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# Algebraic independence of certain power series associated with $d$-adic expansion of real numbers

慶應義塾大学理工学部　田中 孝明 (Taka-aki Tanaka)
Faculty of Science and Technology, Keio Univ.

## 1 Introduction.

Let $\omega>0$ and let $d$ be an integer greater than 1. The number $\omega$ is expressed as a $d$-adic expansion

$$\omega=\sum_{i=-l}^{\infty}\varepsilon_i d^{-i},\quad l=\max\{[\log_d\omega],0\},\quad \varepsilon_i\in\{0,1,\ldots,d-1\},$$

where $[x]$ denotes the largest integer not exceeding the real number $x$. For those $\omega$ having two ways of expression such as $2=1.9999\ldots$ (10-adic), we adopt only the left-hand side expression. Then this expansion is uniquely determined. Let

$$a_k=[\omega d^k]\quad (k=0,1,2,\ldots).$$

It is clear that

$$a_k=\sum_{i=-l}^{k}\varepsilon_i d^{k-i},$$

namely the integer $a_k$ is expressed as the $d$-adic number $\varepsilon_{-l}\varepsilon_{-l+1}\ldots\varepsilon_{k-1}\varepsilon_k$. Hence we see that the sequence $\{a_k\}_{k\geq 0}$ satisfies the recurrence formula

$$a_0=[\omega],\quad a_k=da_{k-1}+\varepsilon_k\quad (k=1,2,3,\ldots).$$

The author [3] proved that the number $\sum_{k=0}^{\infty}\alpha^{a_k}$ is transcendental for any algebraic number $\alpha$ with $0<|\alpha|<1$. In this paper we prove the following algebraic independence result. Let $\omega_1,\ldots,\omega_m>0$. Define

$$f_{id}(z)=\sum_{k=0}^{\infty}z^{[\omega_i d^k]}\quad (i=1,\ldots,m;\ d=2,3,4,\ldots).\tag{1}$$

In what follows, $\mathbb{Q}$ and $\mathbb{R}$ denote the sets of rational and real numbers, respectively.

**Theorem 1.** *If the numbers* $\omega_1, \ldots, \omega_m$ *are linearly independent over* $\mathbb{Q}$, *then the numbers* $f_{id}(\alpha)$ $(i = 1, \ldots, m;\ d = 2, 3, 4, \ldots)$ *are algebraically independent for any algebraic number* $\alpha$ *with* $0 < |\alpha| < 1$.

**Corollary 1.** *If the numbers* $\omega_1, \ldots, \omega_m$ *are linearly independent over* $\mathbb{Q}$, *then the functions* $f_{id}(z)$ $(i = 1, \ldots, m;\ d = 2, 3, 4, \ldots)$ *are algebraically independent over the field* $\mathbb{C}(z)$ *of rational functions.*

EXAMPLE. Let

$$f_{1,d}(z) = \sum_{k=0}^{\infty} z^{d^k}, \qquad f_{2,d}(z) = \sum_{k=0}^{\infty} z^{[\sqrt{2}d^k]},$$

$$f_{3,d}(z) = \sum_{k=0}^{\infty} z^{[\sqrt{3}d^k]}, \qquad f_{4,d}(z) = \sum_{k=0}^{\infty} z^{[\pi d^k]} \qquad (d = 2, 3, 4, \ldots).$$

For example we have

$$\begin{aligned} f_{2,10}(z) &= z + z^{14} + z^{141} + z^{1414} + z^{14142} + z^{141421} + \cdots, \\ f_{3,10}(z) &= z + z^{17} + z^{173} + z^{1732} + z^{17320} + z^{173205} + \cdots, \end{aligned}$$

and

$$f_{4,10}(z) = z^3 + z^{31} + z^{314} + z^{3141} + z^{31415} + z^{314159} + \cdots.$$

Then by Theorem 1 the numbers $f_{i,d}(\alpha)$ $(i = 1, \ldots, 4;\ d = 2, 3, 4, \ldots)$ are algebraically independent for any algebraic number $\alpha$ with $0 < |\alpha| < 1$ since the numbers $1, \sqrt{2}, \sqrt{3}$, and $\pi$ are linearly independent over $\mathbb{Q}$.

Theorem 1 is proved by using the method developed from that of Nishioka used for proving the following:

**Theorem 2** (Nishioka [2, Theorem 1]). *Let*

$$f_d(z) = \sum_{k=0}^{\infty} \sigma_{dk} z^{d^k} \quad (d = 2, 3, 4, \ldots),$$

*where the* $\sigma_{dk}$ $(k = 0, 1, 2, \ldots)$ *are in a finite set of nonzero algebraic numbers for every* $d$. *Then the numbers* $f_d(\alpha)$ $(d = 2, 3, 4, \ldots)$ *are algebraically independent for any algebraic number* $\alpha$ *with* $0 < |\alpha| < 1$.

We further obtain the following, which includes both Theorems 1 and 2.

**Theorem 3.** *Let* $\omega_1, \ldots, \omega_m > 0$. *Define*

$$f_{id}(z) = \sum_{k=0}^{\infty} \sigma_{idk} z^{[\omega_i d^k]} \quad (i = 1, \ldots, m;\ d = 2, 3, 4, \ldots),$$

*where the* $\sigma_{idk}$ $(k = 0, 1, 2, \ldots)$ *are in a finite set of nonzero algebraic numbers for every* $i$ *and for every* $d$. *If the numbers* $\omega_1, \ldots, \omega_m$ *are linearly independent over* $\mathbb{Q}$, *then the numbers* $f_{id}(\alpha)$ $(i = 1, \ldots, m;\ d = 2, 3, 4, \ldots)$ *are algebraically independent for any algebraic number* $\alpha$ *with* $0 < |\alpha| < 1$.

Theorem 3 implies the following result, which also includes Theorem 1.

**Theorem 4.** *Let* $\omega_1, \ldots, \omega_m > 0$ *and* $\eta_1, \ldots, \eta_m \in \mathbb{R}$. *Define*

$$f_{id}(z) = \sum_{k=0}^{\infty} z^{[\omega_i d^k + \eta_i]} \quad (i = 1, \ldots, m;\ d = 2, 3, 4, \ldots).$$

*If the numbers* $\omega_1, \ldots, \omega_m$ *are linearly independent over* $\mathbb{Q}$, *then the numbers* $f_{id}(\alpha)$ $(i = 1, \ldots, m;\ d = 2, 3, 4, \ldots)$ *are algebraically independent for any algebraic number* $\alpha$ *with* $0 < |\alpha| < 1$.

## 2 Lemmas.

We prepare the notation for stating the lemmas. For any algebraic number $\alpha$, we denote by $\lceil \alpha \rceil$ the maximum of the absolute values of the conjugates of $\alpha$ and by $\mathrm{den}(\alpha)$ the smallest positive integer such that $\mathrm{den}(\alpha) \cdot \alpha$ is an algebraic integer and define

$$\|\alpha\| = \max\{\lceil \alpha \rceil, \mathrm{den}(\alpha)\}.$$

If $\Omega = (\omega_{ij})$ is an $n \times n$ matrix with nonnegative integer entries and if $\boldsymbol{z} = (z_1, \ldots, z_n)$ is a point of $\mathbb{C}^n$ with $\mathbb{C}$ the set of complex numbers, we define the transformation $\Omega : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\Omega \boldsymbol{z} = \left( \prod_{j=1}^{n} z_j^{\omega_{1j}}, \prod_{j=1}^{n} z_j^{\omega_{2j}}, \ldots, \prod_{j=1}^{n} z_j^{\omega_{nj}} \right).$$

Let $\{\Omega^{(k)}\}_{k \geq 0}$ be a sequence of $n \times n$ matrices with nonnegative integer entries. We put

$$\Omega^{(k)} = (\omega_{ij}^{(k)}) \qquad \text{and} \qquad \Omega^{(k)} \boldsymbol{z} = (z_1^{(k)}, \ldots, z_n^{(k)}).$$

In what follows, $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of positive and nonnegative integers, respectively. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{N}_0)^n$, we define $\boldsymbol{z}^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n}$ and $|\lambda| = \lambda_1 + \cdots + \lambda_n$. Let $K$ be an algebraic number field. Let $\{f_1^{(k)}(\boldsymbol{z})\}_{k \geq 0}, \ldots, \{f_m^{(k)}(\boldsymbol{z})\}_{k \geq 0}$ be sequences of power series in $K[[z_1, \ldots, z_n]]$. Let $\chi = (z_1, \ldots, z_n)$ be the maximal ideal generated by $z_1, \ldots, z_n$ in the ring $K[[z_1, \ldots, z_n]]$. In what follows, $c_1, c_2, \ldots$ denote positive constants independent of $k$.

**Lemma 1** (cf. Nishioka [2, Theorem 2]). *Assume that*

$$f_i^{(k)}(\boldsymbol{z}) \to f_i(\boldsymbol{z}) \quad \text{as} \quad k \to \infty$$

*with respect to the topology defined by* $\chi$ *for any* $i$ $(1 \le i \le m)$. *Suppose that all the* $f_i^{(k)}(\boldsymbol{z})$ $(k \ge 0)$, $f_i(\boldsymbol{z})$ $(1 \le i \le m)$ *converge in the n-polydisc* $\{\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_j| < r \ (1 \le j \le n)\}$. *If* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ *is a point of* $K^n$ *with* $0 < |\alpha_j| < \min\{1, r\}$ $(1 \le j \le n)$ *and if the following three properties are satisfied, then the values* $f_1^{(0)}(\boldsymbol{\alpha}), \ldots, f_m^{(0)}(\boldsymbol{\alpha})$ *are algebraically independent.*

(I) *There exists a sequence* $\{\rho_k\}_{k \ge 0}$ *of positive numbers such that*

$$\lim_{k \to \infty} \rho_k = \infty, \qquad \omega_{ij}^{(k)} \le c_1 \rho_k, \qquad \log |\alpha_j^{(k)}| \le -c_2 \rho_k.$$

(II) *If we put*

$$f_i^{(0)}(\boldsymbol{\alpha}) = f_i^{(k)}(\Omega^{(k)} \boldsymbol{\alpha}) + b_i^{(k)} \quad (1 \le i \le m),$$

*then* $b_i^{(k)} \in K$ *and*

$$\log \|b_i^{(k)}\| \le c_3 \rho_k \quad (1 \le i \le m).$$

(III) *For any power series* $F(\boldsymbol{z})$ *represented as a polynomial in* $z_1, \ldots, z_n, f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z})$ *with complex coefficients of the form*

$$F(\boldsymbol{z}) = \sum_{\lambda, \mu = (\mu_1, \ldots, \mu_m)} a_{\lambda, \mu} \boldsymbol{z}^{\lambda} f_1(\boldsymbol{z})^{\mu_1} \cdots f_m(\boldsymbol{z})^{\mu_m},$$

*where* $a_{\lambda,\mu}$ *are not all zero, there exists a* $\lambda_0 \in (\mathbb{N}_0)^n$ *such that if* $k$ *is sufficiently large, then*

$$|F(\Omega^{(k)} \boldsymbol{\alpha})| \ge c_4 |(\Omega^{(k)} \boldsymbol{\alpha})^{\lambda_0}|.$$

Although Theorem 2 of Nishioka [2] requires the assumption that the coefficients of $f_i^{(k)}(\boldsymbol{z})$ are in a finite set $S \subset K$ for all $i$ and $k$, it can be weakened as in Lemma 1, which is proved by the almost same way as in the proof of Theorem 2 of Nishioka [2].

**Lemma 2** (Nishioka [2]). *Let* $f(\boldsymbol{z}) = \sum_{\lambda_1, \ldots, \lambda_n} c_{\lambda_1, \ldots, \lambda_n} z_1^{\lambda_1} \cdots z_n^{\lambda_n} \in \mathbb{C}[[z_1, \ldots, z_n]]$ *converge around the origin. If* $\boldsymbol{z}$ *is sufficiently close to the origin, then*

$$\sum_{\lambda \ge H} |c_{\lambda_1, \ldots, \lambda_n}| \cdot |z_1|^{\lambda_1} \cdots |z_n|^{\lambda_n} \le \gamma^{H+1} \max_{1 \le i \le n} |z_i|^H,$$

*where* $\gamma$ *is a positive constant depending on* $f(\boldsymbol{z})$.

The following lemma is originally due to Masser [1] and improved by Nishioka [2].

**Lemma 3** (Masser [1], Nishioka [2]). *Let* $b_1 > \cdots > b_n \ge 2$ *be pairwise multiplicatively independent integers. Let* $\theta = \log b_1$ *and* $\theta_j = \theta / \log b_j$ $(1 \le j \le n)$. *Suppose that for each* $\alpha$ *in a finite set* $A$ *we are given real numbers* $\lambda_{1\alpha}, \ldots, \lambda_{n\alpha}$, *not all zero, and define the sequence*

$$S_\alpha(k) = \sum_{j=1}^{n} \lambda_{j\alpha} b_j^{[\theta_j k]} \quad (k = 0, 1, 2, \ldots).$$

*If $\{k_l\}_{l\geq 1}$ is an increasing sequence of positive integers with $\{k_{l+1}-k_l\}_{l\geq 1}$ bounded, then there exists a positive number $\delta$ such that*

$$R(\delta)=\{k_l \mid \min_{\alpha\in A}|S_\alpha(k_l)|\geq \delta b_1^{k_l}\}=\{m_l\}_{l\geq 1}, \qquad m_l<m_{l+1},$$

*is an infinite set and $\{m_{l+1}-m_l\}_{l\geq 1}$ is bounded.*

Using Lemma 3, we have the following:

**Lemma 4.** *Let $b_1,\ldots,b_n$ be integers as in Lemma 3 and let $\theta_1,\ldots,\theta_n$ be defined in Lemma 3. Let $\omega_1,\ldots,\omega_m>0$ be linearly independent over $\mathbb{Q}$. Then there exist an infinite set $\Lambda$ of positive integers, a sequence $\{\delta(l)\}_{l\geq 1}$ of positive numbers, and a total order $\succ$ in $(\mathbb{N}_0)^{mn}$ such that if $\lambda=(\lambda_{ij})\succ\mu=(\mu_{ij})$ with $|\lambda|=\lambda_{11}+\cdots+\lambda_{mn}, |\mu|=\mu_{11}+\cdots+\mu_{mn}\leq l$, then*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\lambda_{ij}[\omega_i b_j^{[\theta_j q]}]-\sum_{i=1}^{m}\sum_{j=1}^{n}\mu_{ij}[\omega_i b_j^{[\theta_j q]}]\geq \delta(l)b_1^q$$

*for all sufficiently large $q\in\Lambda$. Moreover, any subset $S$ of $(\mathbb{N}_0)^{mn}$ has the minimal element with respect to the total order $\succ$.*

**Lemma 5** (Nishioka [2]). *Let $d$ be an integer greater than 1 and let*

$$f_l(z)=\sum_{h=0}^{\infty}s_h^{(l)}z^{d^{lh}} \quad (l=1,2,\ldots),$$

*where the coefficients $s_h^{(l)}$ are nonzero complex numbers. Then $f_l(z)$ $(l=1,2,\ldots)$ are algebraically independent over $\mathbb{C}(z)$.*

## 3 Proof of Theorems 1 and 4.

*Proof of Theorem* 1. Let

$$D=\{d\in\mathbb{N}\mid d\neq a^n\ (a,n\in\mathbb{N},\ n\geq 2)\}.$$

Then

$$\mathbb{N}\setminus\{1\}=\bigcup_{d\in D}\{d,d^2,\ldots\},$$

which is a disjoint union since any two distinct elements of $D$ are multiplicatively independent by the definition of $D$. Let $d_1>\cdots>d_n$ be elements of $D$ and let $\boldsymbol{z}=(z_{11},\ldots,z_{m1},\ldots,z_{1n},\ldots,z_{mn})$, where $z_{11},\ldots,z_{m1},\ldots,z_{1n},\ldots,z_{mn}$ are distinct variables. For any $i$ $(1\leq i\leq m)$ and for any $d_j\in D$ $(1\leq j\leq n)$, we define the sequence $\{r_k^{(i,j)}\}_{k\geq 0}$ by

$$r_0^{(i,j)}=1, \quad r_k^{(i,j)}=[\omega_i d_j^k] \quad (k\geq 1) \tag{2}$$

and define

$$f_{ijl0}(\boldsymbol{z}) = \sum_{h=0}^{\infty} \alpha^{r_{lh}^{(i,j)} - d_j^{lh}} z_{ij}^{d_j^{lh}} \quad (1 \le i \le m,\ 1 \le j \le n,\ 1 \le l \le t).$$

Letting $\boldsymbol{\alpha} = (\alpha, \dots, \alpha, \dots, \alpha, \dots, \alpha)$, we have

$$f_{ijl0}(\boldsymbol{\alpha}) = \sum_{h=0}^{\infty} \alpha^{r_{lh}^{(i,j)}} = \alpha + \sum_{h=1}^{\infty} \alpha^{[\omega_i d_j^{lh}]} = f_{id_j^l}(\alpha) - \alpha^{[\omega_i]} + \alpha,$$

where $f_{id}$ is defined by (1). Hence it suffices to prove the algebraic independency of the values $f_{ijl0}(\boldsymbol{\alpha})$ $(1 \le i \le m,\ 1 \le j \le n,\ 1 \le l \le t)$. For the purpose we apply Lemma 1.

Put $b_j = d_j^{t!}$, $\theta = \log b_1$, and $\theta_j = \theta / \log b_j$ $(1 \le j \le n)$. Noting that

$$0 \le r_{lh+t![\theta_j q]}^{(i,j)} - r_{t![\theta_j q]}^{(i,j)} d_j^{lh} \le d_j^{lh} - 1 \quad (1 \le i \le m),$$

we put

$$\begin{aligned} \Sigma_q &= \left( \alpha^{r_{lh+t![\theta_j q]}^{(i,j)} - r_{t![\theta_j q]}^{(i,j)} d_j^{lh}} \right)_{1 \le i \le m,\ 1 \le j \le n,\ 1 \le l \le t,\ h \ge 0} \\ &\in \prod_{h=0}^{\infty} \prod_{j=1}^{n} \prod_{l=1}^{t} \{1, \alpha, \dots, \alpha^{d_j^{lh}-1}\}^m \end{aligned}$$

for any $q \in \Lambda$ with the $\Lambda$ defined in Lemma 4. Since the right-hand side is a compact set, there exists a converging subsequence $\{\Sigma_{q_k}\}_{k \ge 1}$ of $\{\Sigma_q\}_{q \in \Lambda}$, where $q_1$ will be chosen sufficiently large. Let

$$\lim_{k \to \infty} \Sigma_{q_k} = \left( \alpha^{s_h^{(i,j,l)}} \right)_{1 \le i \le m,\ 1 \le j \le n,\ 1 \le l \le t,\ h \ge 0}$$

and define

$$f_{ijlk}(\boldsymbol{z}) = \sum_{h=0}^{\infty} \alpha^{r_{lh+t![\theta_j q_k]}^{(i,j)} - r_{t![\theta_j q_k]}^{(i,j)} d_j^{lh}} z_{ij}^{d_j^{lh}}$$
$$(1 \le i \le m,\ 1 \le j \le n,\ 1 \le l \le t,\ k \ge 1)$$

and

$$f_{ijl}(\boldsymbol{z}) = \sum_{h=0}^{\infty} \alpha^{s_h^{(i,j,l)}} z_{ij}^{d_j^{lh}} \quad (1 \le i \le m,\ 1 \le j \le n,\ 1 \le l \le t).$$

Then

$$\lim_{k \to \infty} f_{ijlk}(\boldsymbol{z}) = f_{ijl}(\boldsymbol{z}).$$

Define the $mn \times mn$ matrix

$$\Omega^{(k)} = \operatorname{diag}\left([\omega_1 b_1^{[\theta_1 q_k]}], \ldots, [\omega_m b_1^{[\theta_1 q_k]}], \ldots, [\omega_1 b_n^{[\theta_n q_k]}], \ldots, [\omega_m b_n^{[\theta_n q_k]}]\right).$$

We assert first that $\{\Omega^{(k)}\}_{k\geq 1}$, $\boldsymbol{\alpha} = (\alpha, \ldots, \alpha, \ldots, \alpha, \ldots, \alpha)$, and $\rho_k = b_1^{q_k}$ $(k \geq 1)$ satisfy the assumptions (I) and (II) of Lemma 1. Since $b_1 > \cdots > b_n$, we have

$$b_1^{q_k - 1} \leq b_j^{-1} b_1^{q_k} < b_j^{[\theta_j q_k]} \leq b_1^{q_k}$$

and so

$$\frac{1}{2}\left(\min_{1\leq i\leq m} \omega_i\right) b_1^{q_k-1} \leq \left(\min_{1\leq i\leq m} \omega_i\right) b_1^{q_k-1} - 1 < [\omega_i b_j^{[\theta_j q_k]}] \leq b_1^{q_k} \max_{1\leq i\leq m} \omega_i$$

for any $i$ $(1 \leq i \leq m)$, $j$ $(1 \leq j \leq n)$, and for all $k \geq 1$, if $q_1$ is sufficiently large. Hence the assumption (I) is satisfied.

Let $K = \mathbb{Q}(\alpha)$. Then $f_{ijlk}(\boldsymbol{z}) \in K[[\boldsymbol{z}]]$ $(1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq l \leq t,\ k \geq 0)$ and

$$f_{ijlk}(\Omega^{(k)}\boldsymbol{\alpha}) = \sum_{h=0}^{\infty} \alpha^{r^{(i,j)}_{lh+t![\theta_j q_k]}} = f_{ijl0}(\boldsymbol{\alpha}) - \sum_{h=0}^{(t!/l)[\theta_j q_k]-1} \alpha^{r^{(i,j)}_{lh}}$$
$$(1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq l \leq t,\ k \geq 1).$$

Since $r^{(i,j)}_{l(k+1)} > r^{(i,j)}_{lk}$ $(1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq l \leq t)$ for all sufficiently large $k$ by the definition, there is a positive constant $C$ such that $\max_{0\leq h\leq k-1} r^{(i,j)}_{lh} \leq C r^{(i,j)}_{lk}$ $(1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq l \leq t)$ for all $k \geq 1$. Hence

$$\begin{aligned}\log\left\|-\sum_{h=0}^{(t!/l)[\theta_j q_k]-1} \alpha^{r^{(i,j)}_{lh}}\right\| &\leq \log(t!/l)[\theta_j q_k] + \left(\max_{0\leq h\leq (t!/l)[\theta_j q_k]-1} r^{(i,j)}_{lh}\right)\log\|\alpha\| \\ &\leq \left(1 + C(\max_{1\leq i\leq m}\omega_i)\log\|\alpha\|\right)\rho_k,\end{aligned}$$

and the assumption (II) is satisfied.

Therefore, if the assumption (III) is also satisfied, the proof is completed. Noting that $z_{11}, \ldots, z_{m1}, \ldots, z_{1n}, \ldots, z_{mn}$ are distinct variables, we see by Lemma 5 that the functions $f_{ijl}(\boldsymbol{z})$ $(1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq l \leq t)$ are algebraically independent over $\mathbb{C}(z_{11}, \ldots, z_{m1}, \ldots, z_{1n}, \ldots, z_{mn})$. Let

$$F(\boldsymbol{z}) = \sum_{\mu=(\mu_{ij}),\nu=(\nu_{ijl})} a_{\mu,\nu}\boldsymbol{z}^{\mu} f_{111}^{\nu_{111}} \cdots f_{mnt}^{\nu_{mnt}} = \sum_{\lambda=(\lambda_{ij})\in(\mathbb{N}_0)^{mn}} c_\lambda \boldsymbol{z}^{\lambda},$$

where the coefficients $a_{\mu,\nu}$ are not all zero, and let $\lambda_0 = (\lambda^{(0)}_{ij})$ be the minimal element in $(\mathbb{N}_0)^{mn}$ with respect to the total order $\succ$ defined in Lemma 4 among $\lambda$ with $c_\lambda \neq 0$. Let

$l = 2(|\lambda_0| + 1)\left(\left[\frac{\max_{1\le i\le m}\omega_i}{\min_{1\le i\le m}\omega_i}\right] + 1\right) b_1$. If $k$ is sufficiently large, then by Lemma 2

$$\begin{aligned}&\sum_{|\lambda|\ge l} |c_\lambda| \cdot |\alpha|^{\lambda_{11}[\omega_1 b_1^{[\theta_1 q_k]}]} \cdots |\alpha|^{\lambda_{m1}[\omega_m b_1^{[\theta_1 q_k]}]} \cdots |\alpha|^{\lambda_{1n}[\omega_1 b_n^{[\theta_n q_k]}]} \cdots |\alpha|^{\lambda_{mn}[\omega_m b_n^{[\theta_n q_k]}]} \\ \le\ & \gamma^{l+1}\left(|\alpha|^{\frac{1}{2}(\min_{1\le i\le m}\omega_i) b_1^{q_k - 1}}\right)^l \\ \le\ & \gamma^{l+1}|\alpha|^{(\max_{1\le i\le m}\omega_i) b_1^{q_k}(|\lambda_0|+1)}.\end{aligned}$$

Since

$$\begin{aligned}&\lambda_{11}^{(0)}[\omega_1 b_1^{[\theta_1 q_k]}] + \cdots + \lambda_{m1}^{(0)}[\omega_m b_1^{[\theta_1 q_k]}] + \cdots + \lambda_{1n}^{(0)}[\omega_1 b_n^{[\theta_n q_k]}] + \cdots + \lambda_{mn}^{(0)}[\omega_m b_n^{[\theta_n q_k]}] \\ \le\ & |\lambda_0|(\max_{1\le i\le m}\omega_i) b_1^{q_k},\end{aligned}$$

we have

$$\frac{|\sum_{|\lambda|\ge l} c_\lambda(\Omega^{(k)}\boldsymbol{\alpha})^\lambda|}{|(\Omega^{(k)}\boldsymbol{\alpha})^{\lambda_0}|} \le \gamma^{l+1}|\alpha|^{(\max_{1\le i\le m}\omega_i) b_1^{q_k}}$$

if $k$ is sufficiently large. If $|\lambda| < l$ and $\lambda \ne \lambda_0$, then by Lemma 4

$$\frac{|c_\lambda(\Omega^{(k)}\boldsymbol{\alpha})^\lambda|}{|(\Omega^{(k)}\boldsymbol{\alpha})^{\lambda_0}|} \le |c_\lambda| \cdot |\alpha|^{\delta(l) b_1^{q_k}}$$

for all sufficiently large $k$. Therefore

$$|F(\Omega^{(k)}\boldsymbol{\alpha})/(\Omega^{(k)}\boldsymbol{\alpha})^{\lambda_0} - c_{\lambda_0}| \to 0 \quad (k \to \infty),$$

which implies (III), and the proof of the theorem is completed.

*Proof of Theorem* 4. Define

$$g_{id}(z) = \sum_{k=0}^{\infty} \alpha^{[\omega_i d^k + \eta_i] - [\omega_i d^k]} z^{[\omega_i d^k]} \quad (i = 1, \ldots, m;\ d = 2, 3, 4, \ldots).$$

Then

$$\alpha^{[\omega_i d^k + \eta_i] - [\omega_i d^k]} \in \{\alpha^{[\eta_i]}, \alpha^{[\eta_i]+1}\},$$

since $0 \le [\omega_i d^k + \eta_i] - [\omega_i d^k] - [\eta_i] \le 1$ for any $i$, $d$, and for all $k$. By Theorem 3 the numbers $g_{id}(\alpha)$ $(i = 1, \ldots, m;\ d = 2, 3, 4, \ldots)$ are algebraically independent, which implies the theorem.

# References


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