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The quarkonial characterization of weighted spaces and its application

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1 Introduction

In this paper we study the quarkonial characterization of weighted Sobolev spaces on a bounded domain and its application to a generalization of the Sobolev-Lieb-Thirring inequality and the estimate of the Hausdorff dimension of the attractor of a nonlinear equation.

The decomposition of a function on a domain is studied by several mathematicians. The atomic decomposition of a function is an example of such decomposition. But in general the coefficients of the atomic decomposition are not linear functionals of the function. Hence we use the quarkonial decomposition of a function which was investigated by Triebel([7]). In the quarkonial decomposition the coefficients of the decomposition are linear functionals of the function and given by the L^2-inner product with suitable functions. We need this property in applications to a generalization of the Sobolev-Lieb-Thirring inequality on a domain.

In Section 2 we consider the quarkonial characterization of weighted Sobolev spaces on $\mathbb{R}^n$ with A_2-weight. In Section 3 we give results on the quarkonial characterization of weighted Sobolev spaces on a bounded domain Ω. In Sections 4 and 5 we shall give some applications.

We remark that Triebel also studied about the quarkonial characterization of weighted spaces with C^∞ weights which satisfy some conditions. Our method is a modification of his results.

2 The quarkonial characterization of $H^s(w)$ and $L^2(w)$

Let w be a nonnegative locally integrable function on $\mathbb{R}^n$. For $1 < p < \infty$ we say that w is an A_p-weight, that is $w \in A_p$, if $w^{-1/(p-1)}$ is locally integrable on $\mathbb{R}^n$ and w

satisfies the inequality

$$\frac{1}{|Q|}\int_Q w\,dx\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}dx\right)^{p-1} \le C$$

for all cubes $Q \subset \mathbb{R}^n$. An example of A_p-weight is given by $w(x) = |x|^\alpha \in A_p$ where $-n < \alpha < n(p-1)$.

Let $w \in A_2$ and $s > 0$. We define $H^s(w)$ as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{H^s(w)} = \left\{\int_{\mathbb{R}^n} (|(-\Delta)^{s/2} f(x)|^2 + |f(x)|^2) w(x)\,dx\right\}^{1/2},$$

where we define via inverse Fourier transform

$$(-\Delta)^{s/2} f(x) = \mathcal{F}^{-1}(|\xi|^s \hat{f})(x).$$

Let $\psi \in C^\infty(\mathbb{R}^n)$ be a function such that $\psi \ge 0$,

(1) $$\operatorname{supp}\psi \subset \{x \in \mathbb{R}^n : |x| < 2^r\}$$

for some $r \ge 0$, and

$$\sum_{k\in\mathbb{Z}^n} \psi(x-k) = 1 \quad \text{if} \quad x \in \mathbb{R}^n.$$

For $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, k \in \mathbb{Z}^n$, and $\beta \in \mathbb{N}_0^n$ we set

$$\psi^\beta(x) = x^\beta \psi(x) \quad \text{and} \quad \psi_{jk}^\beta(x) = 2^{nj/2}\psi^\beta(2^j x - k).$$

We call $\psi_{jk}^\beta(x)$ a quark(c.f.[7]).

For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$ we define the cube

$$Q_{jk} = \{\,(x_1, \dots, x_n) : \frac{k_i}{2^j} \le x_i < \frac{k_i}{2^j} + \frac{1}{2^j},\ i = 1, \dots, n\}.$$

We can prove the following quarkonial characterization of $H^2(w)$.

Theorem 2.1 *Let $w \in A_2$, $\rho > r$, and $s > 0$. Then there exist $\Psi_{jk}^\beta \in \mathcal{S}(\mathbb{R}^n)$ for $j \in \mathbb{N}_0, k \in \mathbb{Z}^n, \beta \in \mathbb{N}_0^n$ such that for all $f \in H^s(w)$*

$$f = \sum_{\beta\in\mathbb{N}_0^n}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} (f, \Psi_{jk}^\beta)\psi_{jk}^\beta$$

in $H^s(w)$ and

$$\|f\|_{H^s(w)}^2 \approx \sup_{\beta\in\mathbb{N}_0^n} 2^{2\rho|\beta|} \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} 2^{2sj}|(f, \Psi_{jk}^\beta)|^2 \frac{1}{|Q_{jk}|}\int_{Q_{jk}} w\,dx.$$

Next we give a characterization of $L^2(w)$.

For $w \in L^1_{loc}(\mathbb{R}^n)$, $w \geq 0$, we define

$$L^2(w) = \left\{ f : \|f\|^2_{L^2(w)} = \int_{\mathbb{R}^n} |f|^2 w \, dx < \infty \right\}.$$

For $j \in \mathbb{N}_0, k \in \mathbb{Z}^n$, and $\beta \in \mathbb{N}_0^n$ we set

$$\psi^{\beta,1}_{jk}(x) = 2^{nj/2}((-\Delta)\psi^\beta)(2^j x - k).$$

Theorem 2.2 *Let $w \in A_2$ and $\rho > r$. Then there exist $\Psi^\beta_{jk}, \Psi^{\beta,1}_{jk} \in \mathcal{S}(\mathbb{R}^n)$ for $j \in \mathbb{N}_0, k \in \mathbb{Z}^n, \beta \in \mathbb{N}_0^n$ such that for all $f \in L^2(w)$*

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \{ (f, \Psi^\beta_{jk}) \psi^\beta_{jk} + (f, \Psi^{\beta,1}_{jk}) \psi^{\beta,1}_{jk} \}$$

in $L^2(w)$ and

$$\|f\|^2_{L^2(w)} \approx \sup_{\beta \in \mathbb{N}_0^n} 2^{2\rho|\beta|} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \{ 2^{4j} |(f, \Psi^\beta_{jk})|^2 + |(f, \Psi^{\beta,1}_{jk})|^2 \} \frac{1}{|Q_{jk}|} \int_{Q_{jk}} w \, dx.$$

For the proof of Theorems 2.1 and 2.2 we use the characterization of $H^s(w)$ by Frazier-Jawerth's φ-transform([2]) and a modification of the argument in [7].

3 The quarkonial characterization of $H^1_0(\Omega, w)$ and $L^2(\Omega, w)$

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded C^2-domain. We define

$$\tilde{A}_p(\Omega) = \{ w \in L^1_{loc}(\Omega) \ : \ \exists w' \in A_p, \ \ w = w' \text{ on } \Omega \}.$$

For $w \in \tilde{A}_2(\Omega)$ we define $H^1_0(\Omega, w)$ as the completion of $C^\infty_0(\Omega)$ with respect to the norm

$$\|f\|_{H^1_0(\Omega,w)} = \left\{ \int_\Omega (|\nabla f(x)|^2 + |f(x)|^2) w(x) \, dx \right\}^{1/2}.$$

Let $d(x) = \text{dist}\,(x, \partial\Omega)$. We consider the following condition.

Condition (C) There exists a $c > 0$ such that

$$\int_\Omega \frac{|f(x)|^2}{d(x)^2} w(x)\,dx \le c \int_\Omega |\nabla f(x)|^2 w(x)\,dx$$

for all $f \in C_0^\infty(\Omega)$.

We give some examples of weights which satisfy the condition (C).

Example

1. $w(x) = d(x)^a$ where $0 \le a < 1$.
2. $w \in \tilde{A}_2(\Omega)$ where there exist positive constants c_1, c_2 and c_3 such that

$$0 < c_1 \le w(x) \le c_2 \quad \text{for all } x \in \Omega \text{ satisfying } \operatorname{dist}(x, \partial\Omega) \le c_3.$$

Let r be the number in (1). We can prove that there exist $c_1, c_2 > 0$ and $J \in \mathbb{N}_0$ which satisfy the following conditions. For all $j \in \mathbb{N}_0, j \ge J$, there exist lattices

$$\{\frac{\ell_k}{2^j} : \ell_k \in \mathbb{Z}^n,\ k = 1, \dots, M_j\} \subset \Omega$$

such that

$$B_{jk} = \{x : |x - \ell_k 2^{-j}| < 2^{r-j}\} \subset \Omega, \quad \operatorname{dist}(B_{jk}, \partial\Omega) \ge c_1 2^{-j},$$

and

$$\sum_{k=1}^{M_j} \psi(2^j x - \ell_k) = 1 \quad \text{for all } \ x \in \Omega \ \text{ such that } \ \operatorname{dist}(x, \partial\Omega) \ge c_2 2^{-j}$$

For $j \in \mathbb{N}_0, j \ge J, k = 1, \dots, M_j$, $\beta \in \mathbb{N}_0^n$ we define

$$\tilde{\psi}_{jk}^\beta(x) = 2^{nj/2} \psi^\beta(2^j x - \ell_k)$$

and $\tilde{Q}_{jk} = Q_{j\ell_k}$.

We have the following quarkonial characterization of $H_0^1(\Omega, w)$.

Theorem 3.1 *Let $\rho > r$, $w \in \tilde{A}_2(\Omega)$ and w satisfy the condition (C). Then there exist $\tilde{\Psi}_{jk}^\beta \in C_0^\infty(\Omega)$ for $\beta \in \mathbb{N}_0^n, j \in \mathbb{N}_0, j \ge J, k = 1, \dots, M_j$, such that for all $f \in H_0^1(\Omega, w)$*

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{j=J}^{\infty} \sum_{k=1}^{M_j} (f, \tilde{\Psi}_{jk}^\beta) \tilde{\psi}_{jk}^\beta$$

in $H_0^1(\Omega, w)$ and

$$\|f\|_{H_0^1(\Omega,w)}^2 \approx \sup_{\beta \in \mathbb{N}_0^n} 2^{2\rho|\beta|} \sum_{j=J}^{\infty} \sum_{k=1}^{M_j} 2^{2j} |(f, \tilde{\Psi}_{jk}^\beta)|^2 \frac{1}{|\tilde{Q}_{jk}|} \int_{\tilde{Q}_{jk}} w\,dx.$$

For $w \in L^1_{loc}(\Omega)$, $w \geq 0$ we define

$$L^2(\Omega, w) = \left\{ f : \ \|f\|^2_{L^2(\Omega,w)} = \int_\Omega |f|^2 w\, dx < \infty \right\}.$$

For $j \in \mathbb{N}_0, j \geq J, k = 1, \ldots, M_j$, $\beta \in \mathbb{N}_0^n$ we set

$$\tilde{\psi}^{\beta,1}_{jk}(x) = 2^{nj/2}((-\Delta)\psi^\beta)(2^j x - \ell_k).$$

Theorem 3.2 *Let $w \in \tilde{A}_2(\Omega)$ and $\rho > r$. Then there exist $\tilde{\Psi}^\beta_{jk}, \tilde{\Psi}^{\beta,1}_{jk} \in C_0^\infty(\Omega)$ for $\beta \in \mathbb{N}_0^n, j \in \mathbb{N}_0, j \geq J, k = 1, \ldots, M_j$, such that for all $f \in L^2(\Omega, w)$*

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{j=J}^{\infty} \sum_{k=1}^{M_j} \{(f, \tilde{\Psi}^\beta_{jk})\tilde{\psi}^\beta_{jk} + (f, \tilde{\Psi}^{\beta,1}_{jk})\tilde{\psi}^{\beta,1}_{jk}\}$$

in $L^2(\Omega, w)$ and

$$\|f\|^2_{L^2(\Omega,w)} \approx \sup_{\beta \in \mathbb{N}_0^n} 2^{2\rho|\beta|} \sum_{j=J}^{\infty} \sum_{k=1}^{M_j} \{2^{4j}|(f, \tilde{\Psi}^\beta_{jk})|^2 + |(f, \tilde{\Psi}^{\beta,1}_{jk})|^2\} \frac{1}{|\tilde{Q}_{jk}|} \int_{\tilde{Q}_{jk}} w\, dx.$$

For the proof of Theorems 3.1 and 3.2 we use the Whitney decomposition of Ω, the localization of $\|f\|_{H^1_0(\Omega,w)}$, and Theorems 2.1 and 2.2.

4 A generalization of the Sobolev-Lieb-Thirring inequality

In 1976 Lieb and Thirring proved the following Sobolev-Lieb-Thirring inequality([3]).

Theorem 4.1 *Let $n \in \mathbb{N}$. Then there exists a positive constant $c = c(n)$ such that for every family $\{f_i\}_{i=1}^N$ in $H^1(\mathbb{R}^n)$ which is orthonormal in $L^2(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} \rho(x)^{1+2/n}\, dx \leq c \sum_{i=1}^N \int_{\mathbb{R}^n} |\nabla f_i(x)|^2\, dx$$

where

$$\rho(x) = \sum_{i=1}^N |f_i(x)|^2.$$

The Sobolev-Lieb-Thirring inequality has applications to the stability of matter or the estimates of the Hausdorff dimension of the attractor of nonlinear equations.

We can prove the following generalization by the quarkonial characterization of $H_0^1(\Omega, w)$.

Theorem 4.2 *Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded C^2-domain. Suppose that $w \in \tilde{A}_2(\Omega)$, $w^{-n/2} \in \tilde{A}_{n/2}(\Omega)$ and w satisfies the condition (C). Then there exists a positive constant c such that for every family $\{f_i\}_{i=1}^N$ in $L^2(\Omega) \cap H_0^1(\Omega, w)$ which is orthonormal in $L^2(\Omega)$, we have*

$$\int_\Omega \rho(x)^{1+2/n} w(x)\, dx \leq c \sum_{i=1}^N \int_\Omega |\nabla f_i(x)|^2 w(x)\, dx$$

where

$$\rho(x) = \sum_{i=1}^N |f_i(x)|^2.$$

Our method of the proof is a modification of results in [1] and [5].

5 Estimate of the Hausdorff dimension of the attractor of a nonlinear equation

In [6] applications of the Sobolev-Lieb-Thirring inequality to the problem of the estimate of the dimension of the attractor of nonlinear equations are explained. By the method in [6] we give an application of Theorem 4.2 to a nonlinear equation. The equation considered in this section is an example given in [6].

Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded C^2-domain. Suppose that $w \in \tilde{A}_2(\Omega)$, $w^{-n/2} \in \tilde{A}_{n/2}(\Omega)$,

$$c|Q|^{2/n} \leq \frac{1}{|Q|} \int_Q w\, dx$$

for all cubes $Q \subset \Omega$, and w satisfies the condition (C). Let

$$g(s) = \sum_{j=0}^{2p-1} b_j s^j, \quad b_j \in \mathbb{R},\ b_{2p-1} > 0$$

$$\kappa_1 \geq 0, \quad g'(s) \geq -\kappa_1, \quad \text{for all } s \in \mathbb{R}.$$

For $d > 0$ and $u = u(x,t)$ we consider the equation

$$\frac{\partial u}{\partial t} - d\sum_{i=1}^{n} \partial_{x_i}(w(x)\partial_{x_i}u) + g(u) = 0 \quad \text{in } \Omega \times \mathbb{R}_+,$$

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times \mathbb{R}_+,$$

$$u(x,0) = u_0(x) \quad x \in \Omega.$$

We set

$$V = L^2(\Omega) \cap H_0^1(\Omega, w).$$

Theorem 5.1 *Under above notations and assumptions, for any* $u_0 \in L^2(\Omega)$, *there exists a unique solution* u *of the equation such that*

$$u \in L^2(0,T;V)$$

for all $T > 0$ *and*

$$u \in C(\mathbb{R}_+; L^2(\Omega)).$$

The mapping $u_0 \to u(\cdot,t)$ *is continuous in* $L^2(\Omega)$.

Furthermore there exists a maximal attractor $\mathcal{A}$ *which is bounded in* V, *compact and connected in* $L^2(\Omega)$. *Let* m *be the integer such that*

$$m - 1 < c\left(\frac{\kappa_1}{d}\right)^{n/2} \int_\Omega w^{-n/2}\, dx \le m.$$

Then the Hausdorff dimension of $\mathcal{A}$ *is less than or equal to* m.

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