数理解析研究所講究録 1386 巻 2004 年 1-19

Operator-theoretic renormalization group and aspects of the infrared problem in non-relativistic QED

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1 Introduction

One of the key difficulties in the study of non-relativistic QED is the appearance of infrared singularities in the computation of many fundamentally important quantities, such as scattering amplitudes (if computed naively), which originate from the fact that the photon has no mass. The link between zero photon mass, and the infrared pathologies can be explained as follows. By Planck's law, the energy of the photon is proportional to its frequency, and can be arbitrarily small (the lower bound on the kinetic energy for any relativistic particle is given by its rest mass, which is zero in the case of the photon). As a consequence, electrons always form an energetically favorable bound state with an infinite number of low frequency (soft, infrared) photons of small total energy (of order $O(g^2)$, where g is the electron charge, considered as a small parameter), thus establishing a so-called *infraparticle state*. However, the canonical quantization of classical non-relativistic electrodynamics yields a quantum field theory in which the electrons are stricly distinguished from the photons.

The infraparticle state is in most cases *not* a vector in the usual product Hilbert space of the electron L^2-space with the photon Fock space obtained from the canonical quantization procedure, but an element of a so-called infrared representation Hilbert space that is unitarily inequivalent to it. In the case of confined particles, the infrared problem is reduced, due to the localization of the electron wave function, whereas in the case of free electrons, the infrared problems are worst. Although many non-measurable quantities are infrared divergent, it is important to note that measurable quantities, such as scattering cross sections or the infraparticle mass, are infrared finite, and can be computed by a limiting process, in which an artificial infrared regularization in the theory is removed.

For the historical development of the study of infrared problems in QED, we refer to [11, 44, 18, 33, 19, 42, 21, 22, 23]. Among a great number of the recent works in this direction, we in particular mention the works by V. Bach, J. Fröhlich, I. M. Sigal, [3, 4], M. Griesemer, E. H. Lieb, M. Loss, [25], A. Pizzo [39], E. H. Lieb and M. Loss, [35], and H. Spohn et al [36, 37, 41, 32], and furthermore, [24, 30, 1].

Here, we report on [15, 16], which focus on the translation invariant system consisting of a freely propagating electron in $\mathbb{R}^3$ that interacts with the quantized electromagnetic field, and the associated problems of infrared renormalization. We aim at analyzing properties of the infraparticle states, and of the corresponding eigenenergies.

Our method uses the operator-theoretic renormalization group introduced by V. Bach, J. Fröhlich, and I. M. Sigal, [3, 4]. To the same degree as we are interested in furthering our understanding of non-relativistic QED, we are focused on further developing the operator-theoretic renormalization group as a method in functional analysis, [7]. The mathematically rigorous theory of renormalization in quantum field theory, [12, 29], and renormalization group, [10, 13, 20, 40], has a long and successful history, originating in the groundbreaking work of K. Wilson, [43]. Most known methods are tailored for the renormalization of the n-point functions in a quantum field theory, from which the scale dependence of the important physical parameters can be extracted. This is physically satisfying, but one may wish for additional mathematical, structural insight. One would for instance desire a more direct link between the study of a quantum field theory to the traditional theory of Schrödinger operators as a branch of functional analysis.

This is precisely the motivation and impact of the new renormalization group method of Bach, Fröhlich and Sigal, [3, 4, 7]. It is designed for the spectral analysis of quantum field theoretic Hamiltonians, to study questions about the location of spectrum and resonances, about the spectral type in a given spectral interval, about the constructive determination of eigenvalues and the corresponding eigenvectors, etc.. Furthermore, it requires only a very mild combinatorial effort, since the key task is to control relative operator bounds, rather than explicitly evaluating Feynman amplitudes.

2 Definition of the model and statement of main results

We shall here introduce the Pauli-Fierz model for a free electron that interacts with a quantized electromagnetic radiation field, described in the Coulomb gauge. The Hilbert space of states is given by

$$\mathcal{H} = \mathcal{H}_{el} \otimes \mathcal{F} \, ,$$

where $\mathcal{H}_{el} = L^2(\mathbb{R}^3)$ is the Hilbert space accounting for a scalar electron. The Hilbert space of states accounting for the quantized electromagnetic field is given by the Fock space

$$\begin{aligned} \mathcal{F} &= \oplus_{n\geq 0}\mathcal{F}_n \, , \\ \mathcal{F}_n &= \mathrm{Sym}\Big[\Big(L^2(\mathbb{R}^3)\otimes \mathbb{C}^2\Big)^{\otimes n}\Big] \, , \end{aligned}$$

where $\mathcal{F}_n$ is the totally symmetrized n-photon Hilbert space, with $\mathbb{C}^2$ accounting for the two possible polarizations of the photon. We choose a basis of polarization vectors, with indices $+$ or $-$. For $\lambda \in \{+,-\}$ and $f \in L^2(\mathbb{R}^3)$, we introduce creation operators $a^*_\lambda(f) : \mathcal{F}_n \rightarrow \mathcal{F}_{n+1}$ and annihilation operators $a_\lambda(f) : \mathcal{F}_n \rightarrow \mathcal{F}_{n-1}$ on $\mathcal{F}$, which satisfy the canonical commutation relations

$$\big[a_\lambda(f), a^*_\lambda(f')\big] = \langle f, f'\rangle_{L^2} \; , \quad \big[a^\sharp_\lambda(f), a^\sharp_\lambda(f')\big] = 0 \, ,$$

for all $f, f' \in L^2(\mathbb{R}^3)$. Furthermore, there exists a unique unit ray $\Omega_f \in \mathcal{F}$, the Fock vacuum, with $a_\lambda(f)\Omega_f = 0$ for all $f \in L^2(\mathbb{R}^3)$, and $\lambda = \pm$. This defines the operator-valued distributions $a^\sharp_\lambda(k)$, with $k \in \mathbb{R}^3$, such that $a^\sharp_\lambda(f) = \int d^3k f(k) a^\sharp_\lambda(k)$. In second quantized representation,

$$H_f = \sum_\lambda \int dk|k|a^*_\lambda(k)a_\lambda(k) \;\; , \quad P_f = \sum_\lambda \int dk k a^*_\lambda(k)a_\lambda(k)$$

are the Hamilton and the momentum operator of the free photon field

To use the translation invariance of the model, we decompose $\mathcal{H}$ into a direct integral $\mathcal{H} = \int^\oplus_{\mathbb{R}^3} dp\mathcal{H}_p$, where $\mathcal{H}_p$ is the fibre Hilbert space corresponding to conserved total momentum $p \in \mathbb{R}^3$. Every $\mathcal{H}_p$ is isomorphic to $\mathbb{C}^2 \otimes \mathcal{F}$, and invariant under time and space translations. The Hamiltonian of the system can likewise be decomposed into $H(\kappa) = \int^\oplus dpH(p,\kappa)$ on $\mathcal{H}$, where the fibre Hamiltonian on $\mathcal{H}_p$ is given by

$$H(p,\kappa) = \frac{1}{2}\Big(p - P_f - gA_\kappa\Big)^2 + H_f \, .$$

Here, g is the electron charge, σ is the vector of Pauli matrices, and

$$A_\kappa \;\; = \;\; \sum_\lambda \int \frac{dk}{|k|^{1/2}}\chi(\kappa < |k| < 1)\Big\{\epsilon_\lambda(k)a_\lambda(k) + h.c.\Big\} \, .$$

A_κ denotes infrared and ultraviolet regularized quantized electromagnetic vector. The value of $0 < \kappa \ll 1$ can be chosen arbitrarily small, and the polarization vectors $\epsilon_+(k)$, $\epsilon_-(k)$ together with $k \in \mathbb{R}^3$ form an orthogonal basis, for all $k \neq 0$, since we are using the Coulomb gauge. We note that therefore, $P_f A_\kappa = A_\kappa P_f$.

2.1 The main theorem

The main results of [15] characterize the infimum of the spectrum of the fibre Hamiltonian $H(p,\kappa)$. We prove that it consists of a non-degenerate eigenvalue for all $\kappa > 0$. The associated ground state eigenvector $\psi(p,\kappa) \in \mathcal{H}_p$ is a so-called *infraparticle state*, which is a bound state composed from the electron and an infinite number of very low frequency (soft) photons of small total energy. In particular, we prove bounds on the *renormalized infraparticle mass* at fixed conserved momentum p, which are *uniform* in the infrared cutoff κ, as $\kappa \to 0$. We note that due to the absence of positron production in non-relativistic QED, there is no renormalization of g.

Theorem 2.1 *Assume that $g > 0$ is sufficiently small, and that $|p| \leq \frac{1}{20}$. Then, for any $\kappa > 0$,*

$$E(p,\kappa) := \inf \operatorname{spec}_{\mathcal{F}}\Big(H(p,\kappa)\Big)$$

is a non-degenerate eigenvalue. Let $\Omega(p,\kappa) \in \mathcal{F}$ denote its corresponding eigenvector, with normalization condition $\langle \Omega(p,\kappa), \Omega^{(el)} \otimes \Omega_f \rangle = 1$. Then,

$$\begin{aligned} \|\Omega(p,\kappa)\|_{\mathcal{F}} &\sim e^{cg^2p^2|\log\kappa|}, & (1)\\ \Big|\partial^a_{|p|}\Big(E(p,\kappa) - \frac{p^2}{2}\Big)\Big| &\leq cg & \\ \partial^2_{|p|}E(p,\kappa) &< 1 & (2) \end{aligned}$$

for $0 \leq a \leq 2$, and all $\kappa \geq 0$, where all constants c_i are independent of g, $|p|$, and in particular κ.

The upper bounds on $|p|$ can be improved, but not beyond a critical value below 1. This is connected to the fact that if $|p|$ approaches the rest energy of the infraparticle, the infraparticle tends to reduce its kinetic energy by the emission of Cherenkov radiation. It is thus expected that the eigenvalue $E(p,\kappa)$ disappears in this limit, and that instead, a resonance emerges.

The second derivative of $E(p,\kappa)$ with respect to $|p|$ determines the *renormalized infraparticle mass*

$$m(p,\kappa) = \big(\partial^2_{|p|}E(p,\kappa)\big)^{-1}..$$

The key novelty in theorem 2.1 is the *uniformity* of the bounds on $m(p,\kappa)$ with respect to κ, even in the case $p \neq 0$. The cloud of soft photons *increases* the mass, in comparison to the naked mass of the electron.

The vector $\Omega(p,\kappa)$ on $\mathcal{H}_p \cong \mathcal{F}$ represents an *infraparticle state*, consisting of the electron in a bound state with an *infinite* number of soft photons with a small total energy. The divergent bounds in (1) as $\kappa \to 0$ hint to the inexistence of a ground state for all $p \neq 0$. Only in the case $|p| = 0$, $\Psi(0, \kappa)$ converges to an element of $\mathbb{C}^2 \otimes \mathcal{F}$ in the limit $\kappa \to 0$.

This is an instance of the infamous infrared in QED. The deeper structure of this problem has been clarified in the work of J. Fröhlich, [21, 22]. Let $\mathfrak{A}$ denote the $*$-algebra generated by $\{1, a^*(f, \lambda), a(g, \lambda)\}$ for $f, g \in L^2(\mathbb{R}^3)$ and $\lambda \in \{+, -\}$. A state on $\mathfrak{A}$ is a linear functional $\omega : \mathfrak{A} \to \mathbb{C}$ that is positive, $\omega(A^*A) \geq 0$ for all $A \in \mathfrak{A}$, and normalized, $\omega(\mathbf{1}) = 1$. For fixed κ and p, let $\omega_{p,\kappa}$ denote the vector state defined by

$$\omega_{p,\kappa} : \mathfrak{A} \to \mathbb{C} \ , \ \ A \mapsto \big\langle \Psi(p; \kappa), A\Psi(p; \kappa) \big\rangle \, .$$

For the related case of the massless Nelson model, it was proved in [21] that $\omega_p(A) = \lim_{\kappa \to 0} \omega_{p,\kappa}(A)$ is well-defined for all $A \in \mathfrak{A}$, and all $|p|$ sufficiently small. The GNS construction, [21], corresponding to ω_p yields an *infrared representation Hilbert space* $\mathcal{H}_p^{(IR)}$. If $|p| > 0$, the latter carries a representation of the CCR algebra that is unitarily inequivalent to the Fock representation. The same fact is expected to hold for the present system.

2.2 Structure of the proof

The proof uses an extension of the operator-theoretic renormalization group based on the *smooth Feshbach map* of V. Bach, J. Fröhlich and I. M. Sigal, [7, 8, 15]. One considers a certain Banach space $\mathcal{W}$ of generalized Wick kernels, and an embedding H of $\mathcal{W}$ into the bounded operators acting on the Hilbert space $\mathcal{H}_{ref} := Ran(\chi(H_f < 1)) \subset \mathcal{F}$. Furthermore, one makes a careful choice of a polydisc $\mathcal{P} \subset \mathcal{W}$, introduces a renormalization map $\mathcal{R} : \mathcal{P} \to \mathcal{P}$, and studies the dynamical system $(\mathcal{P}, \mathcal{R})$. A key property of $\mathcal{R}$ is that it is contractive on a subspace of $\mathcal{P}$ of codimension two. Using the smooth Feshbach map, one associates $H(p, \kappa)$ to an element $\underline{w}^{(0)} \in \mathcal{P}$, and considers the orbit $\{\underline{w}^{(n)}\}_{n \in \mathbb{N}_0}$ under $\mathcal{R}$ that emanates from this initial condition. In particular, all $H[\underline{w}^{(n)}]$ are mutually isospectral in the sense of the Feshbach theorem, [7]. The intersection of the critical set of $\mathcal{R}$ with this orbit corresponds to the effective Hamiltonian in the scaling limit, $H[\underline{w}^{(\infty)}]$, for which it is trivial to determine the ground state eigenvalue and eigenvector. This is because of the infrared regularization at κ, the scaling limit determines a non-interacting theory. Thus, by isospectrality of the smooth Feshbach map, one reconstructs the corresponding ground state data of $H(p, \kappa)$.

For every $\kappa > 0$, one can then in principle estimate the renormalized infraparticle mass and other quantities of interest, using the recursive bounds generated in the renormalization group iteration. However, the proof of uniform bounds as $\kappa \to 0$ is extremely difficult. In contrast to the models studied in [3, 4], which treated confined electrons in atoms and molecules, the interaction in the translation invariant model is, in the renormalization group context, purely marginal. The key result in [15] is a method to control the renormalization group flow of purely marginal operators. To describe the difficulty, let $\beta_N := \sum_{n=0}^{N} \delta\beta_n$ denote the coefficient of a strictly marginal operator, where $\delta\beta_n$ is its correction under the renormalization map passing from scale $n-1$ to n. Then, despite $0 < |\delta\beta_n| = O(\epsilon)$ with respect to some small parameter ϵ, and all n, $|\beta_N| \leq C\epsilon$, with C uniformly bounded in N. The key idea in [15] is a renormalization group subiteration that controls the almost complete cancellations in the oscillatory sum that defines β_N, based on a strong induction principle that exploits the algebraic concatenation identities satisfied by the smooth Feshbach map. Furthermore, $U(1)$ gauge invariance is used to fundamentally reduce the complexity of the problem, by identifying several a priori independent strictly marginal operators, and is implemented in the form of generalized Ward-Takahashi identities. In the context of the operator-theoretic renormalization group, they are given by an infinite hierarchy of non-perturbative first order differential identities which are preserved by the renormalization map.

2.3 Further results

A subsequent work, [16], investigates the interconnection between spatial and gauge symmetries in the physical system, and the algebraic structure of the Feshbach renormalization group. This allows for the extension of [15] to the case including electron spin. Furthermore, the analysis in [15] is simplified, and rigorously reorganized, in order to render the method more transparent, and more generally applicable.

A non-confining potential in a Schrödinger operator can become confining if the electron is coupled to the quantized electromagnetic field. In a recent collaboration with V. Vougalter and S. A. Vugalter, [17], results about enhanced binding in non-relativistic quantum electrodynamics were established for small g, and spin $\frac{1}{2}$, and the increase of binding energies due to the coupling to the photon field was proved. The first work on enhanced binding was [31], and further works are [17, 14, 28, 2]. Furthermore, in a present joint work of the PI with S. A. Vugalter and J.-M. Barbaroux, [9], binding conditions for N-electron systems were established, for clusters of $N-1$ and one electron, using results of [15] and [25]. In a beautiful work of E. H. Lieb and M. Loss, the general case was recently solved, [35].

3 Elements of the operator-theoretic RG

In this section, we introduce the *smooth Feshbach map*, [7], which generalizes the standard Feshbach map in [3, 4], by replacing projectors with smooth partitions of unity.

3.1 Feshbach Pairs and Smooth Feshbach Map

Let $\mathcal{H}$ denote a separable Hilbert space, and let $0 \le \chi \le 1$ be a selfadjoint operator which, together with $\bar{\chi} := \sqrt{1-\chi^2}$, constitutes a partition of unity, $\chi^2 + \bar{\chi}^2 = \mathbf{1}$. It is very important to note that $\mathrm{Ran}(\chi)$ and $\mathrm{Ran}(\bar{\chi})$ are in general not disjoint.

Definition 3.1 *A pair of closed operators (H, τ) acting on $\mathcal{H}$ is called a* **Feshbach pair corresponding to** χ *if it satisfies* $(\mathbf{FP_1}) \sim (\mathbf{FP_4})$, *[7].*

$(\mathbf{FP_1})$ *$Dom(H) = Dom(\tau) \subset \mathcal{H}$, and $[\chi, \tau] = 0 = [\bar{\chi}, \tau]$.*

$(\mathbf{FP_2})$ *χ and $\bar{\chi}$ map $Dom(H)$ to itself.*

$(\mathbf{FP_3})$ *Let $\omega := H - \tau$. The operators τ and $H_{\bar{\chi}} = \tau + \bar{\chi}\omega\bar{\chi}$ are bounded invertible on* $\mathrm{Ran}(\bar{\chi})$.

$(\mathbf{FP_4})$ *Let $\bar{R} := H_{\bar{\chi}}^{-1}$, and let $H_{\bar{\chi}} = U|H_{\bar{\chi}}|$ denote the polar decomposition of $H_{\bar{\chi}}$ on* $\mathrm{Ran}(\bar{\chi})$. *Then, $\bar{R}$, $|\bar{R}|^{\frac{1}{2}}U^{-1}\bar{\chi}\omega\chi$, and $\chi\omega\,\bar{\chi}|\bar{R}|^{\frac{1}{2}}$ extend to bounded operators on $\mathcal{H}$.*

The set of Feshbach pairs acting on $\mathcal{H}$ corresponding to χ is denoted by $\mathfrak{FP}(\mathcal{H}, \chi)$.
The **smooth Feshbach map** *is defined by*

$$\begin{aligned} F_\chi : \mathfrak{FP}(\mathcal{H}, \chi) &\rightarrow \mathcal{L}(\mathcal{H}) , \\ (H, \tau) &\mapsto \tau + \chi\omega\chi - \chi\omega\bar{\chi}\bar{R}\bar{\chi}\omega\chi , \end{aligned} \tag{3}$$

where $F_\chi(H,\tau)|_{Ran(\chi)} \in \mathcal{B}(Ran(\chi))$. Furthermore,

$$\begin{aligned} Q_\chi : \mathfrak{FP}(\mathcal{H}, \chi) &\rightarrow \mathcal{B}(\mathrm{Ran}(\chi), \mathcal{H}) , \\ (H, \tau) &\mapsto \chi - \bar{\chi}\bar{R}\bar{\chi}\omega\chi , \\ Q_\chi^\sharp : \mathfrak{FP}(\mathcal{H}, \chi) &\rightarrow \mathcal{B}(\mathcal{H}, \mathrm{Ran}(\chi)) , \\ (H, \tau) &\mapsto \chi - \chi\omega\bar{\chi}\bar{R}\bar{\chi} . \end{aligned} \tag{4}$$

are referred to as **intertwining maps.**

The smooth Feshbach map establishes a non-linear, isospectral map between operators on $\mathcal{H}$ and $Ran(\chi)$ in the sense of the following key theorem.

Theorem 3.1 *(Feshbach isospectrality) Assume that $(H, \tau) \in \mathfrak{FP}(\mathcal{H}, \chi)$.*

1. *H is bounded invertible on $\mathcal{H} \Longleftrightarrow F_\chi(H, \tau)$ is bounded invertible on $Ran(\chi)$.*
2. *Let $\psi \in \mathcal{H}$. Then, $H\psi = 0 \Longleftrightarrow F_\chi(H, \tau)\chi\psi = 0$.*
3. *Let $\zeta \in Ran(\chi)$. Then, $F_\chi(H, \tau)\zeta = 0 \Longleftrightarrow HQ_\chi(H, \tau)\zeta = 0$.*

3.2 A Banach Space of effective Hamiltonians

We choose a smooth partition of unity $\chi_1^2 + \bar{\chi}_1^2 = 1$ on $\mathbb{R}_+$, where $\chi_1(x) = 1$ for $x \in [0, \frac{1}{2}]$, $\text{supp}\{\chi_1\} = [0,1]$, and where χ_1 shall be monotonic. Then, we denote the cutoff operator $\chi_1[H_f]$ acting on $\mathcal{F}$ by χ_1, for brevity. We then consider the fixed Hilbert subspace

$$\mathcal{H}_{red} := Ran(\chi_1) = \mathbf{1}(H_f < 1)\mathcal{F} \subset \otimes\mathcal{F} . \tag{5}$$

We focus on a particular class of operators, referred to as *effective Hamiltonians*, characterized by

$$H = T[H_f, P_f] + \chi_1 W \chi_1 - E\chi_1^2 , \tag{6}$$

acting on $\mathcal{H}_{red}$.

The scalar $E \in \mathbb{C}$ is a spectral parameter.

The operator $T[H_f, P_f]$ is the non-interacting term in the effective Hamiltonian, defined via spectral calculus by a function $T \in C^2(I \times B_1)$, where $I = [0,1]$, and B_1 is the unit ball in $\mathbb{R}^3$. It is required to satisfy $T[0,0] = 0$, and clearly, $T[H_f, P_f]$ commutes with H_f, P_f.

We introduce the notation

$$K_i := (k_i, \lambda_i) \ , \ \tilde{K}_i := (\tilde{k}_j, \tilde{\lambda}_j) \in \mathbb{R}^3 \times \{+,-\} , \tag{7}$$

with $i = 1, \dots, M$, $j = 1, \dots, N$, and $M + N \geq 0$, and

$$\begin{aligned}
K^{(M)} &:= (K_1, \dots, K_M) \\
\tilde{K}^{(N)} &:= (\tilde{K}_1, \dots, \tilde{K}_N) \\
K^{(M,N)} &:= (K^{(M)}, \tilde{K}^{(N)}) \\
a^\sharp(K^{(M)}) &:= \prod_{i=1}^{M} a^\sharp(K_i) \\
d\mu_\kappa(K^{(M,N)}) &:= \prod_{i=1}^{M}\prod_{j=1}^{N} \frac{dK_i\, h_\kappa(|k_i|)}{|k_i|^{1/2}} \frac{d\tilde{K}_j\, h_\kappa(|\tilde{k}_j|)}{|\tilde{k}_j|^{1/2}} .
\end{aligned} \tag{8}$$

Here, $\int dK = \sum_\lambda \int dk$, and $a^\sharp(K) = a^\sharp_\lambda(k)$. Then, the interaction term in the effective Hamiltonian is given by $W = \sum_{M+N\geq 1} W_{M,N}$. The operator

$$W_{M,N}[H_f, P_f] = \int_{B_1^{M+N}} d\mu_\kappa(K^{(M,N)})\, a^*(K^{(M)})\, w_{M,N}\left[H_f, P_f; K^{(M,N)}\right]\, a^\sharp(\tilde{K}^{(N)})$$

acts on $\mathcal{H}_{red}$, and is called a *generalized Wick monomial* of degree (M, N).

The integral kernels $w_{M,N}$ are referred to as *generalized Wick kernels*, and commute with H_f, P_f. $w_{M,N}$ is fully symmetric with respect to $K_1, \dots, K_M$ and $\tilde{K}_1, \dots, \tilde{K}_N$.

Let

$$\underline{X} := (X_0, X) \in I \times B_1 \;, \;\; X = (X_1, X_2, X_3) \tag{9}$$

denote the spectral variables corresponding to (H_f, P_f). Using the multiindex $\underline{a} := (a_0, \dots, a_3)$ with $a_i \in \mathbb{N}_0$ and $|\underline{a}| := \sum_{j=0}^{3} a_j$, we shall write

$$\partial_{\underline{X}} := (\partial_{X_0}, \nabla_X) \;, \;\; \nabla_X := (\partial_{X_1}, \partial_{X_2}, \partial_{X_3}) \;, \;\; \partial_{\underline{X}}^{\underline{a}} := (\partial_{X_0}^{a_0}, \dots, \partial_{X_3}^{a_3}) \;. \tag{10}$$

We introduce the norms

$$\begin{aligned} \|w_{M,N}\|_{M,N} &:= \sup_{\underline{X} \in I \times B_1} \sup_{K^{(M,N)}} \left| w_{M,N}[\underline{X}; K^{(M,N)}] \right| \;, \\ \|w_{M,N}\|_{M,N}^{\sharp} &:= \sum_{0 \le |\underline{a}| \le 2} \|\partial_{\underline{X}}^{\underline{a}} w_{M,N}\|_{M,N} + \sup_{(k,\lambda) \in K^{(M,N)}} \|\partial_{|k|} w_{M,N}\|_{M,N} \;, \end{aligned} \tag{11}$$

and define

$$\mathfrak{W}_{M,N}^{\sharp} := \left\{ w_{M,N} \,\middle|\, \|w_{M,N}\|_{M,N}^{\sharp} < \infty \right\} \;, \tag{12}$$

which is the Banach space of generalized Wick kernels of degree (M, N).

Our next task is to accommodate sums of Wick monomials $w_{M,N}$ with $M + N \ge 1$. To this end, we choose $\xi \in (0, 1)$, and introduce the Banach space

$$\mathfrak{W}_{\ge 1}^{\sharp} := \bigoplus_{M+N \ge 1} \mathfrak{W}_{M,N}^{\sharp} \;,$$

consisting of all sequences $\underline{w} = (w_{M,N})_{M+N \ge 1}$ with

$$\|\underline{w}\|_{\xi, \ge 1}^{\sharp} := \sum_{M+N \ge 1} (2\pi^{\frac{1}{2}} \xi)^{-(M+N)} \|w_{M,N}\|_{M,N}^{\sharp} < \infty \;.$$

In the special case $M + N = 0$, we have

$$\mathfrak{W}_{0,0}^{\sharp} \;=\; \left\{ w_{0,0} \in C^2(I \times B_1) \,\middle|\, \|w_{0,0}\|_{0,0}^{\sharp} := \sum_{0 \le |\underline{a}| \le 2} \sup_{\underline{X} \in I \times B_1} |\partial_{\underline{X}}^{\underline{a}} w_{0,0}| < \infty \right\} .$$

We note that in contrast to $M + N \ge 1$, $w_{0,0}$ only possesses a scalar, but no spinor part. For the system in discussion, this suffices, due to spatial rotation and reflection invariance. The decomposition

$$\begin{aligned} w_{0,0}[\underline{X}] &= w_{0,0}[\underline{0}] \chi_1^2[X_0] + T[\underline{X}] \\ T[\underline{X}] &:= w_{0,0}[\underline{X}] - w_{0,0}[\underline{0}] \chi_1^2[X_0] \end{aligned}$$

induces a natural bijection

$$\mathfrak{W}^{\sharp}_{0,0} = \mathbb{C} \oplus \mathfrak{T}$$

with

$$\mathfrak{T} := \left\{ T \in C^2(I \times B_1) \middle| T(0,0) = 0, \|T\|_{\mathfrak{T}} := \sum_{1 \le |\underline{a}| \le 2} \sup_{\underline{X} \in I \times B_1} |\partial^{\underline{a}}_{\underline{X}} T| < \infty \right\}.$$

In our discussion, $\mathfrak{W}^{\sharp}_{0,0}$ and $\mathbb{C} \oplus \mathfrak{T}$ will not be distinguished. In accordance to our notational conventions, we have $W_{0,0}[w_{0,0}] := w_{0,0}[H_f, P_f] \in \mathcal{B}(\mathcal{H}_{red})$.

Assembling all of the above, we obtain the Banach space

$$\mathfrak{W}^{\sharp}_{\ge 0} := \bigoplus_{M+N \ge 0} \mathfrak{W}^{\sharp}_{M,N} = \mathbb{C} \oplus \mathfrak{T} \oplus \mathfrak{W}^{\sharp}_{\ge 1} \tag{13}$$

endowed with the norm

$$\|\underline{w}\|^{\sharp}_{\xi} := |w_{0,0}[\underline{0}]| + \|T\|_{\mathfrak{T}} + \|\underline{w}\|^{\sharp}_{\xi, \ge 1} . \tag{14}$$

Every sequence

$$\underline{w} := \left(E, T, \{w_{M,N}\}_{M+N \ge 1}\right) \in \mathfrak{W}^{\sharp}_{\ge 0}$$

defines an operator

$$\begin{aligned} H[\underline{w}] &= \sum_{M+N \ge 0} W_{M,N}[\underline{w}] \\ &= E\chi_1^2 + T[\underline{X}] + \sum_{M+N \ge 1} \chi_1 W_{M,N}[w_{M,N}] \chi_1 . \end{aligned} \tag{15}$$

of the form (6).

Theorem 3.2 *For any $0 < \xi < 1$, the map $H : \mathfrak{W}^{\sharp}_{\ge 0} \to \mathcal{B}(\mathcal{H}_{red})$ is an injective embedding, the subspace $H(\mathfrak{W}^{\sharp}_{\ge 0}) \subseteq \mathcal{B}(\mathcal{H}_{red})$ is closed, and $\|H[\underline{w}]\|_{op} \le \|\underline{w}\|^{\sharp}_{\xi}$.*

For the proof, we refer to [7].

3.3 A Polydisc of effective Hamiltonians

The effective Hamiltonians in our applications depend holomorphically on a spectral parameter. Let $\mathfrak{W}_{\ge 0}$ denote the Banach space of analytic functions on

$$\tilde{D} := \left\{ z \in \mathbb{C} \middle| |z - \frac{1}{10}| \le \frac{1}{10} \right\} \tag{16}$$

with values in $\mathfrak{W}^\sharp_{\geq 0}$, endowed with the norm

$$\|\underline{w}[\,\cdot\,]\|_\xi := \sup_{z\in\tilde{D}} \|\underline{w}[z]\|^\sharp_\xi \, . \tag{17}$$

The Banach space of analytic families $\tilde{D} \to H(\mathfrak{W}^\sharp_{\geq 0})$, $z \mapsto H(\underline{w}[z])$ is denoted by $\mathfrak{W}_{\geq 0}$. Whenever the dependence on the conserved momentum p is emphasized, we shall write $\underline{w}[z;p]$ for $\underline{w}[z]$.

For ϵ, $\delta \ll 1$, $\lambda < \frac{5}{8}$, $p \leq \frac{1}{20}$, and $z \in \tilde{D}$, we define the polydisc

$$\begin{aligned}\mathrm{Pol}_{g,p,\xi}(\epsilon,\delta,\lambda) := \Big\{\underline{w}[\,\cdot\,] = (E[\,\cdot\,], T[\,\cdot\,], (w_{M,N}[\,\cdot\,])_{M+N\geq 1}) \\ \text{satisfying } (\mathbf{P_1}) \sim (\mathbf{P_4})\Big\} \subset \mathfrak{W}_{\geq 0}\end{aligned}$$

($\mathbf{P_1}$) The quantity $E[z;p] = -w_{0,0}[z;p;\underline{X} = 0] \in \mathbb{C}$ is a holomorphic function of $z \in \tilde{D}$, and satisfies $|E[z;p] - z| < \epsilon$.

($\mathbf{P_2}$) $T \in \mathfrak{T}$ has the following structure. For $\underline{X} = (X_0, X) \in I \times B_1$,

$$T[z;p;\underline{X}] = X_0 + \chi_1[X_0]\Big(\beta_p[z]X^{\|} + \gamma_p[z]X^2 + \zeta_p[z;\underline{X}]\Big)\chi_1[X_0] \, , \tag{18}$$

where $X^{\|} := X \cdot n_p$.

The complex coefficients $\beta_p[z], \gamma_p[z] \in \mathbb{C}$ are holomorphic functions of $z \in \tilde{D}$. Furthermore, they transform like scalars under spatial rotations, and thus depend only on the radial part $|p|$ of p. The estimates

$$\Big|\beta_p[z] + |p|\Big| \, , \quad \Big|(\partial_{|p|}\beta_p)[z] + 1\Big| < \delta \tag{19}$$

hold, and

$$\gamma_p[z] \in [-\epsilon, \lambda] + i[-\epsilon, \epsilon] \, . \tag{20}$$

Furthermore,

$$\Big|\partial_z\beta_p[z]\Big| \, , \quad \Big|\partial_z\gamma_p[z]\Big| < \epsilon^{\frac{3}{2}} \, . \tag{21}$$

The function $\zeta_p[z;\underline{X}]$ is real analytic in $X \in B_1$, holomorphic in z. It is a higher order term, which is ϵ-small on $I_1 := \mathrm{supp}\chi_1\bar{\chi}_1$, and which satisfies some less good estimates on the spectral overlap region $I_0 := [0,1] \setminus I_1$ (for reasons we shall not elaborate upon here, this is not problematic).

(P$_3$) The generalized Wick kernels $w_{M,N}[\underline{X}; z; p; K^{(M,N)}]$, $M + N \geq 1$, are real analytic functions of $X \in B_1$, holomorphic in $z \in \tilde{D}$. Let Y stand for $|k_i|, |\tilde{k}_j|$, for $i = 1, \ldots, M$, $j = 1, \ldots, N$, or z. Then, the following bounds are satisfied.

1. The case $M + N = 1$. These will be referred to as *marginal kernels*

$$\begin{aligned} \|\partial^b_{|p|}\partial^{\underline{a}}_{\underline{X}} w_{M,N}\|_{M,N}\ ,\ \ \|\partial^b_{|p|}\partial_Y w_{M,N}\|_{M,N} &< \epsilon_0^b \epsilon^{1-b} \xi^{M+N} \\ \|1_{|X|\leq X_0 \epsilon I_0 - |k|} \partial^b_{|p|}\partial_Y w_{M,N}\|_{M,N} &< \epsilon_0^b \epsilon^{\frac{3}{2}-b} \xi^{M+N} \end{aligned} \tag{22}$$

for $0 \leq |\underline{a}| \leq 2$, and $0 \leq b \leq 1$.

2. The case $M + N \geq 2$. These will be referred to as *irrelevant kernels.*

$$\begin{aligned} \|\partial^b_{|p|}\partial^{\underline{a}}_{\underline{X}} w_{M,N}\|_{M,N} &< \epsilon_0^b \epsilon^{\frac{7}{4}-\frac{|\underline{a}|}{4}-b} \xi^{M+N} \\ \|\partial^b_{|p|}\partial_Y w_{M,N}\|_{M,N} &< \epsilon_0^b \epsilon^{\frac{3}{2}-b} \xi^{M+N} \end{aligned} \tag{23}$$

for $0 \leq |\underline{a}| \leq 2$, and $0 \leq b \leq 1$.

(P$_4$) The elements of $\underline{w}$ are interrelated by an infinite hierarchy of non-perturbative identities, the *generalized Ward-Takahashi identities.* For all $M + N \geq 0$, they link $w_{M,N}$ to $w_{M',N'}$ with $|M - M'| + |N - N'| = 1$.

3.4 The Renormalization Transformation

In this section, we define the renormalization map, [7]. It depends explicitly on a parameter $0 < \rho < 1$, which we fix to be $\rho = \frac{1}{2}$ (but for notational transparence, we will continue writing ρ).

Given $\underline{w}[z] \in \mathfrak{W}_{\geq 0}$, for $z \in \tilde{D}$, we consider the composition of the following three operations.

(F) A decimation of degrees of freedom associated to states in $\mathbb{C}^2 \otimes \mathcal{F}$ with photon energies between ρ and 1, implemented by the Feshbach pair

$$(H[\underline{w}[z]], \alpha[\underline{w}[z]]H_f) \in \mathfrak{FP}(\mathcal{H}_{red}, \chi_\rho)\ , \tag{24}$$

and the smooth Feshbach map $F_{\chi_\rho[H_f]}(H[\underline{w}[z]], \alpha[\underline{w}[z]])$ on $Ran(\chi_\rho[H_f])$. The complex function $\alpha[\underline{w}[\,\cdot\,]]$ is determined by the implicit equation

$$\alpha[\underline{w}[z]] = \left\langle \partial_{H_f} F_{\chi_\rho}(H[\underline{w}[z]], \alpha[\underline{w}[z]]H_f) \right\rangle_{\Omega_f}\ .$$

It is analytic on $\tilde{D}$, and

$$\left\{\alpha[\underline{w}[z]]\Big|z \in \tilde{D}\right\} \subset D_\epsilon(1) .$$

(S) A unitary scaling transformation, whereby

$$Ran(\chi_\rho[H_f]) \rightarrow \mathcal{H}_{red} \quad \text{and} \quad \chi_\rho[H_f] \mapsto \chi_1[H_f] ,$$

followed by multiplication with $\frac{1}{\rho\alpha[\underline{w}[z]]}$.

(E) An analytic transformation $E_{\rho,\alpha}$ of the spectral parameter $z \in \tilde{D}$ in $\underline{w}[z]$.

Using the composition (**E**) $\circ$ (**S**) $\circ$ (**F**), $H[\underline{w}[z]]$ is mapped to a *renormalized effective Hamiltonian* $H[\underline{\hat{w}}[\hat{z}]]$ acting on $\mathcal{H}_{red}$. Our specific choice of $\alpha[\underline{w}[z]]$, and of the rescaling map, have been made such that the leading marginal operator in $H[\underline{\hat{w}}[\hat{z}]]$ is again H_f, as required by the definition of the polydisc. The correspondence

$$\mathcal{R}_\rho : \underline{w}[z] \mapsto \underline{\hat{w}}[\hat{z}]$$

defines the *renormalization map.*

4 The renormalization group flow

The operator-theoretic RG corresponds to the discrete dynamical system

$$\left(\mathrm{Pol}_{g,p,\xi}(\epsilon_0, \delta_0, \lambda_0), \mathcal{R}_\rho\right) ,$$

for a suitable choice of parameters $\epsilon_0, \delta_0, \lambda_0$.

4.1 Main theorems of the operator-theoretic RG

The first step in the construction is provided by Theorem 4.1, which establishes an isospectral correspondence between the fiber Hamiltonian $H(p, \kappa)$ and an effective Hamiltonian.

Theorem 4.1 *Let $z \in \tilde{D}$, choose some small $\xi \ll 1$, and assume that $\frac{g}{\xi} \ll 1$ is suffiently small. Then, for $\tilde{e} := \frac{p^2}{2} + g^2\langle\Omega_f, A_\kappa^2\Omega_f\rangle$,*

$$\left(H(p, \kappa) - \tilde{e} - z, \tilde{\alpha}[z]H_f\right) \in \mathfrak{F}\mathfrak{P}(\mathbb{C}^2 \otimes \mathcal{F}, \chi_1[H_f]) .$$

In particular, there exist parameters

$$\epsilon_0 = \frac{200g}{\xi} \ll 1 \, , \ \ \delta_0 \leq g \, , \ \ \lambda_0 < \frac{5}{8} \, ,$$

and

$$\underline{w}^{(0)}[z] \in \mathrm{Pol}_{g,p,\xi}\Big(\epsilon_0, \delta_0, \lambda_0\Big) \, ,$$

such that

$$H[\underline{w}^{(0)}[z]] = \frac{1}{\tilde{\alpha}[z]} F_{\chi_1[H_f]}\Big(H(p,\kappa) - \tilde{e} - z \, , \, \tilde{\alpha}[z] H_f\Big) \ \ \mathit{on} \ \mathcal{H}_{red} \, .$$

The function $\tilde{\alpha}[\,\cdot\,]$ *is analytic on* $\tilde{D}$*, and defined by the implicit relation*

$$\tilde{\alpha}[z] = \Big\langle \partial_{H_f} F_{\chi_\rho}\Big(H(p,\kappa) - z \, , \, \tilde{\alpha}[z] H_f\Big)\Big\rangle \, . \tag{25}$$

Next, Theorem 4.2 provides control over a single application of $\mathcal{R}_\rho$ on a polydisc $\mathrm{Pol}_{g,p,\xi}(\epsilon, \delta, \lambda)$, and establishes that

$$\mathcal{R}_\rho[\mathrm{Pol}_{g,p,\xi}(\epsilon, \delta, \lambda)] \subseteq \mathrm{Pol}_{g,p,\xi}(\hat{\epsilon}, \hat{\delta}, \hat{\lambda})$$

for $(\hat{\epsilon}, \hat{\delta}, \hat{\lambda})$ satisfying the bounds (27).

Theorem 4.2 *Let* $\xi \ll 1$ *be as in Theorem 4.1, and assume*

$$|p| \leq \frac{1}{20} \, , \ \ \rho = \frac{1}{2} \, , \ \ \epsilon_0 := \frac{200g}{\xi} \, , \ \ \epsilon \leq \epsilon_0 \, , \ \ \lambda < \frac{5}{8} \, . \tag{26}$$

Then, for ϵ_0 *sufficiently small,*

$$\mathcal{R}_\rho : \mathrm{Pol}_{g,p,\xi}(\epsilon, \delta, \lambda) \longrightarrow \mathrm{Pol}_{g,p,\xi}(\hat{\epsilon}, \hat{\delta}, \hat{\lambda})$$

with

$$\begin{aligned} \hat{\epsilon} &= \max\left\{\frac{17}{18}\,\epsilon \, , \, 3g|p| + 50g\lambda + \epsilon^{3/2}\xi\right\} \\ \hat{\delta} &\leq \delta + \epsilon \\ \hat{\lambda} &\leq \rho\lambda + \epsilon \, . \end{aligned} \tag{27}$$

The approach to the proof is very close to [3, 4, 7], but it is now necessary to give a much more careful account on terms originating from the non-vanishing overlaps $\chi\bar{\chi}$. The Ward-Takahashi identities are used to reduce the number of independent purely marginal operators, by showing that the coefficient $\beta_p[z]$ of the operator $P_f^{\parallel}$ in $T[z; H_f, P_f]$ (cf. $(\mathbf{P_1})$)

in the definition of $\mathrm{Pol}_{g,p,\xi}(\epsilon,\delta,\lambda)$) *completely determines* $w_{0,1}$ and $w_{1,0}$, which are the only purely *marginal interaction kernels* of the theory.

However, the estimates in Theorem 4.2 nevertheless only control a single application of $\mathcal{R}_\rho$, and are not strong enough to prove uniform boundedness of δ under repeated applications of $\mathcal{R}_\rho$.

The latter is, however, provided by Theorem 4.3, which yields the desired uniform bounds by invoking a strong induction argument that involves a recursive application of Theorem 4.2.

Theorem 4.3 *Let* $\underline{w}^{(0)} \in \mathrm{Pol}_{g,p,\xi}(\epsilon_0,\delta_0,\lambda_0)$, *as in Theorem 4.1. Assume that for* $0 \leq k < n$,

$$\underline{w}^{(k)} = \mathcal{R}_\rho^{(k-1)} \circ \cdots \circ \mathcal{R}_\rho^{(0)}[\underline{w}^{(0)}] \in \mathrm{Pol}_{g,p,\xi}(\epsilon_k,\delta_k,\lambda_k)$$

where $(\epsilon_k,\delta_k,\lambda_k)$ *and* $(\epsilon_{k+1},\delta_{k+1},\lambda_{k+1})$ *pairwise satisfy (27), and in particular, that*

$$\delta_k \leq 2\delta_0 \ ,$$

for all $0 \leq k < n$. *Then, it follows that*

$$\underline{w}^{(n)} \in \mathrm{Pol}_{g,p,\xi}(\epsilon_n, 2\delta_0, \lambda_n) \ .$$

The key to proving Theorem 4.3 is to bound an oscillatory sum that determines the purely marginal operators of the theory, by the algebraic composition identities satisfied by the smooth Feshbach map.

Our key result is Theorem 4.4, which states that the renormalization map $\mathcal{R}_\rho$ is contractive on a subset of $\mathrm{Pol}_{g,p,\xi}(\epsilon_0,\delta_0,\lambda_0)$ of codimension 2. This result is established by combining Theorems 4.2 and 4.3.

Theorem 4.4 *Let* $N_\kappa := \left\lceil \frac{\log\kappa}{\log\rho} \right\rceil$. *Assume that* $\{\underline{w}^{(0)},\underline{w}^{(1)},\ldots,\underline{w}^{(n)}\}$ *is the orbit of length* $n+1$ *generated by* $\mathcal{R}_\rho$ *with initial condition provided by Theorem 4.1. Then,*

$$\underline{w}^{(n)} \in \mathrm{Pol}_{g,p,\xi}\big(\epsilon_n,\delta_n,\lambda_n\big)$$

with

$$\begin{aligned} \lambda_n &\leq (2-\epsilon_0)^{-n}\lambda_0 \\ \epsilon_n &\leq \max\left\{ \left(\frac{17}{18}\right)^n \epsilon_0 \, , \, 2|p|\epsilon_0(1+2\delta_0)\mathbf{1}_{n\leq N_\kappa} \right\} \\ \delta_n &\leq 2\delta_0 \ . \end{aligned} \tag{28}$$

Hence, in particular,

$$\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \epsilon_n = 0 \ ,$$

and

$$\epsilon_n \leq 2\epsilon_0 \ \ , \ \ \delta_n \leq 2\delta_0 \ ,$$

uniformly in n, *as* $n \to \infty$.

4.2 Proof of Theorem 2.1

The bounds asserted in Theorem 2.1 are immediately obtained from the renormalization group flow by the identities

$$\begin{aligned} \partial_{|p|}^{a+1}E(p,\kappa) &= \lim_{n\to\infty} \partial_{|p|}^{a}\beta_p^{(n)}[0] \ , \ a=0,1 \\ \|\Omega(p,\kappa)\|_{\mathcal{F}}^2 &= \lim_{n\to\infty} \tilde{\alpha}[E(p,\kappa)] \prod_{k=0}^{n} \alpha[\underline{w}^{(n)}[e_n]] \end{aligned} \tag{29}$$

where e_n is the image of $E(p,\kappa)$ under n-fold renormalization of the spectral parameter. $E(p,\kappa)$ is determined by the renormalization group flow in the same manner as in [7].

Acknowledgements

I am deeply grateful to my Ph.D. advisor, Prof. J. Fröhlich, for his support, advice, and generosity. It is a great pleasure to thank Prof. K. R. Ito, Prof. I. Ojima, and Prof. Y. Takahashi for their great kindness and warm hospitality during our visit in Kyoto. The author is supported by a Courant Instructorship, and in part by a grant from the NYU Research Challenge Fund Program.

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