数理解析研究所講究録 1384 巻 2004 年 109-114

A new approach to higher dimensional continued fractions

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§0. Definition of CFS. Let $s \geqq 1$ be an integer, and

$$\alpha_1=\theta_{1-s},\ \alpha_2=\theta_{1-s}\theta_{2-s},\ \ldots,\ \alpha_s=\theta_{1-s}\theta_{2-s}\cdots\theta_0 \in \mathbf{R}\setminus\{0\}$$

be given s real numbers. Choosing a lattice point

$$\underline{a}_1={}^t(a_1^{(1)},a_1^{(2)},\ldots,a_1^{(s)})\in \mathbf{Z}^s,$$

we can define a number $\theta_1 \in \mathbf{R}$ by

$$1=a_1^{(1)}\theta_{1-s}+a_1^{(2)}\theta_{1-s}\theta_{2-s}+\cdots+a_1^{(s)}\theta_{1-s}\theta_{2-s}\cdots\theta_0+\theta_{1-s}\theta_{2-s}\cdots\theta_0\theta_1.$$

If $\theta_1\neq 0$, then, choosing $\underline{a}_2={}^t(a_2^{(1)},a_2^{(2)},\ldots,a_2^{(s)})\in \mathbf{Z}^s$, we can define θ_2 by

$$1=a_2^{(1)}\theta_{2-s}+a_2^{(2)}\theta_{2-s}\theta_{3-s}+\cdots+a_2^{(s)}\theta_{2-s}\theta_{3-s}\cdots\theta_1+\theta_{2-s}\theta_{3-s}\cdots\theta_1\theta_2.$$

Repeating the procedure, we have a system of equalities

$$(\#)\quad \begin{aligned} &1=a_1^{(1)}\theta_{1-s}+a_1^{(2)}\theta_{1-s}\theta_{2-s}+\cdots+a_1^{(s)}\theta_{1-s}\theta_{2-s}\cdots\theta_0+\theta_{1-s}\theta_{2-s}\cdots\theta_0\theta_1.\\ &1=a_2^{(1)}\theta_{2-s}+a_2^{(2)}\theta_{2-s}\theta_{3-s}+\cdots+a_2^{(s)}\theta_{2-s}\theta_{3-s}\cdots\theta_0+\theta_{2-s}\theta_{3-s}\cdots\theta_1\theta_2,\\ &1=a_3^{(1)}\theta_{3-s}+a_3^{(2)}\theta_{3-s}\theta_{4-s}+\cdots+a_3^{(s)}\theta_{3-s}\theta_{4-s}\cdots\theta_0+\theta_{3-s}\theta_{4-s}\cdots\theta_2\theta_3,\\ &\cdots\cdots\cdots\cdots\cdots\cdots, \end{aligned}$$

which will be referred to as a continued fractional system (abbr. CFS) for

$$\underline{\alpha} := (\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_s) = (\theta_{1-s},\ \theta_{1-s}\theta_{2-s},\ \ldots,\ \theta_{1-s}\theta_{2-s}\cdots\theta_0).$$

If (#) given above is one of the CFSs for the $\underline{\alpha}$, we write

$$\langle \underline{a}_1,\ \underline{a}_2,\ \underline{a}_3,\ \ldots\rangle \in \mathrm{CFS}(\underline{\alpha}).$$

We say that the (#) is non-singular if $\theta_n\neq 0$ for all $n\geqq 1-s$, and converging if it is non-singular and

$$\lim_{n\to\infty} \theta_1\cdots\theta_n = 0.$$

We can define a class of deterministic or/and non-deterministic algorithms to get converging (or diverging) CFSs for given $\underline{\alpha}$.

§1. A formula of CFS and Polybonacci words. If $s=1$, then (#) can be written as

$$1=a_1\theta_0+\theta_0\theta_1,\ 1=a_2\theta_1+\theta_1\theta_1,\ 1=a_3\theta_2+\theta_2\theta_3,\ \ldots,$$

which implies $\theta_0=1/(a_1+\theta_1)=1/(a_1+1/(a_2+\theta_2))$, so that we have formal expression

$$\theta_0=1/(a_1+1/(a_2+1/(a_3+\cdots)))=[0;a_1,a_2,a_3,\ldots].$$

Taking $s=2$, we can write

$$1=a_1\theta_{-1}+b_1\theta_{-1}\theta_0+\theta_{-1}\theta_0\theta_1,\ 1=a_2\theta_0+b_2\theta_0\theta_1+\theta_0\theta_1\theta_2,\ 1=a_3\theta_1+b_3\theta_1\theta_2+\theta_1\theta_2\theta_3,\ \ldots,$$

so that

$$\alpha_1=\theta_{-1}=\frac{1}{a_1+(b_1+\theta_1)\theta_0}=\frac{1}{a_1+\frac{b_1+\theta_1}{a_2+(b_2+\theta_2)\theta_1}}=\frac{1}{a_1+\frac{b_1+\frac{1}{a_3+(b_3+\theta_3)\theta_2}}{a_2+\frac{b_2+\theta_2}{a_3+(b_3+\theta_3)\theta_2}}}=\cdots$$

$$=\frac{1}{a_1+\frac{b_1+\frac{1}{a_3+\frac{b_3+\frac{1}{a_5+\frac{b_5+\theta_5}{a_6+(b_6+\theta_6)\theta_5}}}{a_4+\frac{b_4+\frac{1}{a_6+(b_6+\theta_6)\theta_5}}{a_5+\frac{b_5+\theta_5}{a_6+(b_6+\theta_6)\theta_5}}}}}{a_2+\frac{b_2+\frac{1}{a_4+\frac{b_4+\frac{1}{a_6+(b_6+\theta_6)\theta_5}}{a_5+\frac{b_5+\theta_5}{a_6+(b_6+\theta_6)\theta_5}}}}{a_3+\frac{b_3+\frac{1}{a_5+\frac{b_5+\theta_5}{a_6+(b_6+\theta_6)\theta_5}}}{a_4+\frac{b_4+\frac{1}{a_6+(b_6+\theta_6)\theta_5}}{a_5+\frac{b_5+\theta_5}{a_6+(b_6+\theta_6)\theta_5}}}}}}=\cdots,$$

which implies formal expressions

$$\alpha_1=\frac{1}{a_1+\frac{b_1+\frac{1}{a_3+\frac{b_3+\frac{1}{a_5+\frac{b_5}{a_6}}}{a_4+\frac{b_4+\frac{1}{a_6}}{a_5+\frac{b_5}{a_6}}}}}{a_2+\frac{b_2+\frac{1}{a_4+\frac{b_4+\frac{1}{a_6}}{a_5+\frac{b_5}{a_6}}}}{a_3+\frac{b_3+\frac{1}{a_5+\frac{b_5}{a_6}}}{a_4+\frac{b_4+\frac{1}{a_6}}{a_5+\frac{b_5}{a_6}}}}}}\;\vdots$$

$$\alpha_2=\theta_{-1}\theta_0=\cfrac{1}{b_1+\cfrac{1}{a_1+\cfrac{a_3+\cfrac{b_3+\cfrac{1}{a_5+\cfrac{b_5}{a_6}}}{a_4+\cfrac{b_4+\cfrac{1}{a_6}}{a_5+\cfrac{b_5}{a_6}}}}{a_2+\cfrac{b_2+\cfrac{1}{a_4+\cfrac{b_4+\cfrac{1}{a_6}}{a_5+\cfrac{b_5}{a_6}}}}{a_3+\cfrac{b_3+\cfrac{1}{a_5+\cfrac{b_5}{a_6}}}{a_4+\cfrac{b_4+\cfrac{1}{a_6}}{a_5+\cfrac{b_5}{a_6}}}}}}} \cdots \times \cfrac{1}{b_2+\cfrac{1}{a_2+\cfrac{a_4+\cfrac{b_4+\cfrac{1}{a_6}}{a_5+\cfrac{b_5}{a_6}}}{a_3+\cfrac{b_3+\cfrac{1}{a_5+\cfrac{b_5}{a_6}}}{a_4+\cfrac{b_4+\cfrac{1}{a_6}}{a_5+\cfrac{b_5}{a_6}}}}}} \cdots$$

In the denominator of the expression of α_1, if we read symbols a_n, b_n $(n\geq 1)$ along vertical lines from the upside to the downside of the fraction from the left, we get a word

$$w=a_1|b_1a_2|a_3b_2a_3|b_3a_4a_4b_3a_4|a_5b_4a_5b_4a_5a_5b_4a_5|b_5a_6a_6b_5a_6a_6b_5a_6b_5a_6a_6b_5a_6|\ldots.$$

We can show that the resulting word

a ba aba baaba ababaaba baabaababaaba ...,

obtained from w by forgetting all the indices, is the Fibonacci word; from this fact, we can consruct the formulae for α_1, α_2. Moreover, if we denote by $\alpha_1^{(n)}$, $\alpha_2^{(n)}$ the rational functions of a_n, b_n $(n\geq 1)$ obtained by the truncations of the formulae (of depth n), respectively, for α_1, α_2, then we can show that ${}^t(\alpha_1^{(n)}, \alpha_2^{(n)})$ coincides with the nth convergent of the continued fraction (abbr. CF) of dimension s=2:

$$[0;\ \underline{a}_1^*,\ \underline{a}_2^*,\ \ldots,\ \underline{a}_n^*,\ \ldots],$$

where $\underline{a}_1^*:={}^t(0,a_1)$, $\underline{a}_n^*:={}^t(b_{n-1},a_n)$ $(n\geq 2)$. Thus, the formal expressions for α_1, α_2 converge if and only if the CF converges. In addition, we can show that the convergence of

$$\langle \underline{a}_1,\ \underline{a}_2,\ \underline{a}_3,\ \ldots\rangle \in \mathrm{CFS}(\underline{\alpha}),\ \underline{\alpha}={}^t(\alpha_1,\alpha_2),\ \underline{a}_n={}^t(a_n,b_n)\in \mathbf{Z}^2,$$

implies that the continued fraction $[0;\ \underline{a}_1^*,\ \underline{a}_2^*,\ \ldots,\ \underline{a}_n^*,\ \ldots]$ converges to $\underline{\alpha}$ under a geometric condition on the CFS. We can extend such results to any dimension $s\geq 1$: The continued fraction corresponding to

$$\langle \underline{a}_1,\ \underline{a}_2,\ \underline{a}_3,\ \ldots\rangle \in \mathrm{CFS}(\underline{\alpha}),\ \underline{\alpha}\in \mathbf{R}^s,\ \underline{a}_n={}^t(a_n^{(1)},a_n^{(2)},\ldots,a_n^{(s)})\in \mathbf{Z}^s$$

is

$$[\underline{0};\ \underline{a}_1^*,\ \underline{a}_2^*,\ \ldots,\ \underline{a}_n^*,\ \ldots]\in \mathbf{R}^s,$$

$$\underline{a}_n^*:={}^t(0,\ldots,0,a_1^{(n)},\ldots,a_{n-1}^{(2)},a_n^{(1)}),\ 1\leq n<s,$$

$$\underline{a}_n^*:={}^t(a_{n-s+1}^{(s)},\ldots,a_{n-1}^{(2)},a_n^{(1)}),\ n\geq s;$$

and the convergents of $[\underline{0};\ \underline{a}_1^*,\ \underline{a}_2^*,\ \ldots,\ \underline{a}_n^*,\ \ldots]$ can be constructed by using the Polybonacci word of degree s>1; and the CF converges to $\underline{\alpha}$ under a geometric condition on the CFS. The Polyboacci word of degree 1 is a periodic word aaa..., and the usual CF with s=1 can be considered as a degenerated expression; the convergence of a CFS and that of its corresponding CF happen together if s=1.

§2. A formula of CFS and linear independence. Let

$$\langle \underline{a}_1, \underline{a}_2, \underline{a}_3, \ldots\rangle \in CFS(\underline{\alpha}),\ \underline{a}_n \in \mathbf{Z}^s,\ \underline{\alpha}=(\alpha_1, \alpha_2, \ldots, \alpha_s) \in \mathbf{R}^s,$$

and let

$$A_n := \begin{bmatrix} {}^t\underline{0} & 1 \\ E_s & -\underline{a}_n \end{bmatrix},\ U_n=(u_{n-s+j}{}^{(i)})_{0\leq i\leq s,\, 0\leq j\leq s} := A_1A_2\cdots A_n \in M_{s+1}(\mathbf{Z})\ (U_0:=E)$$

Then

$$u_n{}^{(0)}+u_n{}^{(1)}\alpha_1+u_n{}^{(2)}\alpha_2+\cdots+u_n{}^{(s)}\alpha_s=\theta_{1-s}\theta_{2-s}\cdots\theta_0\theta_1\cdots\theta_n\ (n\geq -s),$$

where the right-hand side turns out to be the empty product(=1) for n=-s. Hence, any CFS for $\underline{\alpha}=(\alpha_1, \alpha_2, \ldots, \alpha_s)$ is non-singular if s+1 numbers 1, α_1, ..., α_s are linearly independent over $\mathbf{Q}$; the converse does not hold.

§3. The periodicity of CFS. For simplicity, we take s=2, and consider an algebraic number α satisfying $1=\alpha+\alpha^2+\alpha^3$. Then we have a purely periodic CFS for ${}^t(\alpha, \alpha^2)$ consisting of the copies of one equality $1=\alpha+\alpha^2+\alpha^3$. If we consider a cubic irational α satisfying $1=a\alpha+b\alpha^2-\alpha^3$ $(a, b\in\mathbf{Z})$. Then we get a purely periodic CFS for ${}^t(\alpha, \alpha^2)$:

$$* \ 1=a\alpha+b\alpha\alpha+\alpha\alpha(-\alpha),$$
$$1=a\alpha+(-b)\alpha(-\alpha)+\alpha(-\alpha)\alpha,$$
$$* \ 1=(-a)(-\alpha)+(-b)(-\alpha)\alpha+(-\alpha)\alpha\alpha,$$

where the asterisques indicate the period. Note that not only $\underline{a}_n\in\mathbf{Z}^2$, but also the totality of the equations are periodic.

We remark that, for instance, a CFS for ${}^t(\alpha, \alpha^2)$ can be periodic even for a quadratic irrational α; such a degenerated phenomenon can be found even for continued fractions.

(Ex. 1) $\underline{\alpha}={}^t(\sqrt{2}-1, (\sqrt{2}-1)^2)$.

$$CFS(\underline{\alpha}) \ni \langle \underline{a}, \underline{a}, \underline{a}, \ldots\rangle,\ \underline{a}={}^t(1,3)$$
$$\underline{\alpha}=[\underline{0}; \underline{a}^*, \underline{b}^*, \underline{b}^*, \underline{b}^*, \ldots],\ \underline{a}^*={}^t(0,1),\ \underline{b}^*={}^t(3,1).$$

Notice that both the CFS $\langle \underline{a}, \underline{a}, \underline{a}, \ldots\rangle$ and the CF $[\underline{0}; \underline{a}^*, \underline{b}^*, \underline{b}^*, \underline{b}^*, \ldots]$ converge; while the covergence of the CF is not exponential.

We can show that the purely periodicity of a CFS (#) for ${}^t(\alpha_1, \alpha_2, \ldots, \alpha_s)= {}^t(\alpha, \alpha^2, \ldots, \alpha^s)$ implies that α is of degree at most s+1. We remark that, for any purely periodic CFF for ${}^t(\alpha, \alpha^2)$, $\ell=2$ never occurs, where ℓ is the length of the shortest period.

(Ex.2). Suppose a CFF for ${}^t(\alpha,\alpha^2)$ is purely periodic and $\ell=6$. Then

$$\begin{vmatrix} -1+a_1^{(1)}\alpha+a_1^{(2)}\alpha^2 & 1 & 0 & 0 & 0 \\ -\alpha+a_2^{(1)}\alpha^2 & a_2^{(2)} & 1 & 0 & 0 \\ -\alpha^2 & a_3^{(1)} & a_3^{(2)} & 1 & 0 \\ 0 & -1 & a_4^{(1)} & a_4^{(2)} & 1 \\ 0 & 0 & -1 & a_5^{(1)} & a_5^{(2)}+\alpha \end{vmatrix} = 0,$$

which says that α is an algebraic number of degree at most three.

§4. Approximation of an algebraic number of degree s+1 by those of degree less than s+1. It is clear that any periodic CFS with $|\theta_n|<1$ $(n\geq n_0)$ is converging. Hence, using the formula for CFS (#) in Section 2, we can construct a series of polynomials $f_n\in\mathbb{Z}[x]$ (deg $f_n=s$) such that one of the roots of f_n approximates a given algebric unit α of degree s+1 with $|\alpha|<1$.

(Ex.3) Let $\alpha\fallingdotseq 0.5436890128$ be the reciprocal of the Pisot number whose characteristic polynomial is x^3-x^2-x-1. We put

$$A_n:=\begin{bmatrix} {}^t\underline{0} & 1 \\ E_2 & -\underline{a} \end{bmatrix},\quad \underline{a}={}^t(1,1),$$

$$U_n = (u_{n+j-2}^{(i)})_{0\leq i\leq 2,\,0\leq j\leq 2} = A^n,$$

$$f_n(x):=u_n^{(0)}+u_n^{(1)}x+u_n^{(2)}x^2 \in\mathbb{Z}[x].$$

Then $f_n(\alpha)=\alpha^{n+2}\to 0$ $(n\to\infty)$, so that the cubic irrational α can be approximated by α_n as n tends to infinity, where α_n is a root of the quadratic polynomial f_n.

§5. Periodic continued fractions related to algebraic numbers of degree s+1. We can construct a periodic CF for $\underline{\alpha}={}^t(\alpha,\alpha^2,\ldots,\alpha^s)$ for any algebraic unit α of degree s+1 having the minimum magnitude among the conjugates, i.e., $|\alpha|<|\beta|$ for all algebraic conjugates β different from α. The method is simple:

(1) from the minimal polynomial of α, we can find a periodic CFF $\langle\underline{a}_1,\underline{a}_2,\underline{a}_3,\ldots\rangle$ for the $\underline{\alpha}$ (cf. Section 2).

(2) the CF corresponding to the CFF (cf. Section 1) is what we are looking for.

Note that the CF obtined by our method can not be admissible if $N(\alpha)=(-1)^{s+1}$.

In the following examples, ω denotes the degree of the exponent of the convergence of the CF given there. In other words, the mesure of simultaneous approximation is equal to, or bigger than ω. We can show $\omega=1-\log|\beta|/\log|\alpha|$, where β is a conjugate of α such that $|\beta|=\min\{|\gamma|;\ \gamma\in\{\text{the conjugates of } \alpha\}\setminus\{\alpha\}\}$, which implies

(1) $\omega>0$ for our CF, so that the CF always converge,

(2) if α is the reciprocal of a Pisot number, then $\omega>1$, i.e., the convergence of the CF is exponential.

(Ex.4) trivial case: s=2, $1=\alpha+\alpha^2+\alpha^3$ (α as in (Ex.3).), $N(\alpha)=1$.

CFS($\underline{\alpha}$)∋ $\langle\overset{*}{\underline{a}}\rangle=\langle\underline{a}, \underline{a}, \underline{a}, \ldots\rangle$, $\underline{a}={}^t(1,1)$.

$$\underline{\alpha} = \begin{bmatrix} 0;\ 0, & \overset{*}{1} \\ 0;\ 1, & 1 \end{bmatrix}$$: <u>admissible</u>, $\omega=3/2$ (best possible).

We do not need to apply our CFS for finding a periodic CF in (Ex.4); this CF can be obtained by applying the usual Jacobi-Perron algoritm (abbr. JPA). In the following examples, JPA is not useful, since the CFs are not admissible.

(Ex.5) α^{-1}:a totally real Pisot: s=2, $1=2\alpha+\alpha^2-\alpha^3$, $N(\alpha)=-1$. $\alpha\fallingdotseq 1/2.2469796904\fallingdotseq 0.4450418679$, $\alpha'\fallingdotseq -1/0.8019377356\fallingdotseq -1.246979604$, $\alpha''\fallingdotseq 1/0.554958132\fallingdotseq 1.801937736$.

CFS($\underline{\alpha}$)∋ $\langle\overset{*}{\underline{a}}, \underline{b}, \overset{*}{\underline{c}}\rangle$, $\underline{a}={}^t(2,1)$, $\underline{b}={}^t(2,-1)$, $\underline{c}={}^t(-2,-1)$.

$$\underline{\alpha} = \begin{bmatrix} 0;\ 0, & \overset{*}{1}, & -1, & \overset{*}{-1} \\ 0;\ 2, & 2, & -2, & 2 \end{bmatrix}$$: non-admissible, $\omega\fallingdotseq 1.272638189$.

(Ex.6) α^{-1}:a totally real non-Pisot: s=2, $1=\alpha+3\alpha^2-\alpha^3$, $N(\alpha)=-1$.

$\alpha\fallingdotseq 1/2.170086487$, $\alpha'\fallingdotseq -1/1.481194304$, $\alpha''\fallingdotseq 1/0.3111078175$.

CFS($\underline{\alpha}$)∋ $\langle\overset{*}{\underline{a}}, \underline{b}, \overset{*}{\underline{c}}\rangle$, $\underline{a}={}^t(1,3)$, $\underline{b}={}^t(1,-3)$, $\underline{c}={}^t(-1,-3)$.

$$\underline{\alpha} = \begin{bmatrix} 0;\ 0, & \overset{*}{3}, & -3, & \overset{*}{-3} \\ 0;\ 1, & 1, & -1, & 1 \end{bmatrix}$$: non-admissible. Since $\omega=\fallingdotseq 0.4929459914<1$

the convergence of the CF is not exponential.

(Ex.7) α^{-1}:a complex Pisot: s=3, $\alpha=\delta-1$ ($\delta=2^{1/4}$) The minimal polynomial of α^{-1} is $x^4-4x^3-6x^2-4x-1$, $N(\alpha)=-1$.

$\alpha\fallingdotseq 0.189207115$, $|\alpha'|\fallingdotseq|\delta i-1|$, $|\alpha''|\fallingdotseq|-\delta i-1|$, $|\alpha'''|=-\delta-1$, $i=\sqrt{-1}$.

CFS($\underline{\alpha}$)∋ $\langle\overset{*}{\underline{a}}\rangle$, $\underline{a}={}^t(4,6,4)$.

$$\underline{\alpha} = \begin{bmatrix} 0;\ 0, & 0, & \overset{*}{4} \\ 0;\ 0, & 6, & 6 \\ 0;\ 4, & 4, & 4 \end{bmatrix}$$: non-admissible! $\omega\fallingdotseq 1.264690581$.

In the last example, $\omega<1+1/s=1.333\ldots$, such a phenomenon that ω does not attain $1+1/s$ even for complex Pisot case is common for all $s\geq 3$, which comes from a theorem of Minkowski; in this sense, s=1,2 are exceptional cases.