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Estimating Fourier coefficients of Siegel modular forms

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1. Introduction

A famous theorem of Deligne –previously the Ramanujan-Petersson conjecture– states that the m-th Fourier coefficients (or equivalently the m-th Hecke eigenvalues) of a normalized cuspidal Hecke eigenform of integral weight $k \geq 2$ on $SL_2(\mathbf{Z})$ (and also on certain congruence subgroups) are bounded by a constant times $m^{\frac{k-1}{2}+\epsilon}$, for any $\epsilon > 0$.

In the seventies, Resnikoff and Saldaña made a remarkable conjecture on the growth of the Fourier coefficients of a Siegel cusp form of arbitrary genus n which can be viewed as a generalization of the Ramanujan-Petersson conjecture in genus 1. Little if any motivation was given. In fact, the authors computed several hundreds of coefficients of the unique "normalized" cusp form of weight 10 in genus 2 and argued that these data supported their conjecture. However, as was proved in the eighties, this cusp form is the Saito-Kurokawa lift of a form in genus 1, and it turned out that the Saito-Kurokawa lifts on the contrary do not satisfy the conjecture.

In fact, the conjecture of Resnikoff and Saldaña for $n > 1$ is not known in a single case. There are also known counterexamples for $n > 2$ and for small weights w.r.t. the genus constructed by Freitag using theta series with spherical harmonics.

In this short note we would like to survey how one can contribute a little bit to the clarification of the conjecture in two different, maybe opposite ways. First, we would like to motivate why one could expect that the conjecture should hold at least "generically". Secondly, we would like to indicate some more concrete counterexamples to the conjecture for arbitrarily large n and arbitrarily large weights w.r.t. n.

For more detaills the reader is referred to [1] and the literature given there.

2. The conjecture of Resnikoff and Saldaña

For $n \in \mathbf{N}$ we let $\Gamma_n = Sp_n(\mathbf{Z}) \subset GL_{2n}(\mathbf{Z})$ be the Siegel modular group of genus n and denote by $\mathcal{H}_n = \{Z \in M_n(\mathbf{C}) \,|\, Z' = Z, \Im(Z) > 0\}$ the Siegel upper half-space of genus n. Recall that Γ_n operates on $\mathcal{H}_n$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \circ Z = (AZ+B)(CZ+D)^{-1}.$$

For $k \in \mathbf{N}$ we let $S_k(\Gamma_n)$ be the space of Siegel cusp forms of weight k on Γ_n, i.e. the complex vector space of holomorphic functions $F : \mathcal{H}_n \to \mathbf{C}$ such that

$$F((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^k F(Z)$$

for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$ and with a Fourier expansion

$$F(Z) = \sum_{T>0} A(T) e^{2\pi i tr(TZ)}$$

where T runs over all positive definite, symmetric, half-integral matrices of size n.

Conjecture (Resnikoff-Saldaña). *For all $F \in S_k(\Gamma_n)$ one has*

(1) $$A(T) \ll_{\epsilon,F} (\det T)^{\frac{k}{2} - \frac{n+1}{4} + \epsilon} \qquad (\epsilon > 0)$$

where the constant implied in $\ll_{\epsilon,F}$ only depends on ϵ and F.

Remarks. i) There exists a theory of Hecke operators in genus $n > 1$, too, and $S_k(\Gamma_n)$ has a basis of Hecke eigenforms. However, for $n > 1$ the eigenvalues are not "proportional" (in any known reasonable sense) to the Fourier coefficients.

ii) Conjecture (1) is not known for a single F if $n > 1$. The best general results towards (1) known so far –after the "trivial" Hecke bound with exponent $\frac{k}{2}$– are

$$A(T) \ll_{\epsilon,F} (\det T)^{\frac{k}{2} - c_n + \epsilon} \qquad (\epsilon > 0)$$

where

$$c_n := \begin{cases} 13/36, & \text{if } n = 2 \text{ (Kohnen, 1993)} \\ 1/4, & \text{if } n = 3 \text{ (Breulmann, 1998)} \\ 1/2n + (1 - 1/n)\alpha_n, & \text{if } n > 3,\ k > n+1 \text{ (Böcherer-Kohnen, 1993)} \\ 1/2n + (1 - 1/n)\alpha_n, & \text{if } n > 3,\ k = n+1 \text{ (Bringmann, 2004).} \end{cases}$$

Here we have put $\alpha_n^{-1} := 4(n-1) + 4[\frac{n-1}{2}] + \frac{2}{n+2}$. In particular, the discrepancy between (1) and the actual status of our knowledge for general n and an arbitrary F is as immense as almost possible.

3. Some motivation

Suppose that $F \neq 0$. Let

$$D_F(s) := \sum_{\{T>0\}/GL_n(\mathbf{Z})} |A(T)|^2 \epsilon(T)^{-1} (\det T)^{-s} \qquad (\Re(s) >> 0)$$

be the Rankin-Dirichlet series attached to F where $GL_n(\mathbf{Z})$ operates on positive definite matrices T of size n in the usual way by $T \mapsto T[U] := U'TU$ and $\epsilon(T)$ is the number of U with $T[U] = U$.

According to results of Andrianov, Böcherer-Raghavan and Maass the series $D_F(s)$ has a meromorphic continuation to the entire complex plane with first pole (of residue an absolute constant times the square of the Petersson norm of F) occurring at $s = k$. Since $D_F(s)$ has non-negative coefficients, a classical result of Landau therefore implies that $D_F(s)$ in fact converges for $\Re(s) > k$.

Using the well-known formula for the abscissa of convergence of an ordinary Dirichlet series in conjunction with the asymptotic growth of the class number

$$m^{\frac{n-1}{2}-\epsilon} <<_\epsilon \#\{T > 0 \mid \det(2T) = m\}/GL_n(\mathbf{Z}) <<_\epsilon m^{\frac{n-1}{2}+\epsilon} \qquad (m \to \infty;\ \epsilon > 0)$$

due to Kitaoka-Siegel, and *assuming* that the coefficients $|A(T)|^2\epsilon(T)^{-1}$ are of "equal growth", one would therefore expect the bound

$$|A(T)|^2\epsilon(T)^{-1} <<_{\epsilon,F} m^{k+\epsilon-\frac{n-1}{2}-1} \qquad (\det(2T) = m)$$

which implies (1) since $\epsilon(T)$ by reduction theory is universally bounded.

4. Some counterexamples

i) *Counterexamples coming from theta series:*

These examples are essentially due to Freitag. Let S be a positive definite, symmetric, even integral unimodular matrix of size n (such an S exists if and only if $8|n$) and put

$$\vartheta_S(Z) := \sum_{G \in M_n(\mathbf{Z})} (\det G)\, e^{\pi i tr(S[G]Z)} \qquad (Z \in \mathcal{H}_n).$$

Then $\vartheta_S \in S_{1+n/2}(\Gamma_n)$.

Suppose that S has no integral automorphisms of determinant -1. Then ϑ_S is not identically zero (its Fourier coefficient of index S is not zero), hence there exist infinitely many T with $\det T \to \infty$ such that $A(T) \neq 0$ (otherwise $D_F(s)$ would be entire).

Now observe that $S[G] = 2T$ implies that $(\det G)^2 = \det(2T)$ from which in turn it follows that $A(T)$ is an integral multiple of $\sqrt{\det(2T)}$. Therefore ϑ_S does not satisfy (1) which would predict $A(T) << (\det T)^{1/4+\epsilon}$.

ii) *Counterexamples coming from Saito-Kurokawa lifts (n=2):*

Recall the following

Theorem (Andrianov, Eichler-Zagier, Maass, 1981). *Let k be even and let $f \in S_{2k-2}(\Gamma_1)$ be a normalized Hecke eigenform with Hecke L-series $L(f,s)$. Then there exists a Hecke eigenform $F \in S_k(\Gamma_2)$ such that the spinor zeta function $Z_F(s)$ of F equals*

$$Z_F(s) = \zeta(s-k+1)\zeta(s-k+2)L(f,s).$$

In particular, $Z_F(s)$ has a pole at $s = k$.

Now let F be as in the Theorem, let $D < 0$ be a discriminant and denote by $\mathbf{H}(D)$ the class group of Γ_1-classes of positive definite, symmetric, half-integral, primitive $(2,2)$-matrices of discriminant D. For $T \in \mathbf{H}(D)$ put

$$R_T(s) := \sum_{m \geq 1} A(mT) m^{-s} \qquad (\Re(s) > k+1).$$

According to Andrianov, one can always find a $D < 0$ and a character χ of $\mathbf{H}(D)$ such that

$$L(s-k+2,\chi) \sum_{\nu=1}^{h(D)} \chi(T_\nu) R_{T_\nu}(s) = A(\chi) Z_F(s) \qquad (\Re(s) > k+1) \tag{2}$$

where $T_1, \cdots, T_{h(D)}$ are representatives of $\mathbf{H}(D)$ and such that

$$A(\chi) := \sum_{\nu=1}^{h(D)} \chi(T_\nu) A(T_\nu) \neq 0.$$

Assume now that (1) would hold for F. Then

$$A(mT_\nu) << m^{k-3/2+\epsilon} \qquad (\epsilon > 0)$$

for all $m \geq 1$ and all ν. Therefore the left-hand side of (2) would converge for $\Re(s) > k - 1/2$, a contradiction.

iii) *Counterexamples coming from Ikeda lifts ($n \geq 2$):*

Recall the following

Theorem (Ikeda, 1999). *Suppose that $n \equiv k \pmod 2$ and let $f \in S_{2k}(\Gamma_1)$ be a normalized Hecke eigenform. Then there exists a Hecke eigenform $F \in S_{k+n}(\Gamma_{2n})$ such that its standard zeta function $L_{st}(F,s)$ equals*

$$L_{st}(F,s) = \zeta(s) \prod_{j=1}^{2n} L(f, s+k+n-j).$$

Moreover, the Fourier coefficients $A_{f,n}(T)$ of F are given by

$$A_{f,n}(T) = c(|D_{T,0}|) f_T^{k-1/2} \prod_{p | f_T} \tilde{F}_p(T; \alpha_p)$$

where $D_T = (-1)^n \det(2T) = D_{T,0} f_T^2$ *(with* $D_{T,0}$ *a fundamental discriminant and* $f_T \in \mathbf{N}$*),* $c(|D_{T,0}|)$ *is the* $|D_{T,0}|$*-th Fourier coefficient of a Hecke eigenform of weight* $k+1/2$ *and level 4 in the so-called "plus space" corresponding to* f *under the Shimura correspondence,* $\tilde{F}_p(T;X)$ *is a certain symmetric Laurent polynomial attached to* T *and* p *and finally* α_p *is the* p*-Satake parameter of* F *(properly normalized).*

Suppose now that $n \equiv 1 \pmod 4$ and let T_0 be a positive definite, symmetric, even integral unimodular matrix of size $2n-2$. One can then prove that (in obvious notation)

$$\tilde{F}_p(\mathcal{T} \oplus \frac{1}{2}T_0; X) = \tilde{F}_p(\mathcal{T}; X)$$

for any positive definite, symmetric, half-integral matrix $\mathcal{T}$ of size 2. Indeed, this follows from certain local formulas for $\tilde{F}_p$ due to Kitaoka and from the fact that the lattice corresponding to $\frac{1}{2}T_0$ is hyperbolic.

Therefore we see in particular that

$$A_{f,n}(\mathcal{T} \oplus \frac{1}{2}T_0) = A_{f,1}(\mathcal{T}).$$

Since

$$(\det(\mathcal{T} \oplus \frac{1}{2}T_0))^{\frac{k+n}{2} - \frac{2n+1}{4}} = (\det \mathcal{T})^{\frac{k}{2} - \frac{1}{4}}$$

we thus find that the non-validity of (1) for F follows from ii), since the Ikeda lift for $n = 1$ coincides with the Saito-Kurokawa lift.

References

[1] W. Kohnen: On the growth of Fourier coefficients of certain special cusp forms. To appear in Math. Z.

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