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A Vista of Mean Zeta-Values. II

BY YOICHI MOTOHASHI

This is a continuation of our former article [5]. We shall use the same set of notations as there. In the last section of [5] we proposed a few central problems on mean values of automorphic L-functions. One of them is to establish an explicit spectral decomposition for the mean square of the L-function attached to a given irreducible Γ-automorphic representation of G, with $\Gamma=\mathrm{PSL}(2,\mathbb{Z})$ and $G=\mathrm{PSL}(2,\mathbb{R})$. The aim of the present article is to indicate briefly a method with which one may settle this problem in the most difficult situation; that is, the case of unitary principal series representations or Maass wave forms. As a consequence, it is now strongly suggested that the inner-product procedure, which was initiated by A. Good and greatly enhanced by M. Jutila, should be the right way to pursue further, if we wish to establish anything like a unified theory of mean values of automorphic L-functions. The Kirillov map $\mathcal{K}$ defined by [5, (5.3)] has turned out to be a key implement, indeed as we envisaged.

Thus, let V be an irreducible subspace of $L^2(\Gamma\backslash G)$ whose spectral data is ν. As it is to be in the unitary principal series, ν is pure imaginary. Let $\phi(\cdot,\alpha)\in U_\nu$ be such that

$$\mathcal{K}\phi(u)=\begin{cases} u^\alpha\exp(-2\pi u) & \text{for } u\ge 0,\\ 0 & \text{for } u<0,\end{cases} \tag{1}$$

with $\alpha>0$. This is possible, for $\mathcal{K}$ is surjective and the member on the right side is obviously in $\mathcal{H}=L^2(\mathbb{R}^\times,\pi^{-1}d^\times)$. Let

$$\phi(\mathrm{g},\alpha)=\sum_p c_p\phi_p(\mathrm{g}),\quad \phi_p(\mathrm{g})=\phi_p(\mathrm{g};\nu), \tag{2}$$

where $c_p=c_p(\nu,\alpha)$. We may choose an orthonormal base $\{\varphi_p\}$ of V such that

$$\varphi_p(\mathrm{g})=\sum_{n\ne0}\frac{\varrho_V(n)}{\sqrt{|n|}}\mathcal{A}^{\mathrm{sgn}(n)}\phi_p(\mathrm{a}[|n|]\mathrm{g}). \tag{3}$$

We put

$$\varphi(\mathrm{g},\alpha)=\sum_p c_p\varphi_p(\mathrm{g}), \tag{4}$$

We shall later prove that

$$\varphi(\mathrm{g},\alpha)=\sum_{n\ne0}\frac{\varrho_V(n)}{\sqrt{|n|}}\mathcal{A}^{\mathrm{sgn}(n)}\phi(\mathrm{a}[|n|]\mathrm{g},\alpha), \tag{5}$$

provided α is sufficiently large. Given this, we note that

$$\mathcal{A}^{\mathrm{sgn}(n)}\phi(\mathrm{a}[|n|]\mathrm{n}[x]\mathrm{a}[y],\alpha) = \exp(2\pi inx)\mathcal{K}\phi(ny,\alpha). \tag{6}$$

Thus

$$\varphi(\mathrm{n}[x]\mathrm{a}[y],\alpha) = y^\alpha \sum_{n=1}^{\infty} \varrho_V(n) n^{\alpha-\frac{1}{2}} \exp(2\pi in(x+iy)). \tag{7}$$

This brings us to a situation very much similar to the one with holomorphic cusp forms (see [3]).

Now, we shall prove (5). To this end, we compute explicitly the coefficients c_p: The unitaricity of $\mathcal{K}$ gives

$$\begin{aligned} c_p &= \langle \phi, \phi_p \rangle_{U_\nu} = \langle \mathcal{K}\phi, \mathcal{K}\phi_p \rangle_{\mathcal{H}} \\ &= \frac{1}{\pi}\int_0^\infty u^{\alpha-1}\exp(-2\pi u)\overline{\mathcal{A}^{+}\phi_p(\mathrm{a}[u])}du. \end{aligned} \tag{8}$$

On noting that the Jacquet transform is essentially equal to the Whittaker function (or the confluent hypergeometric function) save for a simple factor, the formula 7.621(3) of [2] becomes relevant here. It implies that

$$c_p = (-1)^p 2^{-2\alpha}\pi^{-\nu-\alpha-\frac{1}{2}} \frac{\Gamma\left(\alpha+\nu+\frac{1}{2}\right)\Gamma\left(\alpha-\nu+\frac{1}{2}\right)}{\Gamma(\frac{1}{2}-\nu+p)\Gamma(\alpha+1-p)}. \tag{9}$$

Or one may argue as follows: The bounds (4.3) and (4.5) of [1] imply that the integral is a regular function of ν in a neighbourhood of the imaginary axis. Let us suppose temporarily that $\mathrm{Re}\,\nu$ is negative but small. Then we see, by the first equation of (2.16) in [1], that

$$\begin{aligned} c_p &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{(\xi^2+1)^{\frac{1}{2}-\nu}} \left(\frac{\xi+i}{\xi-i}\right)^{-p} \int_0^\infty u^{\alpha+\nu-\frac{1}{2}} \exp(-2\pi u(1+i\xi))du d\xi \\ &= \frac{1}{\pi}(2\pi)^{-\nu-\alpha-\frac{1}{2}}\Gamma\left(\alpha+\nu+\frac{1}{2}\right)\int_{-\infty}^{\infty} \frac{(1+i\xi)^{-\nu-\alpha-\frac{1}{2}}}{(\xi^2+1)^{\frac{1}{2}-\nu}} \left(\frac{\xi+i}{\xi-i}\right)^{-p} d\xi, \end{aligned} \tag{10}$$

where $\arg(1+i\xi)$ varies from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$ as ξ runs over $\mathbb{R}$ from $-\infty$ to ∞. This integral can be computed by the argument given on p. 47 of [4], whence we obtain (9).

In particular, we find that

$$c_p \ll (|p|+1)^{-\alpha-\frac{1}{2}}, \tag{11}$$

as $|p|$ tends to infinity, and $\nu \in i\mathbb{R}$ is bounded. Thus, indeed $\phi \in U_\nu$ if $\alpha > 0$, and ϕ becomes smoother if we take α larger. Invoking the uniform bound

$$\mathcal{A}^\delta\phi_p(\mathrm{a}[y]) \ll (|p|+|\nu|+1)y^{-\frac{1}{2}}\exp\left(-\frac{y}{|\nu|+|p|+1}\right), \tag{12}$$

we see that (11) confirms (5). This bound is proved in [1, Section 4].

Now, we shall move to an inner-product argument: Let $\tau(\theta)$ be a smooth function supported on a small neighbourhood of $\theta = 0$, and

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \tau(\theta)d\theta = 1. \tag{13}$$

Let m be a positive integer, and $\operatorname{Re} s > 1$. Put

$$f(\mathrm{g}) = y^s \exp(2\pi m i(x+iy))\tau(\theta). \tag{14}$$

Further, put

$$\mathcal{P}f(\mathrm{g}) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f(\gamma \mathrm{g}), \quad \Gamma_\infty = \Gamma \cap N, \tag{15}$$

which is in $L^2(\Gamma\backslash G)$.

With this, consider the inner-product

$$\langle \mathcal{P}f, |\varphi|^2 \rangle_{\Gamma\backslash G}. \tag{16}$$

Let us assume that α is sufficiently large. The unfolding argument gives

$$\langle \mathcal{P}f, |\varphi|^2 \rangle_{\Gamma\backslash G} = \frac{1}{\pi}\int_0^\infty \int_0^1 y^{s-2} \exp(2\pi m i(x+iy)) \times \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \tau(\theta)|\varphi(\mathrm{n}[x]\mathrm{a}[y]\mathrm{k}[\theta],\alpha)|^2 d\theta dx dy. \tag{17}$$

Thus

$$\lim_\tau \langle \mathcal{P}f, |\varphi|^2 \rangle_{\Gamma\backslash G} = \frac{1}{\pi}\int_0^\infty \int_0^1 y^{s-2} \exp(2\pi m i(x+iy))|\varphi(\mathrm{n}[x]\mathrm{a}[y],\alpha)|^2 dx dy, \tag{18}$$

where the support of τ tends to 0. The expression (7) implies readily that

$$\sum_{n=1}^\infty \frac{\varrho_V(n)\overline{\varrho_V(n+m)}}{(n+m)^s(1+m/n)^{\alpha-\frac{1}{2}}} = \frac{\pi(4\pi)^{s+2\alpha-1}}{\Gamma(s+2\alpha-1)} \lim_\tau \langle \mathcal{P}f, |\varphi|^2 \rangle_{\Gamma\backslash G}. \tag{19}$$

With this, we may use the argument of [3, Section 1] and attain the inner sum of the expression

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\varrho_V(n)\overline{\varrho_V(n+m)}}{n^u(n+m)^v} \hat{g}\left(\log\left(1+\frac{m}{n}\right)\right), \tag{20}$$

where $\hat{g}$ is the Fourier transform of g in the context same as in [1]. In fact, it suffices for us to multiply both sides by the factor

$$m^{-u-v+\xi}\Gamma(u+v-\xi)\cdot\frac{1}{2\pi i}\int_{\mathrm{Im}\,t=-c}\frac{\Gamma(\frac{3}{2}-u-\alpha+it)}{\Gamma(v+\frac{3}{2}-\alpha-\xi+it)}g(t)dt, \tag{21}$$

with $c>0$ sufficiently large, and integrate with respect to ξ along an appropriate vertical line. Provided α is sufficiently large and $\mathrm{Re}\,(u+v)>\mathrm{Re}\,\xi>1$, the necessary absolute convergence holds throughout our procedure. Inserting the thus obtained expression into (20), we find that (20) admits an expression in terms of $\langle\mathcal{P}f,|\varphi|^2\rangle_{\Gamma\backslash G}$, provided $\mathrm{Re}\,(u+v)>2$.

The expression (20) is of course related to the non-diagonal part of

$$\int_{-\infty}^{\infty}\left|L_V\left(\frac{1}{2}+it\right)\right|^2 g(t)dt, \tag{22}$$

where

$$L_V(s)=\sum_{n=1}^{\infty}\frac{\varrho_V(n)}{n^s},\quad \mathrm{Re}\,s>1. \tag{23}$$

The inner-product (16) is spectrally decomposed according to the spectral structure of $L^2(\Gamma\backslash G)$. The limit in τ of the decomposition should converge termwise, so do we believe. Then, the left side of (19) admits a spectral decomposition, from which a complete spectral decomposition of (22) ought to transpire.

The above argument appears to extend to bigger groups, at least formally. In our mind is the situation with $G=\mathrm{SL}(3,\mathbb{R})$ and $\Gamma=\mathrm{SL}(3,\mathbb{Z})$. There the minimal parabolic Eisenstein series generates a product of 6 values of the Riemann zeta-function, apart from an unimportant factor. Thus it could be surmised that a mean value of the product of 12 values of the zeta-function is related to the Γ-automorphic structure of G, with this particular combination of G and Γ.

References

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