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Irrationality of certain Lambert series

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1 Introduction and the results

For any fixed $q \in \mathbb{C}$ with $|q| > 1$ and $z \in \mathbb{C}$, the q-logarithmic function $L_q(z)$ and the q-exponential $E_q(z)$ are defined by

$$L_q(z) := \sum_{n=1}^{\infty} \frac{z^n}{q^n - 1} = \sum_{n=1}^{\infty} \frac{z}{q^n - z} \quad (|z| < |q|),$$

$$E_q(z) := 1 + \sum_{n=1}^{\infty} \frac{z^n}{(q-1)\cdots(q^n-1)} = \prod_{n=1}^{\infty} \left(1 + \frac{z}{q^n}\right),$$

respectively. Bézivin [2] showed that the numbers 1, $E_q^{(k)}(\alpha_i)$ ($i = 1, \ldots, m$, $k = 0, 1, \ldots, l$) are linearly independent over $\mathbb{Q}$, where $q \in \mathbb{Z} \setminus \{0, \pm 1\}$ and $\alpha_i \in \mathbb{Q}^{\times}$ satisfy $\alpha_i \neq -q^{\mu}$ and $\alpha_i \neq \alpha_j q^{\nu}$ for all $\mu, \nu \in \mathbb{Z}$ with $\mu \geq 1$ and $i \neq j$. This implies that

$$\sum_{n=1}^{\infty} \frac{1}{q^n + \alpha} \notin \mathbb{Q},$$

where $q \in \mathbb{Z} \setminus \{0, \pm 1\}$ and $\alpha \in \mathbb{Q}^{\times}$ with $\alpha \neq -q^i \, (i \geq 1)$. Under the same conditions on q and α, Borwein [3], [4] obtained irrationality measures for the numbers $\sum_{n=1}^{\infty} 1/(q^n + \alpha)$ and $\sum_{n=1}^{\infty} (-1)^n/(q^n + \alpha)$. These results include the irrationality of $L_2(1) = \sum_{n=1}^{\infty} 1/(2^n - 1)$ proved by Erdös [10]. Furthermore, Bundschuh and Väänaänen [6], and Matala-Aho and Väänänen [11] obtained quantitative irrationality results for the values of the q-logarithm both in the Archimedean and p-adic cases. In [7], Duverney generalized certain results obtained by Borwein [3], [4], and Bundschuh and Väänänen [6]. Recently, Van Assche [15] gave irrationality measures for the numbers $L_q(1)$ and $L_q(-1)$ by using little q-Legendre polynomials. In this paper, we prove irrationality results for certain Lambert series, which in particular implies the linear independence of the numbers 1, $L_q(1)$, $L_q(-1)$ with $q \in \mathbb{Z} \setminus \{0, \pm 1\}$ by developing Borwein's idea in [4].

Let R_n be a binary recurrence defined by

$$R_{n+2} = A_1 R_{n+1} + A_2 R_n \, (n \geq 0), \quad A_1, A_2 \in \mathbb{Q}^{\times}, \; R_0, R_1 \in \mathbb{Q}.$$

André-Jeannin [1] proved for some R_n the irrationality of the value of the function $f(x) = \sum_{n=1}^{\infty} x^n/R_n$ at a nonzero rational integer x in the disk of convergence of f, which gave the first proof of the irrationality of the numbers $\sum_{n=1}^{\infty} 1/F_n$ and $\sum_{n=1}^{\infty} 1/L_n$, where F_n and L_n are Fibonacci numbers and Lucas numbers, respectively. Prévost [13] extended this result to any rational x in the domain of meromorphy of f. Recently, Matala-aho and Prévost [12] obtained for some type of R_n irrationality measures for the number $\sum_{n=1}^{\infty} \gamma^n/R_{an}$, where γ belongs to an imaginary quadratic field, and $a > 0$ is an integer. We will prove for some R_n the irrationality of the numbers $\sum_{n=1}^{\infty} \gamma^n/R_{an+b}$ and $\sum_{n=1}^{\infty} \gamma^n/R_{an+b}R_{a(n+1)+b}$, where $a > 0$, $b \geq 0$ are integers and γ is a certain number in a real quadratic field(see Corollaries 2 and 3, below).

For an algebraic number α, we denote by $\overline{|\alpha|}$ the maximum of absolute values of its conjugates and by denα the least positive integer such that α·denα is an algebraic integer. We define generalized Pisot number α by algebraic integer α satisfying $|\alpha| > 1$ and $|\alpha^\sigma| < 1$ for any $\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$ with $\alpha^\sigma \neq \alpha$. We put $\mathbb{N} = \{0, 1, 2, \dots\}$.

Theorem 1. *Let* **K** *be either* $\mathbb{Q}$ *or an imaginary quadratic field. Assume that* q *is an integer in* **K** *with* $|q| > 1$ *and* $\{a_n\}$ *a periodic sequence in* **K** *of period two, not identically zero. Then*

$$\theta = \sum_{n=1}^{\infty} \frac{a_n}{1 - q^n} \notin \mathbf{K}.$$

Corollary 1. *Let* $q \in \mathbb{Z}$ *with* $|q| \geq 2$ *and* $\{a_n\}, \{b_n\}$ *be periodic sequences in* $\mathbb{Q}$ *of period two, not identically zero. Then the numbers*

$$1, \qquad \sum_{n=1}^{\infty} \frac{a_n}{q^n - 1}, \qquad \sum_{n=1}^{\infty} \frac{b_n}{q^n - 1}$$

are linearly independent over $\mathbb{Q}$ *if and only if* $\{a_n\}$ *and* $\{b_n\}$ *are linearly independent over* $\mathbb{Q}$.

Proof. This follows immediately from Theorem 1.

Example 1. *Let* $q \in \mathbb{Z}$ *with* $|q| \geq 2$. *Then*

$$1, \qquad L_q(1) = \sum_{n=1}^{\infty} \frac{1}{q^n - 1}, \qquad L_q(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{q^n - 1} = \sum_{n=1}^{\infty} \frac{-1}{q^n + 1}$$

are linearly independent over $\mathbb{Q}$.

Theorem 2. *Let* q *be a quadratic generalized Pisot number,* γ *a unit in* $\mathbb{Q}(q)$ *with* $\overline{|\gamma|} \leq 1$, *and* $\alpha \in \mathbb{Q}(q)^\times$ *with* $(\mathrm{den}(q^l\alpha))^4 < |q|$ *for some* $l \in \mathbb{N}$. *Then*

$$\xi = \sum_{n=1}^{\infty} \frac{\gamma^n}{1 - \alpha q^n} \notin \mathbb{Q}(q),$$

provided that $\alpha q^n \neq 1$ *for all* $n \geq 1$.

In the following Corollaries 2 and 3, we consider the binary recurrences $\{R_n\}_{n\geq 0}$ defined by

$$R_{n+2} = A_1R_{n+1} + A_2R_n, \quad A_1, A_2 \in \mathbb{Z}\setminus\{0\},\ R_0,\ R_1 \in \mathbb{Z}.$$

We suppose that $R_n \neq 0$ for all $n \geq 1$, the corresponding polynomial $\Phi(X) = X^2 - A_1X - A_2$ is irreducible in $\mathbb{Q}[X]$, and $\triangle = A_1^2 + 4A_2 > 0$. We can write R_n as

$$R_n = g_1\rho_1^n + g_2\rho_2^n\,(n \geq 0), \quad g_1, g_2 \in \mathbb{Q}(\rho_1)^\times, \tag{1}$$

where ρ_1 and ρ_2 are the roots of $\Phi(X)$. We may assume $|\rho_1| > |\rho_2|$, since $\triangle > 0$.

For $a, b \in \mathbb{N}$ with $a \neq 0$, we define

$$R(z) = \sum_{n=1}^{\infty} \frac{z^n}{R_{an+b}} \quad (\,|z| < |\rho_1|^a\,).$$

This function can be extended to a meromorphic function on the whole complex plane $\mathbb{C}$ with poles $\{(\rho_1^{n+1}/\rho_2^n)^a\,|\,n \geq 0\}$, since

$$\sum_{n=1}^{\infty} \frac{z^n}{1-\alpha q^n} = \sum_{m=1}^{\infty} \frac{\alpha^{-m}z}{z-q^m} \quad (|z| < |q|)$$

for any complex numbers q and α with $|q| > 1$ and $|\alpha| \geq 1$, and so

$$\sum_{n=1}^{\infty} \frac{z^n}{R_{an+b}} = \sum_{n=1}^{i} \frac{z^n}{R_{an+b}} - \frac{z^{i+1}}{g_1\rho_1^{ai+b}} \sum_{n=0}^{\infty} \frac{(-(g_2/g_1)(\rho_2/\rho_1)^{ai+b})^n}{z - \rho_1^a(\rho_1/\rho_2)^{an}}, \tag{2}$$

where i is chosen as $\left|(g_2/g_1)(\rho_2/\rho_1)^{ai+b}\right| < 1$. We denote the function again by $R(z)$.

Corollary 2. *Let R_n be a binary recurrence given by (1) and $a, b \in \mathbb{N}$ with $a \neq 0$. Assume that g_1/g_2 and ρ_1/ρ_2 are units in $\mathbb{Q}(\rho_1)$ and $\gamma \in \mathbb{Q}(\rho_1)^\times$ is not a pole of $R(z)$ with $(\mathrm{den}(\rho_1^a/\gamma))^4 < |\rho_1/\rho_2|^a$. Then we have $R(\gamma) \notin \mathbb{Q}(\rho_1)$.*

Proof. Apply Theorem 2 to the last sum in (2).

Example 2. *Let F_n and L_n be Fibonacci numbers and Lucas numbers defined by $F_{n+2} = F_{n+1} + F_n\,(n \geq 0)$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n\,(n \geq 0)$, $L_0 = 2$, $L_1 = 1$, respectively. Then for every $a, b \in \mathbb{N}$ with $a \neq 0$,*

$$\sum_{n=1}^{\infty} \frac{1}{F_{an+b}},\quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_{an+b}},\quad \sum_{n=1}^{\infty} \frac{1}{L_{an+b}},\quad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{an+b}} \notin \mathbb{Q}(\sqrt{5}).$$

André-Jeannin[1] *proved that each of these numbers is irrational. We remark that the numbers $\sum_{n=1}^{\infty} 1/F_{2n+1}$ and $\sum_{n=1}^{\infty} 1/L_{2n}$ are transcendental* (cf. [8], [9]).

Example 3. *Let F_n be Fibonacci numbers. Then for every $a, b \in \mathbb{N}$ with $a \neq 0$,*

$$\sum_{n=1}^{\infty} \frac{1}{F_{(2a-1)n+b}F_{(2a-1)(n+1)+b}}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_{2an+b}F_{2a(n+1)+b}} \notin \mathbb{Q}(\sqrt{5}).$$

The same holds for Lucas numbers. We put

$$T_l := \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+l}}, \quad T_l^* := \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+l}} \ (l \geq 1).$$

Then Brousseau [5] *and* Rabinowitz [14] *proved that*

$$T_{2l} = \frac{1}{F_{2l}} \sum_{n=1}^{l} \frac{1}{F_{2n-1}F_{2n}}, \quad T_{2l+1} = \frac{1}{F_{2l+1}} \left(T_1 - \sum_{n=1}^{l} \frac{1}{F_{2n}F_{2n+1}} \right),$$

$$T_l^* = \frac{1}{F_l} \left(\frac{1-\sqrt{5}}{2} l + \sum_{n=1}^{l} \frac{F_{n-1}}{F_n} \right),$$

so that $T_{2l} \in \mathbb{Q}$ and $T_l^* \in \mathbb{Q}(\sqrt{5}) \setminus \mathbb{Q}$ for all $l \geq 1$. We see that $T_{2l+1} \notin \mathbb{Q}(\sqrt{5})$ for all $l \geq 0$, since the first sum in this example with $a = 1$, $b = 0$ implies

$$T_1 = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} \notin \mathbb{Q}(\sqrt{5}).$$

2 Lemmas

For the proof of theorems, we prepare some lemmas. Let $\{a_m\}_{m \geq 1}$ be a periodic sequence of complex numbers of period two, not identically zero. We put

$$\theta = \sum_{m=1}^{\infty} \frac{a_m}{1-q^m},$$

where $q \in \mathbb{C}$ with $|q| > 1$. We start with the integral

$$F_n(q) = \frac{1}{2\pi i} \int_{|t|=1} \frac{(-1/t) \prod_{k=1}^{2n} (1-q^k/t)}{\prod_{k=1}^{n} (1-q^{2k}t)} \sum_{m=1}^{\infty} \frac{a_m}{1-q^m/t} dt, \tag{3}$$

which is a variant of that used by Borwein [4]. We note that the integrand is meromorphic in t provided $|q| > 1$. We use the notations

$$[n]_q! := \frac{(1-q^n)(1-q^{n-1}) \cdots (1-q)}{(1-q)^n}, \quad [0]_q := 1,$$

$$\begin{bmatrix} n \\ i \end{bmatrix}_q := \frac{[n]_q!}{[i]_q![n-i]_q!} \in \mathbb{Z}[q].$$

In what follows, we denote $c_1, c_2, \ldots$ positive constants independent of n.

Lemma 1.

$$\begin{aligned} F_n(q) &= \sum_{i=1}^{n} \frac{\prod_{k=1}^{2n}(1-q^{k+2i})}{\prod_{\substack{k=1\\k\neq i}}^{n}(1-q^{2k-2i})} \left(\theta - \sum_{m=1}^{2i} \frac{a_m}{1-q^m}\right) \\ &- \frac{1}{(2n-1)!}\left(\prod_{k=1}^{2n}(t-q^k)\prod_{k=1}^{n}(1-q^{2k}t)^{-1}\sum_{m=1}^{\infty}\frac{a_m}{t-q^m}\right)^{(2n-1)}\Bigg|_{t=0} \end{aligned} \tag{4}$$

Proof. This can be proved by using the residue theorem similarly as the proof of Lemma 1 in [4].

We put $D_n(q) := \prod_{k=n+1}^{2n}(1-q^{2k})$. Then we have

$$|D_n(q)| \leq c_1 |q|^{3n^2+n}. \tag{5}$$

Lemma 2.

$$D_n(q)F_n(q) = A_n(q)\theta + B_n(q), \tag{6}$$

where $A_n(q)$, $B_n(q) \in \mathbb{Z}[a_1, a_2, q]$.

Proof. Since

$$\frac{1}{\prod_{\substack{k=1\\k\neq i}}^{n}(1-q^{2k-2i})} = \frac{q^{i(i-1)}}{\prod_{k=1}^{i-1}(q^{2k}-1)\prod_{k=1}^{n-i}(1-q^{2k})},$$

we have by (4)

$$\begin{aligned} F_n(q) &= \frac{1}{\prod_{k=1}^{n-1}(1-q^{2k})}\sum_{i=1}^{n}(-1)^{i-1}q^{i(i-1)}\begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_{q^2}\prod_{k=1}^{2n}(1-q^{k+2i})\left(\theta - \sum_{m=1}^{2i}\frac{a_m}{1-q^m}\right) \\ &-\sum_{\substack{\lambda,\mu,\nu\geq 0\\ \lambda+\mu+\nu=2n-1}} \frac{1}{\lambda!\mu!\nu!}\left(\prod_{k=1}^{2n}(t-q^k)\right)^{(\lambda)}\Bigg|_{t=0}\left(\prod_{k=1}^{n}(1-q^{2k}t)^{-1}\right)^{(\mu)}\Bigg|_{t=0}\left(\sum_{m=1}^{\infty}\frac{a_m}{t-q^m}\right)^{(\nu)}\Bigg|_{t=0} \end{aligned}$$

with

$$\left(\prod_{k=1}^{2n}(t-q^k)\right)^{(\lambda)}\Bigg|_{t=0} = \lambda!(-1)^{2n-\lambda}\sum_{\substack{\lambda_1+\cdots+\lambda_{2n}=2n-\lambda\\ \lambda_i=0,1}} q^{\lambda_1+2\lambda_2+\cdots+2n\lambda_{2n}},$$

$$\left(\prod_{k=1}^{n}(1-q^{2k}t)^{-1}\right)^{(\mu)}\Bigg|_{t=0} = \mu!\sum_{\substack{\mu_1+\cdots+\mu_n=\mu\\ \mu_i\geq 0}} q^{2(\mu_1+2\mu_2+\cdots+n\mu_n)},$$

$$\left(\sum_{m=1}^{\infty}\frac{a_m}{t-q^m}\right)^{(\nu)}\Bigg|_{t=0} = -\nu!\sum_{m=1}^{\infty}\frac{a_m}{(q^{\nu+1})^m} = \nu!(a_1q^{\nu+1}+a_2)\frac{1}{1-q^{2(\nu+1)}}.$$

Hence we get

$$\begin{aligned} F_n(q) &= \frac{1}{\prod_{k=1}^{n-1}(1-q^{2k})}\sum_{i=1}^{n}(-1)^{i-1}q^{i(i-1)}\begin{bmatrix} n-1\\ i-1\end{bmatrix}_{q^2}\prod_{k=1}^{2n}(1-q^{k+2i})\left(\theta-\sum_{m=1}^{2i}\frac{a_m}{1-q^m}\right) \\ &+ \sum_{\substack{\lambda+\mu+\nu=2n-1\\ \lambda,\mu,\nu\geq 0}} Q_{\lambda\mu\nu}(q)\frac{1}{1-q^{2(\nu+1)}} \end{aligned} \tag{7}$$

with $Q_{\lambda\mu\nu}(q)$ a polynomial in $\mathbb{Z}[a_1,a_2,q]$ for all $\lambda,\mu,\nu\geq 0$. Here we note that

$$\prod_{k=1}^{2n}(1-q^{k+2i})\sum_{m=1}^{2i}\frac{a_m}{1-q^m}\in\mathbb{Z}[a_1,a_2,q], \qquad i=1,2,\ldots,n,$$

and each of $\prod_{k=1}^{n-1}(1-q^{2k})$ and $1-q^{2l}$ ($l=1,\cdots,2n$) divides $D_n(q)$ in $\mathbb{Z}[q]$. Therefore the lemma follows from (7).

Lemma 3. *For large n, we have*

$$0<|F_n(q)|\leq c_3|q|^{-3n^2-2n}. \tag{8}$$

Proof. Similarly to the proof of Lemma 4 in [4], the residue theorem applied exterior to the circle $|t|=1$ shows that

$$F_n(q)=\sum_{m=2n+1}^{\infty}I_m, \qquad I_m=a_m\frac{\prod_{k=1}^{2n}(1-q^{k-m})}{\prod_{k=1}^{n}(1-q^{2k+m})}$$

for large n. Since $|I_m|\leq c_2|q|^{-n^2-n(m+1)}$, we get the upper bound for $|F_n(q)|$. Furthermore, if $a_1\neq 0$, it follows that,

$$F_n(q)=a_1\frac{\prod_{k=1}^{2n}(1-q^{k-2n-1})}{\prod_{k=1}^{n}(1-q^{2k+2n+1})}\left(1+\sum_{l=1}^{\infty}b_{nl}\right)$$

with

$$b_{nl}=\frac{a_{l+1}}{a_1}\prod_{k=1}^{n}\left(\frac{1-q^{2k+2n+1}}{1-q^{2k+2n+l+1}}\right)\prod_{k=1}^{2n}\left(\frac{1-q^{k-2n-l-1}}{1-q^{k-2n-1}}\right),$$

where $|b_{nl}|\le c_4|q^{-n}|^l$. Hence we have $F_n(q)\ne 0$, since $\sum_{l=1}^{\infty}|b_{nl}|<1$ for large n. The proof is similar in the case of $a_1=0, a_2\ne 0$.

3 Proofs of Theorems

Proof of Theorem 1. Let $\mathbf{K}$, q, and $\{a_m\}$ be as in Theorem 1. We may suppose that a_1 and a_2 are integers in $\mathbf{K}$. Assume that $\theta\in\mathbf{K}$ and let $d=\mathrm{den}\theta$. Then by (5),(6), and (8), we have

$$0<d\,|A_n(q)\theta+B_n(q)|\le dc_5|q|^{-n}$$

for large n; which is a contradiction.

Proof of Theorem 2. Let q, α, and γ be as in Theorem 2. Since

$$\sum_{m=1}^{\infty}\frac{\gamma^m}{1-\alpha q^l q^m}=\gamma^{-l}\left(\sum_{m=1}^{\infty}\frac{\gamma^m}{1-\alpha q^m}-\sum_{m=1}^{l}\frac{\gamma^m}{1-\alpha q^m}\right)\quad (l\ge 1),$$

we can assume that α is a generalized Pisot number, by replacing α by $q^l\alpha$ with suitable l. We modify Borwein's integral in [4] as follows:

$$G_n(q,\alpha,\gamma)=\frac{1}{2\pi i}\int_{|t|=1}\prod_{k=1}^{n-1}\left(\frac{1-\alpha q^k/t}{1-q^k t}\right)\frac{-1/t}{1-q^n t}\sum_{m=1}^{\infty}\frac{\gamma^m}{1-\alpha q^m/t}dt.$$

Theorem 2 can be proved by replacing $F_n(q)$ by $G_n(q,\alpha,\gamma)$ in Lemmas.

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