g^{-1})X \in \Sigma_{reg}\}$, $Y_2 := \{(X, gL) \in \mathcal{G} \times (G/L) | \mathrm{Ad}(g^{-1})X \in \Sigma_{reg}\}$ and $\alpha(X, g) = \mathrm{Ad}(g^{-1})X$, $\alpha'(X, g) = (X, gL)$, $\alpha''(X, g) = X$. Denote by ξ_1 the irreducible L-equivariant local system $\alpha^*(\mathcal{E})$ on Y_1 (with respect to the action of L on Y_1 given by $x.(X, g) = (X, gx^{-1})$). The L-equivariance of ξ_1 implies the existence of a unique irreducible local system ξ_2 on Y_2 such that $(\alpha')^*\xi_2 = \xi_1$. Since α'' is a Galois covering with Galois group $W_G(\Sigma)$, the stabilizer of Σ in $N_G(L)/L$, the sheaf $(\alpha'')_*\xi_2$ is a semi-simple local system on Y. Now G acts on Y by Ad, on Y_1 and Y_2 by Ad on the first coordinate and by left translation on the second coordinate, and on Σ trivially; the morphisms α, α' and α'' are then G-equivariant from which we deduce that $(\alpha'')_*\xi_2$ is G-equivariant. The complex $\mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E}) := K(Y, (\alpha'')_*\xi_2)$ is thus a G-equivariant semi-simple perverse sheaf on $\mathcal{G}$ and each direct summand is G-equivariant. Now as in the situation of [Lus84, 4.5], we show that there is a canonical isomorphism

$$\mathrm{ind}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(K(\Sigma, \mathcal{E})) \xrightarrow{\sim} \mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E}).$$

Notation 3.10. Consider the non-trivial additive character $\Psi : \mathbb{F}_q^+ \to \overline{\mathbb{Q}}_\ell^\times$ fixed in the introduction. We denote by $\mathbf{A}^1$ the affine line over $\mathbb{F}$. Let $h : \mathbf{A}^1 \to \mathbf{A}^1$ be the Artin-Shreier covering defined by $h(t) = t^q - t$. Since h is a Galois covering of $\mathbf{A}^1$ with Galois group $\mathbb{F}_q$, the sheaf $h_*\overline{\mathbb{Q}}_\ell$ is a semi-simple local system on $\mathbf{A}^1$ on which $\mathbb{F}_q$ acts; we denote by $\mathcal{L}_\Psi$ the subsheaf of $h_*\overline{\mathbb{Q}}_\ell$ on which $\mathbb{F}_q$ acts as Ψ^{-1}. There exists an isomorphism $\phi_{\mathcal{L}_\Psi} : F^*\mathcal{L}_\Psi \xrightarrow{\sim} \mathcal{L}_\Psi$ such that for any integer $i \geq 1$, we have $\mathbf{X}_{\mathcal{L}_\Psi, \phi_{\mathcal{L}_\Psi}^{(i)}} = \Psi \circ Tr_{\mathbb{F}_{q^i}/\mathbb{F}_q} : \mathbb{F}_{q^i} \to \overline{\mathbb{Q}}_\ell^\times$, see [Kat80, 3.5.4].

3.11. We are now in position to define the **admissible complexes** (or character sheaves) on $\mathcal{G}$ [Lus87]. Let C be a nilpotent orbit on $\mathcal{G}$ and ζ an irreducible G-equivariant local system on C. One says that the pair (C, ζ) is *cuspidal* if for any proper Levi decomposition $P = LU_P$ in G, we have $(\pi_{\mathcal{P}})_!(K(C, \zeta)|_{\mathcal{P}}) = 0$. By a *cuspidal orbital complex*, we shall mean a complex of the form $K(\mathcal{O}, \mathcal{E})$ with $\mathcal{O} = \sigma + C$, $\mathcal{E} = \overline{\mathbb{Q}}_\ell \boxtimes \zeta$ where (C, ζ) is cuspidal and $\sigma \in z(\mathcal{G})$. By a *cuspidal admissible complex*, we shall mean a complex of the form $K(\Sigma, \mathcal{E})$ whith $\Sigma = z(\mathcal{G}) + C$,

$\mathcal{E} = m^*\mathcal{L}_\Psi \boxtimes \zeta$ where (C,ζ) is cuspidal and $m : z(\mathcal{G}) \to \mathbb{F}$ is a $\mathbb{F}$-linear form. If L is a Levi subgroup of G such that $\mathcal{L}$ supports a cuspidal pair, then we say that L is a *cuspidal Levi subgroup* of G. We say that $(L,\Sigma,\mathcal{E})$ is a *cuspidal datum* of $\mathcal{G}$ if L is a (cuspidal) Levi subgroup of G and if $K(\Sigma,\mathcal{E})$ is a cuspidal admissible complex on $\mathcal{L}$. Finally, we define the *admissible complexes* on $\mathcal{G}$ to be the G-equivariant simple perverse sheaves on $\mathcal{G}$ which are direct summand of the complexes of the form $\mathrm{ind}_\Sigma^\mathcal{G}(\mathcal{E})$ with $(L,\Sigma,\mathcal{E})$ a cuspidal datum of $\mathcal{G}$.

3.12. We have the following fundamental result: let $(L,\Sigma,\mathcal{E})$ and $(L',\Sigma',\mathcal{E}')$ be two cuspidal data of $\mathcal{G}$. Then the complexes $\mathrm{ind}_\Sigma^\mathcal{G}(\mathcal{E})$ and $\mathrm{ind}_{\Sigma'}^\mathcal{G}(\mathcal{E}')$ have a common direct summand if and only if $(L,\Sigma,\mathcal{E})$ and $(L',\Sigma',\mathcal{E}')$ are G-conjugate (i.e. there exists $g \in G$ such that $L' = gLg^{-1}$, $\Sigma' = \mathrm{Ad}(g)\Sigma$ and $\mathrm{Ad}(g)^*\mathcal{E}'$ is isomorphic to $\mathcal{E}$), in which case we have $\mathrm{ind}_\Sigma^\mathcal{G}(\mathcal{E}) \simeq \mathrm{ind}_{\Sigma'}^\mathcal{G}(\mathcal{E}')$.

3.2 Endomorphism algebra of $\mathrm{ind}_\Sigma^\mathcal{G}(\mathcal{E})$

Let $(L,\Sigma,\mathcal{E})$ be a cuspidal datum of $\mathcal{G}$. Let $N_G(\mathcal{E}) := \{n \in N_G(L) |\ \mathrm{Ad}(n)\Sigma = \Sigma,\ \mathrm{Ad}(n)^*\mathcal{E} \simeq \mathcal{E}\}$ and let $\mathcal{W}_G(\mathcal{E})$ be the finite group $N_G(\mathcal{E})/L$. We use the notation of 3.9.

Following [Lus84] and [Lus85, 10.2], we are going to describe the endomorphism algebra $\mathcal{A} := \mathrm{End}(\mathrm{ind}_\Sigma^\mathcal{G}(\mathcal{E}))$ in terms of $\mathcal{W}_G(\mathcal{E})$. Let $w \in \mathcal{W}_G(\mathcal{E})$ and let $\delta_w : Y_2 \xrightarrow{\sim} Y_2$ be the isomorphism defined by $\delta_w(X, gL) = (X, g\dot{w}^{-1}L)$ where $\dot{w}$ denotes a representative of w in $N_G(\mathcal{E})$; the map δ_w does not depend on the choice of the representative $\dot{w}$ of w. We have the following cartesian diagram:

$$\begin{tikzcd}
\Sigma \arrow[d, "\mathrm{Ad}(\dot{w})"'] & Y_1 \arrow[l, "\alpha"'] \arrow[r, "\alpha'"] \arrow[d, "f_{\dot{w}}"'] & Y_2 \arrow[r, "\alpha''"] \arrow[d, "\delta_w"'] & Y \arrow[d, "\|"'] \\
\Sigma & Y_1 \arrow[l, "\alpha"'] \arrow[r, "\alpha'"] & Y_2 \arrow[r, "\alpha''"] & Y
\end{tikzcd}$$

where $f_{\dot{w}}(X,g) = (X, g\dot{w}^{-1})$. From the above diagram we see that any isomorphism $\mathrm{Ad}(\dot{w})^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ induces a canonical isomorphism $\delta_w^*\xi_2 \xrightarrow{\sim} \xi_2$; conversely since $\alpha : Y_1 \to \Sigma_{reg}$ is a trivial principal G-bundle if G acts on Y_1 by left translation on both coordinates and on Σ_{reg} trivially, the functor $\alpha^* : Sh(\Sigma_{reg}) \to Sh_G(Y_1)$ is an equivalence of categories and so any isomorphism $\delta_w^*\xi_2 \simeq \xi_2$ defines a unique isomorphism $\mathrm{Ad}(\dot{w})^*\mathcal{E} \simeq \mathcal{E}$. Using $\alpha''_* \circ \delta_w^* = \alpha''_*$ we identify the one dimensional $\overline{\mathbb{Q}}_\ell$-vector space $\mathcal{A}_w$ of all homomorphisms $\delta_w^*\xi_2 \to \xi_2$ with a subspace of $\mathcal{A}$. From the previous discussion, we have a natural injective $\overline{\mathbb{Q}}_\ell$-linear map $\mathrm{Hom}(\mathrm{Ad}(\dot{w})^*\mathcal{E},\mathcal{E}) \to \mathcal{A}$.

For each $w \in \mathcal{W}_G(\mathcal{E})$, we choose a non-zero element θ_w of $\mathcal{A}_w$. Note that for $w, w' \in \mathcal{W}_G(\mathcal{E})$, we have $\delta_w \circ \delta_{w'} = \delta_{ww'}$. Hence for any $w, w' \in \mathcal{W}_G(\mathcal{E})$, we have $\theta_{w'} \circ \delta_{w'}^*(\theta_w) \in \mathcal{A}_{ww'}$. We thus have a well-defined product on $\bigoplus_{w\in\mathcal{W}_G(\mathcal{E})} \mathcal{A}_w$ given by $\theta_w.\theta_{w'} := \theta_{w'} \circ \delta_{w'}^*(\theta_w)$. This makes $\bigoplus_{w\in\mathcal{W}_G(\mathcal{E})} \mathcal{A}_w$ into a $\overline{\mathbb{Q}}_\ell$-algebra. Then as in [Lus84, Proposition 3.5], we show that $\bigoplus_{w\in\mathcal{W}_G(\mathcal{E})} \mathcal{A}_w \simeq \mathcal{A}$ as $\overline{\mathbb{Q}}_\ell$-algebras.

3.3 F-stable admissible complexes

3.13. Let $(L,\Sigma,\mathcal{E})$ be an F-stable cuspidal datum of $\mathcal{G}$ i.e. $F(L) = L$, $F(\Sigma) = \Sigma$ and $F^*\mathcal{E} \simeq \mathcal{E}$, and let $\phi : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ be an isomorphism. For any $w \in \mathcal{W}_G(\mathcal{E})$, we choose arbitrarily a non-zero element $\theta_w \in \mathcal{A}_w \subset \mathcal{A}$, see previous subsection. We fix an element w of $\mathcal{W}_G(\mathcal{E})$ together with a representative $\dot{w}$ of w in $N_G(\mathcal{E})$. By the Lang-Steinberg theorem there is an element $z \in G$ such that $z^{-1}F(z) = \dot{w}^{-1}$. Let $L_w := zLz^{-1}$ and let $\mathcal{L}_w$ be its Lie algebra. Then L_w and

$\Sigma_w := \mathrm{Ad}(z)\Sigma$ are both F-stable. Let $\mathcal{E}_w$ be the local system $\mathrm{Ad}(z^{-1})^*\mathcal{E}$. We now define an isomorphism $\phi_w : F^*\mathcal{E}_w \xrightarrow{\sim} \mathcal{E}_w$ in terms of ϕ. The automorphism θ_w defines an isomorphism $\mathcal{E} \simeq \mathrm{Ad}(\dot{w})^*\mathcal{E}$ leading to an isomorphism (*) $F^*\mathrm{Ad}(z^{-1})^*\mathcal{E} \simeq F^*\mathrm{Ad}(z^{-1})^*\mathrm{Ad}(\dot{w})^*\mathcal{E}$. Since we have $\mathrm{Ad}(\dot{w}) \circ \mathrm{Ad}(z^{-1}) \circ F = F \circ \mathrm{Ad}(z^{-1})$, the isomorphism (*) gives rise to an isomorphism $h : F^*\mathrm{Ad}(z^{-1})^*\mathcal{E} \simeq \mathrm{Ad}(z^{-1})^*F^*\mathcal{E}$. Then the isomorphism $\phi_w : F^*\mathcal{E}_w \simeq \mathcal{E}_w$ is $\mathrm{Ad}(z^{-1})^*(\phi) \circ h$.

We denote by $\phi^{\mathcal{G}} : F^*\left(\mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E})\right) \xrightarrow{\sim} \mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E})$ the natural isomorphism induced by ϕ and by $\phi_w^{\mathcal{G}}$: $F^*\left(\mathrm{ind}_{\Sigma_w}^{\mathcal{G}}(\mathcal{E}_w)\right) \xrightarrow{\sim} \mathrm{ind}_{\Sigma_w}^{\mathcal{G}}(\mathcal{E}_w)$ the natural isomorphism induced by ϕ_w. As in [Lus85, 10.6], there is a natural isomorphism $j : \mathrm{ind}_{\Sigma_w}^{\mathcal{G}}(\mathcal{E}_w) \xrightarrow{\sim} \mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E})$ such that the following diagram commutes.

$$\begin{tikzcd}
F^*\left(\mathrm{ind}_{\Sigma_w}^{\mathcal{G}}(\mathcal{E}_w)\right) \arrow[r, "F^*(j)"] \arrow[d, "\phi_w^{\mathcal{G}}"] & F^*\left(\mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E})\right) \arrow[d, "\theta_w \circ \phi^{\mathcal{G}}"] \\
\mathrm{ind}_{\Sigma_w}^{\mathcal{G}}(\mathcal{E}_w) \arrow[r, "j"] & \mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E})
\end{tikzcd}$$

As a consequence we get that $\mathbf{X}_{\mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E}),\theta_w \circ \phi^{\mathcal{G}}} = \mathbf{X}_{\mathrm{ind}_{\Sigma_w}^{\mathcal{G}}(\mathcal{E}_w),\phi_w^{\mathcal{G}}}$.

3.14. Let $(L, \Sigma, \mathcal{E})$ be a cuspidal datum of $\mathcal{G}$, let $K^{\mathcal{G}} = \mathrm{ind}_{\Sigma}^{\mathcal{G}}(\mathcal{E})$ and let $\mathcal{A} = \mathrm{End}(K^{\mathcal{G}})$. If A is a simple direct summand of $K^{\mathcal{G}}$, we denote by V_A the abelian group $\mathrm{Hom}(A, K^{\mathcal{G}})$. Then V_A is endowed with a structure of $\mathcal{A}$-module defined by $\mathcal{A} \times V_A \to V_A$, $(a, f) \mapsto a \circ f$; since A is a simple perverse sheaf, the $\mathcal{A}$-module V_A is irreducible. We have a natural isomorphism $\bigoplus_A (V_A \otimes A) \xrightarrow{\sim} K^{\mathcal{G}}$ where A runs over the set of simple components of $K^{\mathcal{G}}$ (up to isomorphism). For any $x \in \mathcal{G}$ and any integer i, it gives rise to an isomorphism (*) $\bigoplus_A (V_A \otimes \mathcal{H}_x^i A) \xrightarrow{\sim} \mathcal{H}_x^i K^{\mathcal{G}}$ under which an element $v \otimes a \in V_A \otimes \mathcal{H}_x^i A$ corresponds to $v_x^i(a)$ where $v_x^i : \mathcal{H}_x^i A \to \mathcal{H}_x^i K^{\mathcal{G}}$ is the morphism induced by $v : A \to K^{\mathcal{G}}$.

Assume now that the datum $(L, \Sigma, \mathcal{E})$ is F-stable and let ϕ be an isomorphism $F^*\mathcal{E} \simeq \mathcal{E}$. The complex $K^{\mathcal{G}}$ is thus F-stable and we denote by $\phi^{\mathcal{G}} : F^*K^{\mathcal{G}} \xrightarrow{\sim} K^{\mathcal{G}}$ the isomorphism induced by ϕ. Let A be an F-stable simple direct summand of $K^{\mathcal{G}}$ together with an isomorphism $\phi_A : F^*A \xrightarrow{\sim} A$. This defines a linear map $\sigma_A : V_A \to V_A$, $v \mapsto \phi^{\mathcal{G}} \circ F^*(v) \circ \phi_A^{-1}$ such that for any $x \in \mathcal{G}^F$ and any integer i, the isomorphism $\sigma_A \otimes (\phi_A)_x^i : V_A \otimes \mathcal{H}_x^i A \xrightarrow{\sim} V_A \otimes \mathcal{H}_x^i A$ corresponds under (*) to $(\phi^{\mathcal{G}})_x^i : \mathcal{H}_x^i K^{\mathcal{G}} \xrightarrow{\sim} \mathcal{H}_x^i K^{\mathcal{G}}$. On the other hand, if B is a simple component of $K^{\mathcal{G}}$ which is not F-stable, then $(\phi^{\mathcal{G}})_x^i$ maps $V_B \otimes \mathcal{H}_x^i B \hookrightarrow \mathcal{H}_x^i K^{\mathcal{G}}$ onto a different direct summand. It follows that,

3.15.
$$\mathbf{X}_{K^{\mathcal{G}},\phi^{\mathcal{G}}} = \sum_A \mathrm{Tr}(\sigma_A, V_A)\mathbf{X}_{A,\phi_A}$$

where A runs over the set of F-stable simple components of $K^{\mathcal{G}}$ (up to isomorphism). If for $w \in \mathcal{W}_G(\mathcal{E})$, we replace $\phi^{\mathcal{G}}$ by $\theta_w \circ \phi^{\mathcal{G}}$ with θ_w as in 3.13 and we keep ϕ_A unchanged, then the formula 3.15 becomes

3.16.
$$\mathbf{X}_{K^{\mathcal{G}},\theta_w \circ \phi^{\mathcal{G}}} = \sum_A \mathrm{Tr}(\theta_w \circ \sigma_A, V_A)\mathbf{X}_{A,\phi_A}.$$

Following [Lus86, 10.4] we deduce that

3.17.
$$\mathbf{X}_{A,\phi_A} = |\mathcal{W}_G(\mathcal{E})|^{-1} \sum_{w \in \mathcal{W}_G(\mathcal{E})} \mathrm{Tr}((\theta_w \circ \sigma_A)^{-1}, V_A)\mathbf{X}_{K^{\mathcal{G}},\theta_w \circ \phi^{\mathcal{G}}}$$

for any F-equivariant complex (A, ϕ_A) with A a simple direct summand of $K^{\mathcal{G}}$.

We use the notation of 3.13; by 3.13 and 3.17 we get that

3.18.

$$\mathbf{X}_{A,\phi_A} = |\mathcal{W}_G(\mathcal{E})|^{-1} \sum_{w\in\mathcal{W}_G(\mathcal{E})} \mathrm{Tr}((\theta_w \circ \sigma_A)^{-1}, V_A)\mathbf{X}_{\mathrm{ind}^{\mathcal{G}}_{\Sigma_w}(\mathcal{E}_w),\phi^{\mathcal{G}}_w}$$

for any F-equivariant admissible complex (A, ϕ_A) with A a simple direct summand of $K^{\mathcal{G}}$.

3.19. Let A be an F-stable admissible complex on $\mathcal{G}$. By 3.12, there is a unique (up to G-conjugacy) cuspidal datum $(L, \Sigma, \mathcal{E})$ of $\mathcal{G}$ such that A is a direct summand of $\mathrm{ind}^{\mathcal{G}}_{\Sigma}(\mathcal{E})$. Hence from Lang's theorem, we may choose $(L, \Sigma, \mathcal{E})$ to be F-stable; we thus have a formula like 3.18 for any F-equivariant admissible complex (A, ϕ_A) on $\mathcal{G}$.

3.20. Let $I(\mathcal{G})$ be a set parametrizing the isomorphic classes of the F-stable admissible complexes on $\mathcal{G}$. For $\iota \in I(\mathcal{G})$, let (A_ι, ϕ_ι) be a corresponding F-equivariant admissible complex on $\mathcal{G}$. Then by the main result of [Lus87], the set $\{\mathbf{X}_{A_\iota,\phi_\iota} | \iota \in I(\mathcal{G})\}$ is a basis of $\mathcal{C}(\mathcal{G}^F)$.

3.4 Twisted induction: a second definition

3.21. Let M be an F-stable Levi subgroup of G and let $\mathcal{M}$ be the Lie algebra of M. We define our twisted induction $R^{\mathcal{G}}_{\mathcal{M}} : \mathcal{C}(\mathcal{M}^F) \to \mathcal{C}(\mathcal{G}^F)$ on each element of a basis $\{\mathbf{X}_{A_\iota,\phi_\iota} | \iota \in I(\mathcal{M})\}$ of $\mathcal{C}(\mathcal{M}^F)$ as in 3.20. Let $\iota \in I(\mathcal{M})$ and let $(L, \Sigma, \mathcal{E})$ be an F-stable cuspidal datum of $\mathcal{M}$ such that A_ι is a direct summand of $\mathrm{ind}^{\mathcal{M}}_{\Sigma}(\mathcal{E})$. Let $\phi : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ be an isomorphism. For $w \in \mathcal{W}_M(\mathcal{E})$, let θ_w be a non-zero element of $\in \mathcal{A}_w \subset \mathrm{End}(\mathrm{ind}^{\mathcal{M}}_{\Sigma}(\mathcal{E}))$. As in 3.18 we have

3.22.

$$\mathbf{X}_{A_\iota,\phi_\iota} = |\mathcal{W}_M(\mathcal{E})|^{-1} \sum_{w\in\mathcal{W}_M(\mathcal{E})} \mathrm{Tr}((\theta_w \circ \sigma_{A_\iota})^{-1}, V_{A_\iota})\mathbf{X}_{\mathrm{ind}^{\mathcal{M}}_{\Sigma_w}(\mathcal{E}_w),\phi^{\mathcal{M}}_w}.$$

Then we define $R^{\mathcal{G}}_{\mathcal{M}}(\mathbf{X}_{A_\iota,\phi_\iota})$ by

3.23.

$$R^{\mathcal{G}}_{\mathcal{M}}(\mathbf{X}_{A_\iota,\phi_\iota}) = |\mathcal{W}_M(\mathcal{E})|^{-1} \sum_{w\in\mathcal{W}_M(\mathcal{E})} \mathrm{Tr}((\theta_w \circ \sigma_{A_\iota})^{-1}, V_{A_\iota})\mathbf{X}_{\mathrm{ind}^{\mathcal{G}}_{\Sigma_w}(\mathcal{E}_w),\phi^{\mathcal{G}}_w}.$$

Definition 3.24. *The induction defined above is called* geometrical induction.

Remark 3.25. (i) Note that the definition of $R^{\mathcal{G}}_{\mathcal{M}} : \mathcal{C}(\mathcal{M}^F) \to \mathcal{C}(\mathcal{G}^F)$ does not depend on the choice of the isomorphisms ϕ_ι with $\iota \in I(\mathcal{M})^F$. Indeed, let $R'^{\mathcal{G}}_{\mathcal{M}}$ be the induction defined on another basis $\{\mathbf{X}_{A_\iota,\phi'_\iota} | \iota \in I(\mathcal{M})^F\}$ and let $\iota \in I(\mathcal{M})^F$. Since A_ι is a simple perverse sheaf, there exists a constant $c \in \overline{\mathbb{Q}}_\ell^\times$ such that $\phi_\iota = c\phi'_\iota$. Let $\sigma'_{A_\iota} : V_{A_\iota} \to V_{A_\iota}$ be defined in terms of $\phi^{\mathcal{M}}$, ϕ'_ι as σ_{A_ι} is defined in terms of $\phi^{\mathcal{M}}$, ϕ_ι. We thus have $\sigma_{A_\iota} = c^{-1}\sigma'_{A_\iota}$. Hence for any $w \in \mathcal{W}_M(\mathcal{E})$, we have $(\theta_w \circ \sigma_{A_\iota})^{-1} = c(\theta_w \circ \sigma'_{A_\iota})^{-1}$ and so from 3.23, we get that $R^{\mathcal{G}}_{\mathcal{M}}(\mathbf{X}_{A_\iota,\phi_\iota}) = cR'^{\mathcal{G}}_{\mathcal{M}}(\mathbf{X}_{A_\iota,\phi'_\iota})$. But since $\mathbf{X}_{A_\iota,\phi_\iota} = c\mathbf{X}_{A_\iota,\phi'_\iota}$, this proves that $R^{\mathcal{G}}_{\mathcal{M}}(\mathbf{X}_{A_\iota,\phi_\iota}) = R'^{\mathcal{G}}_{\mathcal{M}}(\mathbf{X}_{A_\iota,\phi'_\iota})$. It is also clear that the induction $R^{\mathcal{G}}_{\mathcal{M}}$ does not depend on the choice of the isomorphisms $\phi : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ and on the choice of the isomorphisms $\theta_w \in \mathcal{A}_w$. The independent from the choice of the F-stable cuspidal data $(L, \Sigma, \mathcal{E})$ is a little bit more subtle, see remark before 3.28.

(ii) If $(M, \Sigma, \mathcal{E})$ is an F-stable cuspidal datum of $\mathcal{G}$ together with an isomorphism $\phi : F^*\mathcal{E} \simeq \mathcal{E}$, then

$$R^{\mathcal{G}}_{\mathcal{M}}(\mathbf{X}_{K(\Sigma,\mathcal{E}),\phi}) = \mathbf{X}_{\mathrm{ind}^{\mathcal{G}}_{\Sigma}(\mathcal{E}),\phi^{\mathcal{G}}}.$$

(iii) Note that unlike Deligne-Lusztig induction, the definition of geometrical induction does not involve any parabolic subgroup of G.

3.26. The following fact is clear: assume that $X^{\mathcal{G}}_{\mathcal{M}} : \mathcal{C}(\mathcal{M}^F) \rightarrow \mathcal{C}(\mathcal{G}^F)$ is a $\overline{\mathbb{Q}}_\ell$-linear map such that for any F-stable cuspidal datum $(L, \Sigma, \mathcal{E})$ of $\mathcal{M}$ and any isomorphism $\phi : F^*\mathcal{E} \simeq \mathcal{E}$, we have $X^{\mathcal{G}}_{\mathcal{M}}(\mathbf{X}_{\mathrm{ind}^{\mathcal{M}}_{\Sigma}(\mathcal{E}),\phi^{\mathcal{M}}}) = \mathbf{X}_{\mathrm{ind}^{\mathcal{G}}_{\Sigma}(\mathcal{E}),\phi^{\mathcal{G}}}$, then $X^{\mathcal{G}}_{\mathcal{M}} = R^{\mathcal{G}}_{\mathcal{M}}$.

3.27. For any F-stable cuspidal datum $(L, \Sigma, \mathcal{E})$ of $\mathcal{M}$ and any isomorphism $\phi : F^*\mathcal{E} \simeq \mathcal{E}$, we have $R^{\mathcal{G}}_{\mathcal{M}}(\mathbf{X}_{\mathrm{ind}^{\mathcal{M}}_{\Sigma}(\mathcal{E}),\phi^{\mathcal{M}}}) = \mathbf{X}_{\mathrm{ind}^{\mathcal{G}}_{\Sigma}(\mathcal{E}),\phi^{\mathcal{G}}}$.

As a straightforward consequence of 3.27, we get that the geometrical induction is transitive and together with 3.26 we get that the formula 3.23 does not depend on the choice of the cuspidal datum $(L, \Sigma, \mathcal{E})$.

Theorem 3.28. *Assume that q is large enough so that the main result of [Lus90] applies. Then Deligne-Lusztig induction and geometrical induction coincide.*

Outlined of the proof: Since Deligne-Lusztig induction is transitive, by 3.26, it is enough to prove that these two inductions coincide on the characteristic functions of F-equivariant cuspidal admissible complexes. Recall that if $(L, \Sigma, \mathcal{E}) = (L, z(\mathcal{L}) + C, \overline{\mathbb{Q}}_\ell \boxtimes \zeta)$ is an F-stable cuspidal datum of $\mathcal{G}$ together with $\phi_\zeta : F^*\zeta \simeq \zeta$, the corresponding *generalized Green function* $\mathcal{Q}^{\mathcal{G}}_{\mathcal{L},C,\zeta,\phi_\zeta} : \mathcal{G}^F_{nil} \rightarrow \overline{\mathbb{Q}}_\ell$ is defined as the restriction to $\mathcal{G}^F_{nil}$ of $\mathbf{X}_{\mathrm{ind}^{\mathcal{G}}_{\Sigma}(\mathcal{E}),\phi^{\mathcal{G}}}$ where $\phi^{\mathcal{G}}$ is the canonical isomorphism induced by $1 \boxtimes \phi_\zeta : F^*\mathcal{E} \simeq \mathcal{E}$.

Now let $(L, \Sigma, \mathcal{E}) = (L, z(\mathcal{L}) + C, m^*\mathcal{L}_\Psi \boxtimes \zeta)$ be an F-stable cuspidal datum of $\mathcal{G}$ and let $\phi : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ be an isomorphism. Let $\sigma, u \in \mathcal{G}^F$ with σ semi-simple and u nilpotent such that $[\sigma, u] = 0$. Assume that $x \in G^F$ is such that $\mathrm{Ad}(x^{-1})\sigma \in z(\mathcal{L})$. Then put $L_x = xLx^{-1}$ and $\mathcal{L}_x = \mathrm{Lie}(L_x)$. We have $\sigma \in z(\mathcal{L}_x)$ and so L_x is a Levi subgroup of $C^o_G(\sigma)$. Let $C_x = \mathrm{Ad}(x)C$ and let $(\zeta_x, \phi_{\zeta_x})$ be the inverse image of the F-equivariant sheaf $(\mathcal{E}, \phi)$ by $C_x \rightarrow \Sigma$, $v \mapsto \mathrm{Ad}(x^{-1})(\sigma+v)$. Note that the irreducible local system ζ_x is isomorphic to $\mathrm{Ad}(x^{-1})^*\zeta$. Then as in [Lus85, 8.5] we show the following character formula:

$$(1)\quad \mathbf{X}_{\mathrm{ind}^{\mathcal{G}}_{\Sigma}(\mathcal{E}),\phi^{\mathcal{G}}}(\sigma + u) = |C^o_G(\sigma)^F|^{-1} \sum_{x \in G^F | \mathrm{Ad}(x^{-1})\sigma \in z(\mathcal{L})} \mathcal{Q}^{C_{\mathcal{G}}(\sigma)}_{\mathcal{L}_x, C_x, \zeta_x, \phi_{\zeta_x}}(u).$$

The main result of Lusztig [Lus90], giving (in the group case) a comparaison formula between the two-variable Green functions and the generalized Green functions, can be transfered to the Lie algebra case by mean of the isomorphism $\overline{\phi} : \mathcal{G}_{nil} \xrightarrow{\sim} G_{uni}$. Using this comparaison formula together with the character formula (1), we show that $R^{\mathcal{G}}_{\mathcal{L}}(\mathbf{X}_{K(\Sigma,\mathcal{E}),\phi})(\sigma + u) := \mathbf{X}_{\mathrm{ind}^{\mathcal{G}}_{\Sigma}(\mathcal{E}),\phi^{\mathcal{G}}}(\sigma + u) = \mathcal{R}^{\mathcal{G}}_{\mathcal{L}}(\mathbf{X}_{K(\Sigma,\mathcal{E}),\phi})(\sigma + u)$ □

4 Fourier transforms and Deligne-Lusztig induction

In the following, for any F-stable Levi subgroup L of G, the Fourier transforms $\mathcal{F}^{\mathcal{L}} : \mathcal{C}(\mathcal{L}^F) \rightarrow \mathcal{C}(\mathcal{L}^F)$ is taken with respect to $(\mu|_{\mathcal{L}\times\mathcal{L}}, \Psi)$ as in the introduction. In [Let03b], the author has conjectured the following statement:

Conjecture 4.1. *For any F-stable Levi subgroup L of G, we have $\mathcal{F}^{\mathcal{G}} \circ \mathcal{R}_{\mathcal{L}}^{\mathcal{G}} = \epsilon_G \epsilon_L \mathcal{R}_{\mathcal{L}}^{\mathcal{G}} \circ \mathcal{F}^{\mathcal{L}}$ where $\epsilon_G = (-1)^{\mathbb{F}_q-rank(G)}$.*

From now we assume that p **is acceptable and that** q **is large enough** so that Deligne-Lusztig induction coincides with geometrical induction. It is then clear that 4.1 is equivalent to:

Conjecture 4.2. *For any F-stable Levi subgroup L of G supporting an F-equivariant cuspidal admissible complex (K, ϕ), we have $\mathcal{F}^{\mathcal{G}} \circ \mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(\mathbf{X}_{K,\phi}) = \epsilon_G \epsilon_L \mathcal{R}_{\mathcal{L}}^{\mathcal{G}} \circ \mathcal{F}^{\mathcal{L}}(\mathbf{X}_{K,\phi})$.*

We denote by $\mathcal{F}^{\mathcal{G}} : \mathcal{M}_G(\mathcal{G}) \to \mathcal{M}_G(\mathcal{G})$ the Deligne-Fourier transforms with respect to (μ, Ψ) that maps $K \in \mathcal{M}_G(\mathcal{G})$ onto $(pr_2)_!((pr_1)^* K \otimes \mu^* \mathcal{L}_\Psi)[\dim \mathcal{G}]$ where $pr_1, pr_2 : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ are the two projections. Recall that if (K, ϕ) is an F-equivariant complex, then there is a canonical isomorphism $\mathcal{F}(\phi) : F^*(\mathcal{F}^{\mathcal{G}} K) \to \mathcal{F}^{\mathcal{G}} K$ such that $\mathbf{X}_{\mathcal{F}^{\mathcal{G}} K, \mathcal{F}\phi} = (-1)^{\dim \mathcal{G}} |\mathcal{G}^F|^{\frac{1}{2}} \mathcal{F}^{\mathcal{G}}(\mathbf{X}_{K,\phi})$. If L is a Levi subgroup of G supporting a cuspidal pair, then by 1.4 any $\mathbb{F}$-linear form on $z(\mathcal{L})$ is of the form $m_\sigma : z(\mathcal{L}) \to \mathbb{F}$, $z \mapsto \mu(z, \sigma)$ for some $\sigma \in z(\mathcal{L})$. Now from [Lus87], for any cuspidal datum $(L, \Sigma, \mathcal{E}) = (L, z(\mathcal{L}) + C, (m_{-\sigma})^* \mathcal{L}_\Psi \boxtimes \zeta)$ of G where $\sigma \in z(\mathcal{L})$ we have $\mathcal{F}^{\mathcal{L}}(K(\Sigma, \mathcal{E})) \simeq K(\sigma + C, \overline{\mathbb{Q}}_\ell \boxtimes \zeta)$. As a consequence we get that 4.2 is equivalent to:

Conjecture 4.3. *For any F-stable Levi subgroup L of G supporting an F-equivariant cuspidal orbital complex (K, ϕ), we have $\mathcal{F}^{\mathcal{G}} \circ \mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(\mathbf{X}_{K,\phi}) = \epsilon_G \epsilon_L \mathcal{R}_{\mathcal{L}}^{\mathcal{G}} \circ \mathcal{F}^{\mathcal{L}}(\mathbf{X}_{K,\phi})$.*

We want to prove that the statement 4.3 is actually equivalent to:

Conjecture 4.4. *For any F-stable Levi subgroup L of G supporting an F-stable cuspidal pair (C, ζ) and any isomorphism $\phi : F^* \zeta \simeq \zeta$, we have $\mathcal{F}^{\mathcal{G}} \circ \mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(\mathbf{X}_{K(C,\zeta),\phi}) = \epsilon_G \epsilon_L \mathcal{R}_{\mathcal{L}}^{\mathcal{G}} \circ \mathcal{F}^{\mathcal{L}}(\mathbf{X}_{K(C,\zeta),\phi})$.*

Note that 4.4 is a particular case of 4.3. The fact that 4.3 and 4.4 are equivalent comes from the following theorem:

Theorem 4.5. *Let (L, C, ζ) be such that L is an F-stable Levi subgroup of G and (C, ζ) is an F-stable cuspidal pair of $\mathcal{L}$. Then there is a constant $c \in \overline{\mathbb{Q}}_\ell^\times$ such that for any $\sigma \in z(\mathcal{L})^F$ and any $\phi : F^*(K_\sigma) \xrightarrow{\sim} K_\sigma$ where $K_\sigma = K(\sigma + C, \overline{\mathbb{Q}}_\ell \boxtimes \zeta)$, we have*

$$\mathcal{F}^{\mathcal{G}} \circ \mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(\mathbf{X}_{K_\sigma,\phi}) = c \mathcal{R}_{\mathcal{L}}^{\mathcal{G}} \circ \mathcal{F}^{\mathcal{L}}(\mathbf{X}_{K_\sigma,\phi}).$$

About the proof of 4.5: When the variety $z(\mathcal{L})$ is used as a parametrizing set of the cuspidal orbital complexes on $\mathcal{L}$ of the form $K(\sigma + C, \overline{\mathbb{Q}}_\ell \boxtimes \zeta)$, it is denoted by S. Let $\mathcal{Z}_1 = S \times z(\mathcal{L})$ and $\mathcal{Z}_2 = \{(z,z) | z \in z(\mathcal{L})\} \subset S \times z(\mathcal{L})$. Then L acts on $\mathcal{Z}_1 \times C$ and on $\mathcal{Z} \times C$ by the adjoint action on C and trivially on the first coordinate. Consider the following F-stable irreducible local systems: $\mathcal{E}_1 = (\mu_{z(\mathcal{L})})^* \mathcal{L}_\Psi \boxtimes \zeta \in ls_L(\mathcal{Z}_1 \times C)$, where $\mu_{z(\mathcal{L})}$ is the restriction of μ to $z(\mathcal{L}) \times z(\mathcal{L})$, and $\mathcal{E}_2 = \overline{\mathbb{Q}}_\ell \boxtimes \zeta \in ls_L(\mathcal{Z}_2 \times C)$. Let $\sigma \in z(\mathcal{L})^F$, we put $K_{1,\sigma} := K(z(\mathcal{L}) + C, (m_\sigma)^* \mathcal{L}_\Psi \boxtimes \zeta)$ and $K_{2,\sigma} := K_\sigma$ as in 4.5. Clearly we have $(j_{\sigma,\mathcal{L}})^* K_1 = K_{1,\sigma}[\dim S]$ and $(j_{\sigma,\mathcal{L}})^* K_2 = K_{2,\sigma}[\dim S]$ where $j_{\sigma,\mathcal{L}} : \mathcal{L} \to S \times \mathcal{L}$, $x \mapsto (\sigma, x)$. Following [Wal01, Chapter 2], one has a functor $\mathrm{ind}_{S \times \mathcal{L}, \mathcal{P}}^{S \times \mathcal{G}} : \mathcal{M}_L(S \times \mathcal{L}) \to \mathcal{D}_c^b(S \times \mathcal{G})$ generalizing the construction of $\mathrm{ind}_{\mathcal{L} \subset \mathcal{P}}^{\mathcal{G}}$, see 3.7. From [Wal01], the complexes $K_1^{S \times \mathcal{G}} := \mathrm{ind}_{S \times \mathcal{L}, \mathcal{P}}^{S \times \mathcal{G}}(K_1)$ and $K_2^{S \times \mathcal{G}} := \mathrm{ind}_{S \times \mathcal{L}, \mathcal{P}}^{S \times \mathcal{G}}(K_2)$ are simple perverse sheaves on $S \times \mathcal{G}$. More precisely since $\{(s,z) \in \mathcal{Z}_1 | z \in z(\mathcal{L})_{reg}\}$ and $\{(s,z) \in \mathcal{Z}_2 | z \in z(\mathcal{L})_{reg}\}$ are non-empty, we can show

[Wal01], following the strategy of 3.9, that the complexes $K_1^{S\times\mathcal{G}}$ and $K_2^{S\times\mathcal{G}}$ are the perverse extensions of F-stable irreducible local systems on some F-stable locally closed subvarieties of $\mathcal{G}$ in particular $K_1^{S\times\mathcal{G}}$ and $K_2^{S\times\mathcal{G}}$ are both F-stable. Let $\phi_1 : F^*(K_1) \simeq K_1$ and $\phi_2 : F^*(K_2) \simeq K_2$ be two isomorphisms, and let $\phi_1^{S\times\mathcal{G}} : F^*K_1^{S\times\mathcal{G}} \simeq K_1^{S\times\mathcal{G}}$ and $\phi_2^{S\times\mathcal{G}} : F^*K_2^{S\times\mathcal{G}} \simeq K_2^{S\times\mathcal{G}}$ be the two isomorphisms induced respectively by ϕ_1 and ϕ_2. As in the proof of 3.28, one has a "character formula" [Let03b] expressing $\mathbf{X}_{K_1^{S\times\mathcal{G}},\phi_1^{S\times\mathcal{G}}}$ and $\mathbf{X}_{K_2^{S\times\mathcal{G}},\phi_2^{S\times\mathcal{G}}}$ in terms of the generalized Green functions. Hence if we define the Deligne-Lusztig induction $\mathcal{R}_{S\times\mathcal{L}}^{S\times\mathcal{G}} : \mathcal{C}(S^F \times \mathcal{L}^F) \to \mathcal{C}(S^F \times \mathcal{G}^F)$ by $\mathcal{R}_{S\times\mathcal{L}}^{S\times\mathcal{G}}(f)(t,x) = |L^F|^{-1}\sum_{y\in\mathcal{L}^F} S_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(x,y)f(t,y)$ where $S_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}(x,y)$ is as in section 2, then we show that

$$\mathcal{R}_{S\times\mathcal{L}}^{S\times\mathcal{G}}(\mathbf{X}_{K_1,\phi_1}) = \mathbf{X}_{K_1^{S\times\mathcal{G}},\phi_1^{S\times\mathcal{G}}} \text{ and } \mathcal{R}_{S\times\mathcal{L}}^{S\times\mathcal{G}}(\mathbf{X}_{K_2,\phi_2}) = \mathbf{X}_{K_2^{S\times\mathcal{G}},\phi_2^{S\times\mathcal{G}}}.$$

Now one has a Fourier transform $\mathcal{F}^{S\times\mathcal{G}} : \mathcal{C}(S^F \times \mathcal{G}^F) \to \mathcal{C}(S^F \times \mathcal{G}^F)$ given by $\mathcal{F}^{S\times\mathcal{G}}(f)(t,x) = |\mathcal{G}^F|^{-\frac{1}{2}}\sum_{y\in\mathcal{G}^F}\Psi(\mu(y,x))f(t,y)$ and a Deligne-Fourier transforms $\mathcal{F}^{S\times\mathcal{G}} : \mathcal{M}_G(S\times\mathcal{G}) \to \mathcal{M}_G(S\times\mathcal{G})$ given by $\mathcal{F}^{S\times\mathcal{G}}(K) = (p_{13})_!((p_{12})^*K\otimes(p_{23})^*(\mu^*\mathcal{L}_\Psi))[\dim\mathcal{G}]$ where $p_{13}, p_{12} : S\times\mathcal{G}\times\mathcal{G} \to S\times\mathcal{G}$ and $p_{23} : S\times\mathcal{G}\times\mathcal{G} \to \mathcal{G}\times\mathcal{G}$ are the projections. We have the following relation: if (K,ϕ) is an F-equivariant complex on $S\times\mathcal{G}$, then ϕ induces an isomorphism $\mathcal{F}(\phi) : F^*(\mathcal{F}^{S\times\mathcal{G}}K) \xrightarrow{\sim} \mathcal{F}^{S\times\mathcal{G}}K$ such that

$$\mathbf{X}_{\mathcal{F}^{S\times\mathcal{G}}K,\mathcal{F}(\phi)} = (-1)^{\dim\mathcal{G}}|\mathcal{G}^F|^{\frac{1}{2}}\mathcal{F}^{S\times\mathcal{G}}(\mathbf{X}_{K,\phi}).$$

Also the Deligne-Fourier transform commutes with the parabolic induction $\mathrm{ind}_{S\times\mathcal{L},\mathcal{P}}^{S\times\mathcal{G}}$ as it can be seen from [Wal01, Chapter 2], and $\mathcal{F}^{S\times\mathcal{L}}(K_2) \simeq K_1$. Hence (*) $\mathcal{F}^{S\times\mathcal{G}}(K_2^{S\times\mathcal{G}}) \simeq K_1^{S\times\mathcal{G}}$. Since our perverse sheaves $K_1^{S\times\mathcal{G}}$ and $K_2^{S\times\mathcal{G}}$ are simple, when taking the characteristic functions in (*), we finally deduce that there exists a constant c (which does not depend on σ) such that

$$\mathcal{F}^{S\times\mathcal{G}}\left(\mathcal{R}_{S\times\mathcal{L}}^{S\times\mathcal{G}}(\mathbf{X}_{K_2,\phi_2})\right) = c\mathcal{R}_{S\times\mathcal{L}}^{S\times\mathcal{G}}\left(\mathcal{F}^{S\times\mathcal{L}}(\mathbf{X}_{K_2,\phi_2})\right).$$

Restricting this equality to $\{\sigma\}\times\mathcal{G}^F$, we get the required result □

4.6. The previous equivalences shows that, under the assumption "p is acceptable and q is large", we have reduced the study of 4.1 to that of 4.4.

4.7. Now let L be an F-stable Levi subgroup of G supporting an F-stable cuspidal pair (C,ζ). Since the group $\mathcal{W}_G(\zeta)$, defined as in 3.2 with ζ instead of $\mathcal{E}$, is nothing but $W_G(L) := N_G(L)/L$ [Lus84, 9.2], we get that there exists an F-stable G-split Levi subgroup L_o of G which is G-conjugate to L, and $w \in W_G(L_o)$ such that (L,C,ζ) is of the form $((L_o)_w,(C_o)_w,(\zeta_o)_w)$, see 3.13. Put $\Sigma = z(\mathcal{L}) + C$, $\Sigma_o = z(\mathcal{L}_o) + C_o$, $\mathcal{E} = \overline{\mathbb{Q}}_\ell \boxtimes \zeta$ and $\mathcal{E}_o = \overline{\mathbb{Q}}_\ell \boxtimes \zeta_o$. From [Lus87], there exist two constants γ, $\gamma_o \in \overline{\mathbb{Q}}_\ell^\times$ such that for any isomorphisms $\phi : F^*(\zeta) \simeq \zeta$ and $\phi_o : F^*(\zeta_o) \simeq \zeta_o$ we have

$$\mathcal{F}^{\mathcal{L}}(\mathbf{X}_{K(\Sigma,\mathcal{E}),1\boxtimes\phi}) = \gamma\mathbf{X}_{K(C,\zeta),\phi} \text{ and } \mathcal{F}^{\mathcal{L}_o}(\mathbf{X}_{K(\Sigma_o,\mathcal{E}_o),1\boxtimes\phi_o}) = \gamma_o\mathbf{X}_{K(C_o,\zeta_o),\phi_o}.$$

The constant γ is called the *Lusztig's constant* attached to (L,C,ζ) with respect to F. Let $e : W_G(L_o) \to \overline{\mathbb{Q}}_\ell$ be the sign character of $W_G(L_o)$.

Proposition 4.8. *We have:*

$$\mathcal{F}^{\mathcal{G}} \circ \mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(\mathbf{X}_{K(C,\zeta),\phi}) = \epsilon_G\epsilon_L\mathcal{R}_{\mathcal{L}}^{\mathcal{G}} \circ \mathcal{F}^{\mathcal{L}}(\mathbf{X}_{K(C,\zeta),\phi})$$

if and only if $\gamma = \epsilon_G\epsilon_L e(w)\gamma_o$.

The proof of 4.8 uses the fact that Harish-Chandra induction commutes with Fourier transforms; this has been proved at first by Lusztig [Lus87] in the case of cuspidal functions and then by Lehrer in full generality [Leh96]. With the above notation, put $\tilde{\gamma} = \eta_L \sigma_L \gamma$ with $\eta_L = \epsilon_{L/Z_L^o}$ and $\sigma_L = (-1)^{rk(L/Z_L^o)}$. Then the equality $\gamma = \epsilon_G \epsilon_L e(w) \gamma_o$ of 4.8 is equivalent to $\tilde{\gamma} = \tilde{\gamma}_o$. We call $\tilde{\gamma}$ the *modified Lusztig's constant* attached to (L, C, ζ) with respect to F.

From 4.6 and the proposition 4.8 we deduce the following theorem:

Theorem 4.9. *Assume that p is acceptable for G and that q is large enough so that Deligne-Lusztig induction coincides with geometrical induction. Then the following assertions are equivalent:*

(i) The statement 4.1 holds.

(ii) For any F-stable triple $\iota = (L, C, \zeta)$ with L a proper G-split Levi subgroup of G and (C, ζ) a cuspidal pair on $\mathcal{L}$, the modified Lusztig's constant attached to ι does not depend on the Frobenius wF with $w \in W_G(L)$.

5 Lusztig's constants

Remark 5.1. The Lusztig constant attached to an F-stable maximal torus T is equal to $(-1)^{rk(G)} q^{\frac{rk(G)}{2}}$, hence does not depend on the Frobenius wF for $w \in W_G(T)$.

5.2. The statement 4.9(ii) can be easily reduced to the case where G is simple. Then using the classification of the cuspidal data of simple algebraic groups [Lus84], we see that 4.9(ii) reduces to:

Conjecture 5.3. *Assume that:*

(i) G is either semi-simple of type A_n or simple of type B_n, C_n or D_n,

(ii) p is very good for G,

(iii) $\mathcal{G}$ supports an F-stable cuspidal pair (C, ζ).

Then the modified Lusztig's constants attached to (C, ζ) does not depend on the $\mathbb{F}_q$-structure on G for which the induced Frobenius endomorphism stabilizes (C, ζ).

5.4. The statement 5.3 is clear if G is of type either B_n or C_n since in that case any Frobenius endomorphism on G acts trivially on the root system of G. From now we assume that G is simple and supports an F-stable cuspidal pair (C, ζ). We also assume that $p > 3(h_o^G - 1)$ where h_o^G is the Coxeter number of G. In the following we give a formula for the Lusztig's constants attached to (C, ζ).

5.5. We fix an element $u_o \in C^F$ where C is as in 5.3. Under our assumption, we can use Dynkin-Kostant-Springer-Steinberg's theory on nilpotent orbits on $\mathcal{G}$. Hence there exists an F-stable $\mathbb{Z}$-grading $\mathcal{G} = \bigoplus_i \mathcal{G}(i)$ of $\mathcal{G}$ with the following properties:

(i) $u_o \in \mathcal{G}(2)$.

(ii) $\mathcal{P} = \oplus_{i \geq 0} \mathcal{G}(i)$ is the Lie algebra of an F-stable parabolic subgroup P of G and $\mathcal{L} = \mathcal{G}(0)$ is the Lie algebra of an F-stable Levi subgroup L of P.

(iii) $\mathcal{G}(2)$ is stable under the adjoint action of L and $\mathcal{O}_{u_o}^L$ is dense in $\mathcal{G}(2)$.

(iv) $\mathcal{U}_P = \oplus_{i>0} \mathcal{G}(i)$.

(v) The group $C_{U_P}(u_o)$ is unipotent and connected, and the group $C_G(u_o)$ is the semi-direct product of $C_L(u_o)$ and $C_{U_P}(u_o)$ as an algebraic group.

(vi) We have $\mathcal{O}_{u_o}^G \cap \left(\oplus_{i \geq 2} \mathcal{G}(i) \right) = \mathcal{O}_{u_o}^P$.

(vii) The pair (C, ζ) being cuspidal, by [Lus84, 2.8] the element u_o is *distinguished* i.e. the map $\mathrm{ad}(u_o) : \mathcal{G}(0) \to \mathcal{G}(2)$ is bijective. Hence we have $\mathcal{G}(i) = \{0\}$ if i is odd i.e. $\mathcal{U}_P = \bigoplus_{i \geq 2} \mathcal{G}(i)$, and from (iii) we deduce that $C_L^o(u_o) = \{0\}$.

5.6. We now define the generalized Gelfand-Graev functions following [Kaw85]. Let $A_G(u_o) := C_G(u_o)/C_G^o(u_o)$ and let $H^1(F, A_G(u_o))$ be the group of F-conjugacy classes of $A_G(u_o)$. By setting that $1 \in H^1(F, A_G(u_o))$ corresponds to the G^F-orbit of u_o, we have a well-defined parametrization of the G^F-orbits in C^F by $H^1(F, A_G(u_o))$. From 5.5(v), we have $A_G(u_o) \simeq A_L(u_o)$, hence for $z \in H^1(F, A_G(u_o)) \simeq H^1(F, A_L(u_o))$, we can choose an element $u_z \in \mathcal{G}(2)^F$ which is in the G^F-orbit of C^F corresponding to z. Let $\mathcal{U}_P^- = \bigoplus_{i\leq -2} \mathcal{G}(i)$, then for each $z \in H^1(F, A_L(u_o))$, we define a linear additive character $\Psi_z : (\mathcal{U}_P^-)^F \to \overline{\mathbb{Q}}_\ell$ by $\Psi_z(u) = \Psi(\mu(u_z, u))$ where (μ, Ψ) is as in section 4. The corresponding generalized Gelfand-Graev function $\Gamma_z : \mathcal{G}^F \to \overline{\mathbb{Q}}_\ell$ is defined by:

$$\Gamma_z(x) = |U_P^F|^{-1} \sum_{\{g\in G^F | \mathrm{Ad}(g)x \in \mathcal{U}_P^-\}} \Psi_z(\mathrm{Ad}(g)x).$$

The G-equivariant irreducible local system ζ corresponds to a unique F-stable irreducible character (denoted again by ζ) of $A_G(u_o)$ which can be extended to a character on the semi-direct product $A_G(u_o) \rtimes \langle F\rangle$ where $\langle F\rangle$ is the cyclic group generated by the Frobenius F. The restriction to $A_G(u_o).F$ of this extended character is constant on the $A_G(u_o)$-orbits and so leads to a unique function $\tilde{\zeta}$ on $H^1(F, A_G(u_o)) \simeq H^1(F, A_L(u_o))$. We then define a nilpotently supported function $\Gamma_\zeta : \mathcal{G}^F \to \overline{\mathbb{Q}}_\ell$ by:

$$\Gamma_\zeta = \sum_{z\in H^1(F, A_L(u_o))} |z|\tilde{\zeta}(z)\Gamma_z.$$

5.7. By [Lus92, 7.6], the function Γ_ζ is proportional to the characteristic function of the F-equivariant perverse sheaf $(K(C,\zeta), \phi)$ for any ϕ. As a consequence we get that:

$$\mathcal{F}^{\mathcal{G}}(\Gamma_\zeta) = \gamma^F \Gamma_\zeta$$

where γ^F denotes the Lusztig's constant attached to (C, ζ) with respect to F. From the classification of the distinguished parabolic subgroups of G, we can verify that the longest element w_o of $W_G(T)$ (with T a maximal torus of L) normalizes L and $\mathrm{Ad}(w_o)$ maps $\mathcal{G}(2)$ onto $\mathcal{G}(-2)$. As a consequence $\mathcal{O}_{-u_o}^G \cap \mathcal{G}(-2) \neq \emptyset$ and any element of $\mathcal{O}_{-u_o}^G \cap \mathcal{G}(-2)$ is distinguished with associated parabolic subgroup $P^- = LU_P^-$. Let $u_o^* \in \mathcal{O}_{-u_o}^{G^F} \cap \mathcal{G}(-2)^F$. From [Lus92, 6.13] we have $\mathcal{F}^{\mathcal{G}}(\Gamma_\zeta)(u_o^*) = \tilde{\zeta}(1)|1||C_G(u_o)^F|q^{-\frac{\dim C_{\mathcal{G}}(u_o)}{2}}$ where by definition $|1| = |\{x^{-1}F(x) | x \in A_L(u_o)\}|$. Hence by 5.5(v), we deduce that:

$$\gamma^F = \frac{\tilde{\zeta}(1)|1||C_L(u_o)^F|}{\Gamma_\zeta(u_o^*)} q^{-\frac{\dim C_{\mathcal{L}}(u_o)}{2}} q^{\frac{\dim C_{\mathcal{U}_P}(u_o)}{2}}.$$

Hence the computation of γ^F reduces to that of $\Gamma_\zeta(u_o^*)$. For any $z \in H^1(F, A_L(u_o))$ we have:

$$\Gamma_z(u_o^*) = |U_P^F|^{-1} \sum_{g\in (P^-)^F} \Psi_z(\mathrm{Ad}(g)u_o^*) = \sum_{g\in L^F} \Psi_z(\mathrm{Ad}(g)u_o^*).$$

These equalities come from 5.5(vi), 5.5(iii) where (u_o, P) is replaced by (u_o^*, P^-), and the fact that the restriction of Ψ_z to $\bigoplus_{i<-2} \mathcal{G}(i)$ is trivial. We thus get that:

$$\Gamma_\zeta(u_o^*) = \sum_{z\in H^1(F, A_L(u_o))} |z|\tilde{\zeta}(z) \sum_{l\in L^F} \Psi_z(\mathrm{Ad}(l)u_o^*).$$

Let $\mathcal{L}_L : L \to L$, $t \mapsto t^{-1}F(t)$ be the Lang map. Then we have a surjective map

$$\overline{\mathcal{L}} : \mathcal{L}_L^{-1}(C_L(u_o))/C_L(u_o) \to H^1(F, C_L(u_o)) \simeq H^1(F, A_L(u_o))$$

which maps $tC_L(u_o)$ onto the F-conjugacy class of $t^{-1}F(t)$. For $z \in H^1(F, A_L(u_o))$, let $l_z \in L$ be such that $l_z^{-1}F(l_z) = \dot{z}$ where $\dot{z} \in C_L(u_o)$ is a representative of z, and $u_z = \mathrm{Ad}(l_z)u_o$. Then we have a well-defined map $\phi_z : L^F \to \overline{\mathcal{L}}^{-1}(z)$ given by $t \mapsto tl_zC_L(u_o)$. This map is clearly surjective and its fibers are all of cardinality $a_z = |\{h \in C_L(u_o) | h^{-1}\dot{z}F(h) = \dot{z}\}|$.

For $g \in \mathcal{L}_L^{-1}(C_L(u_o))$ and $x \in (\mathcal{U}_P^-)^F$, define ${}^g\Psi_o(x) := \Psi(\mu(\mathrm{Ad}(g)u_o, x)) = \Psi(\mu(u_o, \mathrm{Ad}(g^{-1}(x))) = \Psi_o(\mathrm{Ad}(g^{-1})x)$. We thus have:

$$\sum_{t\in L^F} \Psi_z(\mathrm{Ad}(t)u_o^*) = \sum_{t\in L^F} {}^{tl_z}\Psi_o(u_o^*) = a_z \sum_{l\in\overline{\mathcal{L}}^{-1}(z)} {}^l\Psi_o(u_o^*).$$

We finally deduce that:

$$\Gamma_\zeta(u_o^*) = |C_L(u_o)| \sum_{l\in\mathcal{L}_L^{-1}(C_L(u_o))/C_L(u_o)} \tilde{\zeta}(\overline{\mathcal{L}}(l))\,{}^l\Psi_o(u_o^*).$$

Indeed we have $a_z|z| = |C_L(u_o)|$ since by 5.5(vii), we have $A_L(u_o) = C_L(u_o)$. Note that $\mathcal{L}_L^{-1}(C_L(u_o))/C_L(u_o) = (L/C_L(u_o))^F$. We define the quantity:

$$\sigma_\zeta := \tilde{\zeta}(1)^{-1} \sum_{l\in(L/C_L(u_o))^F} \tilde{\zeta}(\overline{\mathcal{L}}(l))\Psi_o^*(\mathrm{Ad}(l)u_o)$$

where Ψ_o^* is the additive character of $\mathcal{G}(2)^F$ defined by $\Psi_o^*(v) = \Psi(\mu(u_o^*, v))$. Note that σ_ζ does not depend on the choice of the extension of ζ on $A_G(u_o) \rtimes \langle F\rangle$. Since $|1||C_L(u_o)^F| = |C_L(u_o)|$, we thus have:

5.8.
$$\gamma^F = \sigma_\zeta^{-1} q^{\frac{d}{2}}$$

where $d = \dim C_{\mathcal{U}_P}(u_o) - \dim C_{\mathcal{L}}(u_o)$.

From 5.8, we see that to prove 5.3, we are reduced to prove the analogous statement for the constants σ_ζ. The constant σ_ζ is computed explicitely in [DLM97] when G is of type A_n from which we can verify the required property. Hence from 4.9, 5.4 and the computation of the Lusztig constants in the case of $SO_n(\mathbb{F})$ [Wal01], we deduce the two following theorems:

Theorem 5.9. *Assume that $p > 3(h_o^G - 1)$ and that q is large enough so that Deligne-Lusztig induction coincides with geometrical induction. Assume moreover that every simple component of G/Z_G^o of type D_n is either the special orthogonal group $SO_{2n}(\mathbb{F})$ or the adjoint group of type D_n, then 4.1 holds.*

Theorem 5.10. *Assume that p and q are as in 5.9. Let L be an F-stable Levi subgroup of G. Let f be the characteristic function of an F-equivariant L-equivariant simple perverse sheaf (K, ϕ) which is supported by the Zariski closure of an L-orbit of $\mathcal{L}$. If K is a direct summand of the parabolic induction of a cuspidal orbital perverse sheaf supported by a regular orbit, then*

$$\mathcal{F}^{\mathcal{G}}(\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(f)) = \epsilon_G\epsilon_L\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(\mathcal{F}^{\mathcal{L}}(f)).$$

Now a result of [Sho95] says that [Lus90, 1.14], which gives a relation between generalized Green functions and two-variable Green functions, holds for any q whenever the cuspidal Levi subgroup is a maximal torus. Hence from 5.1, we have:

Theorem 5.11. *The statement 5.10 holds for p acceptable and any q whenever the complex K is a direct summand of the parabolic induction of a cuspidal orbital perverse sheaf on the Lie algebra of a maximal torus of G.*

If G is either $GL_n(\mathbb{F})$, or a simple group of type E_8, F_4 or G_2, and if p is good for G, then the only proper Levi subgroups which support a cuspidal pair are the maximal tori. Hence from 5.11 we have:

Corollary 5.12. *Assume that G is either $GL_n(\mathbb{F})$ or a simple group of type either E_8, F_4 or G_2, and that p is good for G, then 4.1 holds.*

References

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Soc. Math. France, Paris, 1982, pp. 5–171.

[Bon02] C. Bonnafé, *A note on centralizers of unipotent elements*, preprint of Laboratoire de mathématiques de Besancon **19** (2002).

[Bor] A. Borel, *Linear algebraic groups*, Springer-Verlag (second enlarged edition).

[Del77] P. Deligne, *Cohomologie étale*, Springer-Verlag, Berlin, 1977, Séminaire de Géométrie Algébrique du Bois-Marie SGA $4\frac{1}{2}$, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier, Lecture Notes in Mathematics, Vol. 569.

[DL76] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) **103** (1976), no. 1, 103–161.

[DLM97] F. Digne, G. I. Lehrer, and J. Michel, *On Gel′fand-Graev characters of reductive groups with disconnected centre*, J. Reine Angew. Math. **491** (1997), 131–147.

[DM91] François Digne and Jean Michel, *Representations of finite groups of Lie type*, Cambridge University Press, Cambridge, 1991.

[Gro73] *Théorie des topos et cohomologie étale des schémas. Tome 3*, Springer-Verlag, Berlin, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, Vol. 305.

[Kat80] Nicholas M. Katz, *Sommes exponentielles*, Société Mathématique de France, Paris, 1980, Course taught at the University of Paris, Orsay, Fall 1979, With a preface by Luc Illusie, Notes written by Gérard Laumon, With an English summary.

[Kaw85] N. Kawanaka, *Generalized Gel′fand-Graev representations and Ennola duality*, Algebraic groups and related topics (Kyoto/Nagoya, 1983), North-Holland, Amsterdam, 1985, pp. 175–206.

[Kaw86] ______, *Generalized Gel′fand-Graev representations of exceptional simple algebraic groups over a finite field. I*, Invent. Math. **84** (1986), no. 3, 575–616.

[Kaz77] D. Kazhdan, *Proof of Springer's hypothesis*, Israel J. Math. **28** (1977), no. 4, 272–286.

[Leh96] G. I. Lehrer, *The space of invariant functions on a finite Lie algebra*, Trans. Amer. Math. Soc. **348** (1996), no. 1, 31–50.

[Let03a] E. Letellier, *Deligne-Lusztig induction for invariant functions on finite Lie algebras of Chevalley's type*, submitted (2003).

[Let03b] ———, *Transformation de fourier des fonctions invariantes sur une algèbre de Lie finie*, Thèse de doctorat, Université de Paris 6 et Sydney University, Juin 2003.

[Lus84] G. Lusztig, *Intersection cohomology complexes on a reductive group*, Invent. Math. **75** (1984), no. 2, 205–272.

[Lus85] George Lusztig, *Character sheaves. II, III*, Adv. in Math. **57** (1985), no. 3, 226–265, 266–315.

[Lus86] ———, *Character sheaves. V*, Adv. in Math. **61** (1986), no. 2, 103–155.

[Lus87] ———, *Fourier transforms on a semisimple Lie algebra over* $\mathbb{F}_q$, Algebraic groups Utrecht 1986, Springer, Berlin, 1987, pp. 177–188.

[Lus90] ———, *Green functions and character sheaves*, Ann. of Math. (2) **131** (1990), no. 2, 355–408.

[Lus92] ———, *A unipotent support for irreducible representations*, Adv. Math. **94** (1992), no. 2, 139–179.

[Sho95] Toshiaki Shoji, *Character sheaves and almost characters of reductive groups. I, II*, Adv. Math. **111** (1995), no. 2, 244–313, 314–354.

[Slo80] Peter Slodowy, *Simple singularities and simple algebraic groups*, Springer, Berlin, 1980.

[Spr69] T. A. Springer, *The unipotent variety of a semi-simple group*, Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), Oxford Univ. Press, London, 1969, pp. 373–391.

[Spr71] ———, *Generalization of Green's polynomials*, Representation theory of finite groups and related topics (Proc. Sympos. Pure Math., Vol. XXI, Univ. Wisconsin, Madison, Wis., 1970), Amer. Math. Soc., Providence, R.I., 1971, pp. 149–153.

[Spr76] ———, *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, Invent. Math. **36** (1976), 173–207.

[SS70] T. A. Springer and R. Steinberg, *Conjugacy classes*, Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69), Springer, Berlin, 1970, pp. 167–266.

[Wal01] Jean-Loup Waldspurger, *Intégrales orbitales nilpotentes et endoscopie pour les groupes classiques non ramifiés*, Astérisque (2001), no. 269, vi+449.