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Renormalization group approach to a generalization of the law of iterated logarithms for one-dimensional (non-Markovian) stochastic chains

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Contents

0 Introduction.

Renormalization group (RG) is, roughly speaking, a discrete dynamical system determined by a map which represents the response of (a set of random) objects in consideration to a change of accuracy of observation, or 'scale transformation', on a parameter space of generating functions of quantities defined on the objects.

We can think of, and there has been work on, various objects, for which the RG approach may be effective. Here we will focus on the simplest object for which the RG method is non-trivial, a class of probability measures on a set of paths (stochastic chains) on $\mathbb{Z}$. We focus on trying to explain the idea and efficiency of RG approach by applying the idea to the simplest object. As we will see, the RG approach focuses on (stochastic and/or approximate) similarity of the object (paths, in our case), while Markov and/or martingale properties are unnecessary, hence the RG will be a complimentary tool to the well-established methods.

One other point about introducing RG approaches to stochastic chains is that, like differential equations and stochastic differential equations, RG can be seen as a differential type equation which determine the object (stochastic chain, in our case) as a solution to a RG equation. In fact, we will see that, given an arbitrary one dimensional RG (with very mild assumption), we can uniquely construct a stochastic chain consistent with the equation.

The RG approach to the simple random walk on $\mathbb{Z}$ has been known in mathematics [7]. There, the RG (decimation) method is used to construct the one-dimensional Brownian motion as a continuum limit of the simple random walks. Our standpoint is to place RG in the center, instead of regarding RG as another method of constructing well-known stochastic processes, and to show that there is a large class of stochastic chains, including simple random walks and self-avoiding paths, for which RG acts naturally, and consequently, to show that a generalization of the law of iterated logarithms hold for such chains.

1 From decimation to renormalization group (RG).

1.1 Decimation on set of paths.

We are interested on the long distance asymptotic behaviors of paths on $\mathbb{Z}$. To consider a typical (very irregular) path as a composition of a backbone path dressed by finer structures of various scales, we put $G_n = 2^{-n}\mathbb{Z}$, $n = 0, 1, 2, \cdots$ (Fig. 1), and consider

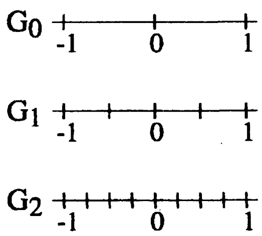

Fig. 1:

paths on G_n's. As a subset of $\mathbb{R}$ ($G_n \subset \mathbb{R}$), G_{n+1} has finer structures than G_n, or G_n can be seen as a set obtained from G_{n+1} by keeping 'typical' points and erasing finer structures.

Choose an $n \in \mathbb{Z}_+$ and w, a path (finite or infinite) on G_n starting from the origin 0. When we say a path on G_n, we mean a sequence of points in G_n such that each neighboring pair of points in the sequence is a neighbor pair in G_n. Namely, a sequence $w = (w(0), w(1), \cdots, w(L(w)))$ in G_n is a path on G_n starting from 0, if $w(0) = 0$, and

$$|w(i) - w(i+1)| = 2^{-n}, \quad i = 0, 1, 2, \cdots, L(w), \tag{1}$$

where $L(w)$ is the length (number of steps) of w. We write $L(w) = \infty$ for a infinite sequence (path of infinite length).

RG approach starts with adding or erasing fine structures of the object in consideration. We adopt the decimation method. Decimation is a map Q_n which maps a path w on G_{n+1} (a path with fine structures) to a path $Q_n w$ on G_n (a path with finest structure omitted), obtained by

(i) omitting from w the points in $G_{n+1} \setminus G_n$,

(ii) and then keeping only 1 point for each consecutive points in the resulting sequence (Fig. 2).

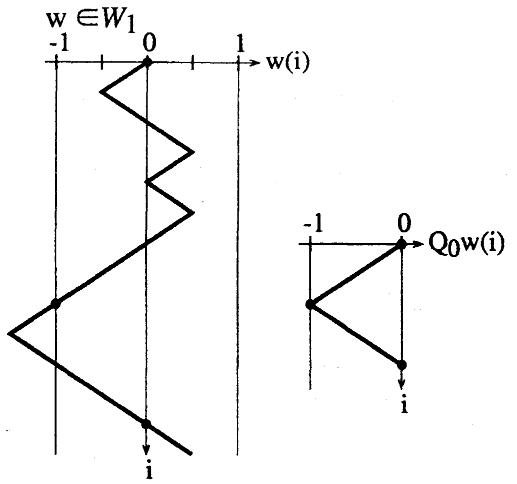

Fig. 2:

Alternatively, for $w \in G_{n+1}$ we define a sequence of hitting times $T_{n,i}$, $i = 0, 1, 2, \cdots$, of G_n, inductively in i, by

$$\begin{aligned} &T_{n,0}(w) = 0, \\ &T_{n,i}(w) = \min\{j > T_{n,i-1}(w) \mid w(j) \in G_n \setminus \{w(T_{n,i-1}(w))\}\}, \\ &\quad \text{if the right hand side exists.} \end{aligned} \tag{2}$$

We define $T_{n,i}$ for those i such that $T_{n,i}$ exists, and we denote the last i by $L(Q_n w)$. Q_n is then defined by

$$(Q_n w)(i) = w(T_{n,i}(w)), \ i = 0, 1, 2, \cdots, L(Q_n w). \tag{3}$$

$L(Q_n w)$ is the length of the path $Q_n w$ on G_n.

Q_n maps a path into paths of shorter steps. To see long time asymptotic properties of paths, we need to consider the inverse operation Q_n^{-1}. Q_n is not one-to-one, hence $Q_n^{-1}(w)$ should be defined as a set of paths on G_{n+1} which are mapped to w by Q_n.

Let w be a path with unit length 1. Then $Q_0^{-1}(w)$ is a set of paths obtained by replacing each step of w by a finite path on G_1 (Fig. 2). Without loss of generality, we may consider the step from 0 to 1 on G_0, and denote by $\tilde{\Omega}_1$ the set of finite paths on G_1 starting at 0 and stopping on first hit at 1, such that is a path segment which may replace a step from 0 to 1 of w to form a path in $Q_0^{-1}(w)$. Then it is easy to see from the definition of decimation procedure that

$$\begin{aligned} \tilde{\Omega}_1 = \{&\text{A finite path on } G_1 \text{ starting from 0 and stopping at first hit at 1,} \\ &\text{which does not hit } \pm 1 \text{ except at the last step}\}. \end{aligned} \tag{4}$$

For any $m \in \mathbb{Z}_+$, the operation Q_m is similar to that of Q_0; the only difference is that the unit spacing is 2^{-m}. Therefore the inverse operation Q_m^{-1} (the operation which adds to finer structures) replaces each step of a path on G_m by a path in $\tilde{\Omega}_1$ scaled in size by 2^{-m}.

To see asymptotic properties of paths, we need to consider paths on G_n with large n. To this end, we denote by $\tilde{\Omega}_n$, the set of finite path on G_n which may replace $(0, 1)$ (i.e., a step from 0 to 1 on G_0) by the n-th scale decimation $Q_0 \circ Q_1 \circ Q_2 \circ \cdots \circ Q_{n-1}$.

Proposition 1

$$\tilde{\Omega}_n = (Q_0 \circ Q_1 \circ \cdots \circ Q_{n-1})^{-1}((0,1))$$

is a set of finite path on G_n starting at 0 and stopping on first hit at 1, which do not hit ± 1 before the final step. ◇

1.2 Generating function and the renormalization group.

In §1.1 we defined decimation as a transformation on sets of paths. On the other hand, a stochastic chain $(X_1, X_2, \cdots)$ determines, for each $k \in \mathbb{Z}_+$, a joint distribution of the first k steps $(X_1, X_2, \cdots, X_k)$ which is a probability measure on the set of length k paths.

To find asymptotic properties of a stochastic chain, we look into the transformations on the corresponding probability measures on sets of paths induced by the decimation of paths. A natural set of paths to be considered first is $\tilde{\Omega}_n$ in Prop. 1. A natural (from RG point of view) probability measure on $\tilde{\Omega}_n$ is induced by the generating function of the length L of paths

$$\Phi_n(z) = \sum_{w \in \tilde{\Omega}_n} b_n(w) z^{L(w)}. \tag{5}$$

The so far arbitrary weight $\{b_n(w) \mid w \in \tilde{\Omega}_n,\ n \in \mathbb{N}\}$ determines the stochastic chain, as we will see later. Here we only assume that $b_n(w)$'s are non-negative, and that the right hand side of (5) has a non-zero radius of convergence.

For $n \in \mathbb{N}$ and $w \in \tilde{\Omega}_{n+1}$ put $w' = Q_n(w) \in \tilde{\Omega}_n$, and for $j = 1, 2, \cdots, L(w')$, consider the path segment of w

$$w_j = (w(T_{n,j-1}(w)), w(T_{n,j-1}(w) + 1), w(T_{n,j-1}(w) + 2), \cdots, w(T_{n,j}(w))). \tag{6}$$

This path segment is a 'fine structure' of the j-th step of $w' = Q_n w$ (Fig. 2), hence is a path on G_{n+1} which starts from $a = w(T_{n,j-1}(w)) \in G_n$ and stops on first hit at a neighboring point $w(T_{n,j}(w)) \in G_n$ such that does not hit points in $G_n \setminus \{a\}$ before it stops. Conversely, a path with such properties can be a path segment (6). Comparing with (4), we see that such a segment is similar to an element in $\tilde{\Omega}_1$. Denoting the similarity correspondence temporarily by $w_j \mapsto \tilde{w}_j \in \tilde{\Omega}_1$, the correspondence

$$w \ \mapsto\ (w', \tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_{L(w')}) \tag{7}$$

is therefore a one-to-one map from $\tilde{\Omega}_{n+1}$ to

$$\{(w', \tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_{L(w')}) \mid w' \in \tilde{\Omega}_n,\ \tilde{w}_j \in \tilde{\Omega}_1,\ j = 1, 2, \cdots, L(w')\}.$$

Here we impose the following condition on $\{b_n(w)\}$, to focus our attention to the cases where the decimation procedure is simplest and most effective.

Condition 1: For all $n \in \mathbb{N}$ and for all $w \in \tilde{\Omega}_{n+1}$,

$$b_{n+1}(w) = b_n(w') \prod_{j=1}^{L(w')} b_1(\tilde{w}_j), \tag{8}$$

where $(w', \tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_{L(w')})$ is as in (7). $\diamond$

We have left $\{b_1(w)\}$ free (except that they are non-negative and the generation function Φ_1 has non-zero radius of convergence), while for $n \geqq 2$, $\{b_n(w)\}$ are completely determined by $\{b_1(w)\}$ through (8).

The simple random walk on $\mathbb{Z}$ corresponds to the case

$$b_n(w) = 1, w \in \tilde{\Omega}_n\,,\ \ n \in \mathbb{N},$$

hence is an example satisfying (8) [1, §5].

The following simple fact is the starting point of everything.

Proposition 2 (RG on the paths on $\mathbb{Z}$) *Assume (8). Then Φ_n, $n \in \mathbb{Z}_+$, is determined by the following recursion relations.*

$$\Phi_{n+1}(z) = \Phi_n(\Phi_1(z)),\ \ n = 1, 2, \cdots, \tag{9}$$

$$\Phi_1(z) = \sum_{k=0}^{\infty} c_k z^k. \tag{10}$$

Here

$$c_k = \sum_{w \in \tilde{\Omega}_1;\ L(w)=k} b_1(w),\ \ k \in \mathbb{Z}_+, \tag{11}$$

are non-negative constants, satisfying $c_0 = c_1 = 0$. $\diamond$

A proof is simple (see [1, §5]).

Prop. 2 further implies

$$\Phi_n = \Phi_1 \circ \cdots \circ \Phi_1, \tag{12}$$

hence

$$\Phi_{n+1}(z) = \Phi_1(\Phi_n(z)) \tag{13}$$

also holds.

We have so far postponed introducing the probability measure related to $\{b_n(w)\}$ and Φ_n on the path set $\tilde{\Omega}_n$. Let x_c be a positive fixed point of $\Phi_1(x)$;

$$\Phi_1(x_c) = x_c\,,\ \ x_c > 0. \tag{14}$$

Then (9) implies

$$\Phi_n(x_c) = x_c\,,\ \ n \in \mathbb{N}. \tag{15}$$

Consider a probability measure determined, for each n, by

$$\mathrm{P}_n[\ \{w\}\ ] = b_n(w) {x_c}^{L(w)-1},\ \ w \in \tilde{\Omega}_n\,. \tag{16}$$

That this determines a probability measure is shown by $x_c > 0$ (positivity) and $\mathrm{P}_n[\ \tilde{\Omega}_n\ ] = \dfrac{1}{x_c}\Phi_n(x_c) = 1$ (normalization), which holds because of (15).

The Laplace transform of distribution of path length L with respect to the path measure P_n is calculated from (16) and (5), to obtain

$$\begin{aligned} &\sum_{k \in \mathbb{Z}_+} e^{-tk} \mathrm{P}_n[\ \{w \in \tilde{\Omega}_n \mid L(w) = k\}\ ] = \sum_{w \in \tilde{\Omega}_n} e^{-tL(w)} \mathrm{P}_n[\ \{w\}\ ] \\ &= \frac{1}{x_c} \sum_{w \in \tilde{\Omega}_n} b_n(w) (e^{-t} x_c)^{L(w)} = \frac{1}{x_c} \Phi_n(e^{-t} x_c). \end{aligned} \tag{17}$$

This explicitly relates the generating function Φ_n in (9) to the path measure P_n, as a Laplace transform of length distribution.

$n \to \infty$ and $L \to \infty$ are related through Tauberian type theorems. In considering asymptotic behaviors, it is natural to normalize L by a scaling factor corresponding to the average growth of L in n. We will see that the appropriate scaling factor is λ^n, where

$$\lambda = \Phi_1'(x_c) = \frac{d\,\Phi_1}{dx}(x_c). \tag{18}$$

Denote by $\tilde{\mathrm{P}}_n$, the distribution of $\lambda^{-n}L$ under P_n;

$$\tilde{\mathrm{P}}_n[\ \{\lambda^{-n}k\}\ ] = \mathrm{P}_n[\ \{w \in \tilde{\Omega}_n \mid L(w) = k\}\ ],\quad k \in \mathbb{Z}_+,. \tag{19}$$

Substituting $t = s\lambda^{-n}$ in (17), we find

$$\sum_{\xi\in\lambda^{-n}\mathbb{Z}_+} e^{-s\xi}\tilde{\mathrm{P}}_n[\ \{\xi\}\ ] = \sum_{k\in\mathbb{Z}_+} e^{-s\lambda^{-n}k}\mathrm{P}_n[\ \{w \in \tilde{\Omega}_n \mid L(w) = k\}\ ] = \frac{1}{x_c}\Phi_n(e^{-\lambda^{-n}s}x_c). \tag{20}$$

We will see that this quantity converges as $n \to \infty$ (See Thm. 5 in §2). This means that Prop. 2 implies asymptotic behaviors of length distribution of paths. We shall call the dynamical system on $\mathbb{R}_+$ determined by the recursion equation (13), the **renormalization group (RG)** (of the sequence of probability measures on paths determined by (16)).

2 Analysis of RG.

We temporarily forget about the probability measures on paths in this section, and look into RG (9), or equivalently, (13), as a one dimensional dynamical system.

Let Φ_1 be a complex analytic function defined by a power series

$$\Phi_1(z) = \sum_{k=0}^{\infty} c_k z^k, \tag{21}$$

satisfying the following.

Condition 2:

(i) The radius of convergence r of (21) is positive.

(ii) $c_0 = c_1 = 0$,

(iii) $c_2 > 0$,

(iv) $c_k \geqq 0,\ k = 3, 4, 5, \cdots$,

(v) $c_k > 0$ for some $k \geqq 3$.

◇

Define a sequence of functions Φ_n, $n = 1, 2, 3, \cdots$, by (13)

$$\Phi_{n+1}(z) = \Phi_1(\Phi_n(z)),\quad n = 1, 2, 3, \cdots,$$

for $z \in \mathbb{C}$ where the right hand side is defined and analytic.

We will point out in §2.1 and §2.2 that this general setting implies asymptotic properties of Φ_n and associated probability measures as $n \to \infty$. We return to path properties with those results in §3.

2.1 Analysis of RG trajectories.

The following is elementary [1, §5].

Proposition 3 *The following hold.*

(i) There exists a unique x_c (positive fixed point) such that

$$\Phi_1(x_c) = x_c, \quad 0 < x_c < r. \tag{22}$$

(ii) $\lambda = \Phi_1'(x_c) > 2$.

$\diamond$

Prop. 3 further leads to the following, also elementary, facts.

Theorem 4 *(i)* $\Phi_n(x_c) = x_c$ *and* $\Phi_n'(x_c) = \lambda^n$, *for* $n = 1, 2, 3, \cdots$.

(ii) For all x satisfying $0 \leqq x < x_c$,

$$\lim_{n\to\infty} \Phi_n(x) = 0, \tag{23}$$

and

$$\overline{\lim_{n\to\infty}}\, 2^{-n} \log \Phi_n(x) < 0. \tag{24}$$

$\diamond$

Except for (24), the claims are straightforward consequences of (22), (13), and uniqueness of positive fixed point, with a similar argument for Prop. 3(i). A proof of (24) needs a little more refined, but still elementary, arguments, using (23), (13) and induction [1, §5]. (An intuitive way to find a proof of (24) is to note that $\Phi_1(x)$ is close to c_2x^2 if x is small.)

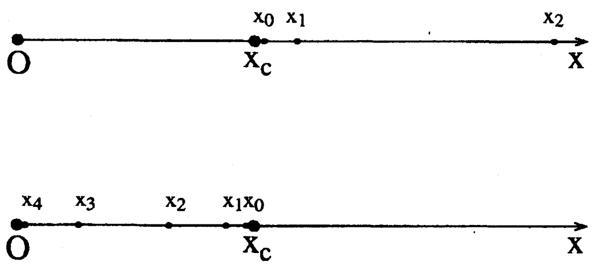

Fig. 3:

For $x > 0$, let us write for simplicity, $x_0 = x$ and $x_n = \Phi_n(x)$, $n \in \mathbb{Z}_+$. In this subsection §2.1, we looked into the trajectories of RG (behavior of sequences $\{x_n\}$ with different x's). (23) says that x_c is a unstable fixed point of Φ_1. (24) says that if $0 \leqq x < x_c$, x_n converges to 0 as fast as e^{-c2^n}. It is also easy to prove that if $x > x_c$, x_n diverges (Fig. 3).

2.2 Asymptotic behavior of distribution of path length.

Theorem 5 *Assume that a sequence of functions $\Phi_n : \mathbb{C} \to \mathbb{C}$, $n = 1, 2, 3, \cdots$, satisfies Condition 2 at the beginning of §2. Put*

$$G_n(s) = \frac{1}{x_c}\Phi_n(e^{-\lambda^{-n}s}x_c), \quad n = 1, 2, 3, \cdots, \tag{25}$$

for those s such that the right hand side is analytic, where λ is as in Prop. 3. Then the following hold.

(i) G_n is the generating function of the Borel probablitiy measure supported on $\mathbb{R}_+$. Namely, G_n is defined on $\mathbf{Re}(s) \geqq 0$ and there exists a Borel probabiity measure $\tilde{\mathrm{P}}_n$ satisfying

$$G_n(s) = \int_0^\infty e^{-s\xi}\tilde{\mathrm{P}}_n[\,d\xi\,], \quad \mathbf{Re}(s) \geqq 0.$$

Furthermore, $\tilde{\mathrm{P}}_n$ converges, as $n \to \infty$, to a Borel probability measure $\tilde{\mathrm{P}}_*$ supported on $\mathbb{R}$.

The generating function

$$G^*(s) = \int_0^\infty e^{-s\xi}\tilde{\mathrm{P}}_*[\,d\xi\,]$$

of $\tilde{\mathrm{P}}_*$ is defined and analytic on $\mathbf{Re}(s) \geqq 0$ and also on $|s| < C_\infty$ for some $C_\infty > 0$, hence there exists $C > 0$ such that

$$\int_{\mathbb{R}_+} e^{C\xi}\tilde{\mathrm{P}}_*[\,d\xi\,] < \infty.$$

$G^*(s)$ is determined by

$$G^{*\prime}(0) = -1, \quad G^*(s) = G_1(-\lambda \log G^*(s/\lambda)). \tag{26}$$

$G_n(s)$ converges as $n \to \infty$ to $G^*(s)$ uniformly on any bounded closed set in $|s| < C_\infty$, hence, in particular, it hods that

$$\lim_{n\to\infty}\int_{\mathbb{R}_+} \xi^p\tilde{\mathrm{P}}_n[\,d\xi\,] = \int_{\mathbb{R}_+} \xi^p\tilde{\mathrm{P}}_*[\,d\xi\,], \quad p > 0\,.$$

(ii) There exist positive constants C and C', such that for any sequence $\{\alpha_n\}$ with positive elements satisfying $\lim\limits_{n\to\infty} 2^{n(1-\nu)/\nu}\alpha_n = \infty$ and $\lim\limits_{n\to\infty} \alpha_n = 0$, it holds that

$$-C \leqq \varliminf_{n\to\infty} {\alpha_n}^{\nu/(1-\nu)} \log \tilde{\mathrm{P}}_n[\,[0, \alpha_n]\,] \leqq \varlimsup_{n\to\infty} {\alpha_n}^{\nu/(1-\nu)} \log \tilde{\mathrm{P}}_n[\,[0, \alpha_n]\,] \leqq -C'\,, \tag{27}$$

and

$$-C \leqq \varliminf_{x\to 0} x^{\nu/(1-\nu)} \log \mathrm{P}_*[\,[0, x]\,] \leqq \varlimsup_{x\to 0} x^{\nu/(1-\nu)} \log \mathrm{P}_*[\,[0, x]\,] \leqq -C', \quad x > 0\,. \tag{28}$$

Here we defined

$$\nu = \frac{\log 2}{\log \lambda}\,. \tag{29}$$

Furthermore, there exist positive constants (independent of ξ and n) C, C', C'' such that for any ξ and n satisfying $\left(\frac{\lambda}{2}\right)^n \xi \geqq C''$,

$$\tilde{\mathrm{P}}_n[\,[0, \xi]\,] \leqq C' e^{-C\xi^{-\nu/(1-\nu)}} \tag{30}$$

holds.

(iii) $\tilde{\mathrm{P}}_*$ *has a* C^∞ *density function* ρ *with respect to the Lebesgue measure;* $\tilde{\mathrm{P}}_*[\,d\xi\,] = \rho(\xi)\,d\xi$. ρ *satisfies*

$$\rho(\xi) = 0,\ \xi < 0, \quad \rho(\xi) > 0,\ \xi > 0\,.$$

(iv) There exists a positive constant b_0 *satisfying the following. For* $b > b_0$ *and* $n \in \mathbb{N}$, *if we put*

$$g_n(\xi) = \frac{1}{\sqrt{2\pi}h_n} e^{-\xi^2/(2h_n^2)},\ \xi \in \mathbb{R}, \quad h_n = b\,\lambda^{-n}\sqrt{n},$$

then

$$\lim_{n\to\infty} \int_{\mathbb{R}} g_n(\xi - \eta)\tilde{\mathrm{P}}_n[\,d\eta\,] = \rho(\xi), \tag{31}$$

uniformly in $\xi \in \mathbb{R}$.

$\diamond$

Idea of Proof. See [1, §B] for a proof of Thm. 5 (and for further detailed results). Here we shall briefly explain why we can expect that RG implies convergence and properties of the probability measures.

That $G_n(s)$ is a generating function of a probability measure is formally obvious, because $\Phi_n(z)$ is a n-th composition of $\Phi_1(z)$, hence its power series expansion has positive coefficients. The crucial point here is that the expansion has a positive radius of convergence uniformly in n (which is proved in a elementary way but needs a careful estimate). Then (25) implies $G_n(s) = \sum_{k=0}^{\infty} e^{-\lambda^{-n}ks} c_{n,k} x_c^{k-1}$, hence we find $\tilde{\mathrm{P}}_n[\,\{\lambda^{-n}k\}\,] = c_{n,k}x_c^{k-1}$. That the total measure is 1 follows from (22).

We can prove a convergence of G_n (in a suitable sense). Then we can take limit of (13) to find (26). Decay of $G^*(s)$ then follows by inductive use of (26), where positivity $G^*(s) \geqq 0$, $s \geqq 0$, is also essential. Tauberian theorems ([1, §A]) then implies a decay estimate at 0 of the corresponding measure $\tilde{\mathrm{P}}_*$, which is (28).

(26) also gives a decay estimate of the characteristic function $\varphi^*(t) = G^*(\sqrt{-1}t)$ at $|t| \to \infty$. Here we need (as a seed of inductive proof) $|\varphi^*(t)| < 1$ in a neighborhood of 0, which follows from the positivity of covariance, which originates from the assumption that there exists 2 or more non-zero terms in the power series expansion of Φ_1. That a characteristic function decays at infinity means that the corresponding probability measure is smooth, in particular, has a density function. A recursion equation for the generating functions imply that for the density function, which, by induction, proves the support property of ρ.

(31) states a speed of local convergence to the limit measure. This follows from a fact that $\varphi_n(t) = G_n(\sqrt{-1}t)$ decays in a similar speed as $\varphi^*(t)$ for $t = O(\lambda^n)$.

$\diamond$

3 RG to stochastic chain.

The argument in §2 is based solely on (13), the RG for Φ_n defined by (21), and is independent of path measures. In this section, we return to the path measure P_n and relate the results in §2 to §1.

Hereafter, we always assume that we are given a set of non-negative constants $\{b_n(w) \mid w \in \tilde{\Omega}_n,\ n \in \mathbb{N}\}$ satisfying (8) (Condition 1), and, defining Φ_n, $n \in \mathbb{N}$, by (5), assume also that Φ_1 satisfies Condition 2 stated at the beginning of §2. We note that Condition 1 determines $\{b_n(w)\}$ for $n \geqq 2$, while $\{b_1(w)\}$ is a set of arbitrary non-negative constants, and then Condition 2 imposes mild conditions on $\{b_1(w)\}$. Thus the restrictions on Φ_1 are very mild, and we have a rich class of stochastic chains for which the following results are applicable. We will give explicit examples in §5.

Since we assume Condition 1 and 2, all the results in §1 and §2 hold.

3.1 Overview of relation between path asymptotics and RG.

Before going into precise statement, let us briefly look into the displacement exponent from the RG point of view.

Denote by E_n, the expectation with respect to P_n in (16). For $p > 0$, (19) implies

$$\mathrm{E}_n[\ (\lambda^{-n}L)^p\ ] = \sum_{k=0}^{\infty}(\lambda^{-n}k)^p \mathrm{P}_n[\ L = k\ ] = \int_{\mathbb{R}_+} \xi^p \tilde{\mathrm{P}}_n[\ d\xi\ ].$$

Thm. 5 implies that this quantity converges;

$$\lim_{n\to\infty} \mathrm{E}_n[\ (\lambda^{-n}L)^p\ ] = c_p\,. \tag{32}$$

Here, Thm. 5 implies $c_p = \displaystyle\int_{\mathbb{R}_+} \xi^p \rho(\xi)\, d\xi > 0$.

P_n is a probability measure on set of paths on $G_n = 2^{-n}\mathbb{Z}$, starting from 0, not hitting -1, and stopping at 1 on first hit. The distribution of number of steps of paths on G_1 from 0 to 1 is equal to that of paths on G_0 from 0 to 2, because the difference is just the difference of step size and is independent of step numbers. Hence if we rescale the step size to 1, we can view P_n as a distribution of step numbers of paths on $\mathbb{Z}$ up to its first hit at $x = 2^n$. That $\mathrm{E}_n[\ (\lambda^{-n}L)^p\ ]$ converges to non-zero value roughly implies that a typical path (under P_n) requires

$$L(w) \fallingdotseq c\lambda^n = cx^{1/\nu}, \quad \left(\nu = \frac{\log 2}{\log \lambda}\right), \tag{33}$$

steps to hit $x = 2^n$. (To be more precise, to derive pathwise asymptotic properties, discussion on averages as in (32) are insufficient. We need to make sure that large deviations from average behaviors are rare. This is implied in the RG argument, such as in the detailed asymptotic behaviors summarized in Thm. 5. For example, (28) (short time estimates) says that paths of short number of steps ($x \to 0$) compared to average are exponentially rare with rate of order $\exp(-Cx^{\nu/(1-\nu)})$. (27) and (30) restates similar ingredients in a form convenient for later use.)

In the RG analysis we regarded position as a parameter n and path length L as a stochastic variable, but to define a stochastic chain, length (discrete time) is the parameter and the position is the stochastic variable. If we change the notation in (33) and write $L(w) = k$ and $x = W_k$, then

$$W_k \doteqdot k^{\nu}, \tag{34}$$

namely, (32) suggests that the displacement exponent is given by $\nu = \dfrac{\log 2}{\log \lambda}$.

3.2 Construction of the stochastic chain associated to RG.

Here we make explicit and rigorous what we overviewed in the previous section §3.1. First, as noted there, P_n is a measure on the set $\tilde{\Omega}_n$ of paths with step size 2^{-n}, for which we may scale the step size to 1 for all n, without any essential changes. We denote the set of rescaled paths by $2^n\tilde{\Omega}_n$. $2^n\tilde{\Omega}_n$ is a set of paths on $\mathbb{Z}$ starting from 0 and stopping on first hit at 2^n, which do not hit -2^n (Fig. 4).

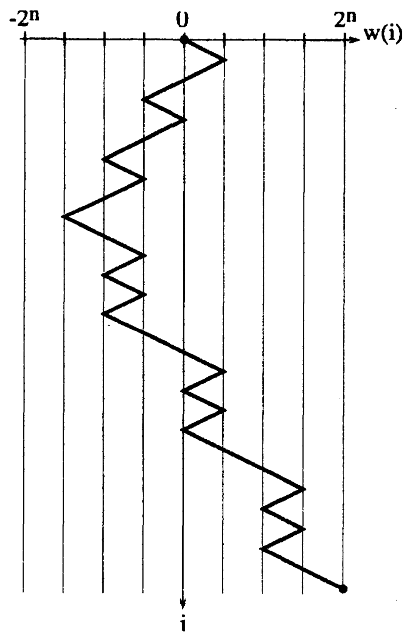

Fig. 4:

To obtain a stochastic chain associated to P_n on $2^n\tilde{\Omega}_n$, we need to consider the following 2 points.

(i) P_n is a measure on a set of paths with fixed endpoints and an unconstrained path length. On the other hand, a stochastic chain by definition requires a distribution of points at each fixed time (i.e., fixed path length). In particular, we need to define probabilities of paths ending at points other than 2^n, consistently with P_n.

(ii) $2^n\tilde{\Omega}_n$ is a set of paths which hits 2^n before -2^n. On the other hand, a long path may hit -2^n before 2^n, hence we must construct a probability measure supoorted also on such paths consistently with P_n.

It turns out that the first point is fixed by the Kolmogorov extension theorem. The RG recursion works as a consistency condition. For the second point, an obviously natural (and in fact the only self-similar) choice is to define a measure on paths hitting -2^n before 2^n as an image measure of reflection at 0 of the paths under P_n, and taking a simple average of the resulting measure and the original P_n. Denote by $2^n\tilde{\Omega}^r_n$ the set of paths each of which is the reflection at 0 of some path in $2^n\tilde{\Omega}_n$. Namely,

$$2^n\tilde{\Omega}^r_n = \{-w \mid w \in 2^n\tilde{\Omega}_n\}. \tag{35}$$

Note that by definition

$$\tilde{\Omega}_n \cap \tilde{\Omega}^r_n = \emptyset.$$

In §1.1, increasing n meant adding finer structures to a path without changing its large scale structure (Fig. 2). In contrast, if we note that a path in $2^{n+1}\tilde{\Omega}_{n+1} \cup 2^{n+1}\tilde{\Omega}^r_n$ hits $\pm 2^n$, we have a map

$$2^{n+1}\tilde{\Omega}_{n+1} \cup 2^{n+1}\tilde{\Omega}^r_{n+1} \to 2^n\tilde{\Omega}_n \cup 2^n\tilde{\Omega}^r_n$$

by assigning a path up to its first hit at $\pm 2^n$. Unlike the decimation in §1.1, this map extracts a first several steps of a given path.

In analogy to P_n on $\tilde{\Omega}_n$, we define a probability measure $\mathrm{P}_{r,n}$ on $\tilde{\Omega}^r_n$ by

$$\mathrm{P}_{r,n}[\ \{w\}\ ] = \mathrm{P}_n[\ \{-w\}\ ], \quad w \in \tilde{\Omega}^r_n. \tag{36}$$

Theorem 6 *Assume that*

$$\{b_n(w) \geqq 0 \mid w \in \tilde{\Omega}_n\ ,\ n \in \mathbb{N}\}$$

satisfies (8) and let $(\tilde{\Omega}_n, \mathrm{P}_n)$, $(\tilde{\Omega}^r_n, \mathrm{P}_{r,n})$, $n \in \mathbb{N}$, be a sequence of probability spaces defined by (5) (14) (16) (35) (36). Then there exists a stochastic chain on $\mathbb{Z}$, $W_0, W_1, W_2, \cdots$, satisfying the following.

For all $k \in \mathbb{Z}_+$ and for all $w = (w(0), w(1), w(2), \cdots, w(k)) \in \Omega_k$,

$$\begin{aligned}
&\mathrm{P}[\ W_j = w(j),\ j = 0,1,2,\cdots,k\ ] \\
&= \frac{1}{2}\mathrm{P}_n[\ \{w' = (w'(0), w'(1), \cdots, w'(L(w'))) \in \tilde{\Omega}_n \mid 2^n w'(j) = w(j)\ ,\ 0 \leqq j \leqq k\}\ ] \\
&+ \frac{1}{2}\mathrm{P}_{r,n}[\ \{w' = (w'(0), w'(1), \cdots, w'(L(w'))) \in \tilde{\Omega}^r_n \mid 2^n w'(j) = w(j)\ ,\ 0 \leqq j \leqq k\}\ ],
\end{aligned} \tag{37}$$

holds for any $n \in \mathbb{N}$ satisfying

$$|w(j)| < 2^n,\ j = 0,1,2,\cdots,k-1. \tag{38}$$

◇

A main ingredient of a proof of Thm. 6 is the Kolmogorov extension theorem. See [1, §C] for details.

4 Generalized law of iterated logarithms.

One of the consequence of RG analysis in §2 on the corresponding stochastic chain constructed in §3 is a generalization of the law of iterated logarithms. The following is the main result.

Theorem 7 *W_k, $k \in \mathbb{Z}_+$, as above, satisfies the following generalized law of iterated logarithms; namely, there exists $C_\pm > 0$ such that*

$$C_- \leqq \overline{\lim_{k\to\infty}} \frac{|W_k|}{\psi(k)} \leqq C_+\,, \text{ a.e..}$$

Here we wrote $\psi(k) = k^\nu(\log\log k)^{1-\nu}$. The constant ν in the exponent of ψ is given by (29); $\nu = \dfrac{\log 2}{\log\lambda}$, where $\lambda = \Phi_1'(x_c)$.

◇

Note that Prop. 3 implies

$$0 < \nu < 1. \tag{39}$$

The original law of iterated logarithms is known to hold for a large class of Markov processes (see, for example, [3, §VIII.5]), where in the proof of the lower bound, Markov property is essentially used. The stochastic chain constructed in §3.2 lacks Markov property in general. The generalized law Thm. 7 is applicable to cases where existing methods and results do not apply.

Idea of a proof of the upper bound of generalized law of iterated logarithms is as follows. For $x \in \mathrm{N}$, put $n = n(x) = \left\lceil \dfrac{\log x}{\log 2} \right\rceil$, or equivalently,

$$2^{n(x)+1} > x \geqq 2^{n(x)}. \tag{40}$$

By considering hitting times of $\pm 2^n$, Thm. 6, and (30) in Thm. 5(ii), we have

$$\tilde{\mathrm{P}}_n[\,[0,\lambda^{-n}k]\,] \leqq C' e^{-C(\lambda^{-n}k)^{-\nu/(1-\nu)}}$$

for all k and n satisfying $2^{-n}k \geqq C''$, where C, C', and C'' are some constants independent of k and n. See [1, §5] for details of the argument. This with (40) and a Borel-Cantelli type argument [1, §2.3] implies

$$\overline{\lim_{k\to\infty}} \frac{|W_k|}{\psi(k)} \leqq C^{-(1-\nu)}, \text{ a.e..}$$

A proof of the lower bound of the generalized law of iterated logarithms is more involved. Considering hitting times of $\pm 2^n$, and then Thm. 5, Thm. 6, and Thm. 5(ii), are used, with an argument in [4, 5] and a theorem of 2nd Borel-Cantelli type [1, §5], to find

$$\mathrm{P}[\,\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty}\{\lambda^{-m}T_m(\log m)^{(1-\nu)/\nu} \leqq (C+\epsilon)^{(1-\nu)/\nu}\}\,] = 1,$$

which, through a standard argument [1, §5] implies the lower bound

$$\overline{\lim_{k\to\infty}} \frac{|W_k|}{\psi(k)} \geqq C^{\nu-1}, \text{ a.e..}$$

See [1, §5] for details.

5 Self-repelling walk on $\mathbb{Z}$.

As examples of stochastic chains for which our results can be applied, we explain a class of chains which we call self-repelling walks [5]. The class continuously interpolates the simple random walk and the self-avoiding walk on $\mathbb{Z}$ in terms of the exponent ν.

It is not difficult to see from the definitions that Φ_1 of the RG for the simple random walk on $\mathbb{Z}$ and the self-avoiding walk on $\mathbb{Z}$ are, respectively [1, §5],

$$\Phi_1(z) = \begin{cases} \dfrac{z^2}{1-2z^2}, & \text{simple random walk,} \\ z^2, & \text{self-avoiding walk.} \end{cases}$$

A simplest interpolation would be, obviously, to define a class of Φ_1 parametrized by $u \in [0,1]$, by

$$\Phi_{1,u}(z) = \frac{z^2}{1-2u^2z^2}, \quad |z| < \frac{1}{u\sqrt{2}}, \tag{41}$$

and define Φ_n, $n = 2,3,4,\cdots$, inductively by

$$\Phi_{n+1,u}(z) = \Phi_{1,u}(\Phi_{n,u}(z)), \quad n \in \mathbb{Z}_+. \tag{42}$$

The case $u = 1$ corresponds to the simple random walk, and the case $u = 0$ to the self-avoiding walk. (The self-avoiding walk on $\mathbb{Z}$ is just a deterministic, straight going motion.) For any $u \in (0,1]$, $\Phi_1 = \Phi_{1,u}$ satisfies the Condition 2 at the beginning of §2, hence all the results of the previous sections hold. The exponent $\nu = \nu_u$ which determines the asymptotic properties, such as the generalized law of iterated logarithms Thm. 7 and the displacement exponent Thm. 8, of the corresponding stochastic chain $W_k = W_{u,k}$, $k \in \mathbb{Z}_+$, is

$$x_{c,u} = \frac{1}{4u^2}(\sqrt{1+8u^2}-1), \quad \lambda_u = \frac{\partial\Phi_{1,u}}{\partial x}(x_{c,u}) = \frac{2}{x_{u,c}} = \sqrt{1+8u^2}+1, \quad \nu_u = \frac{\log 2}{\log\lambda_u}. \tag{43}$$

In particular ν_u is continuous in u. Namely, the class of stochastic chains $W_k = W_{u,k}$, $k \in \mathbb{Z}_+$, $0 \leqq u \leqq 1$, continously interpolates the self-avoiding walk ($\nu = \nu_0 = 1$) and the simple random walk ($\nu_1 = 1/2$) on $\mathbb{Z}$ in terms of the exponent ν_u which determines the asymptotic properties of the chain. Such continous interpolation has not been known. The RG picture, in contrast, gives, as shown above, such interpolation in a most natural way. Comparing with (5) and (41), $\{b_1(w)\}$ also can be obtained in a natural way. However, its explicit form is not simple [5]. The parameter u appears at each turnning point of a path w, but the exponent of u varies with the turning point. It may therefore be not easy to find this model without RG picture. The obtained chains lack Markov properties, in general. The RG method works without Markov properties.

The simple random walk allows 'free' motion of the paths, while in the self-avoiding walk, returning to previously visited points are strictly forbidden. Hence for $0 < u < 1$, we expect a suppression of a path visiting a point more than once. In this sense, we call the obtained class of stochastic chain $W_{u,k}$, $k \in \mathbb{Z}_+$, $0 \leqq u \leqq 1$, self-repelling walks on $\mathbb{Z}$.

A self-repelling walk has a continuum (scaling) limit (self-repelling process) [4], a continuous time non-trivial stochastic process. Detailed properties, corresponding to the asymptotic properties of the self-repelling walk, are known [4]. (In fact, some estimates are slightly easier, becasue of self-similarity, hence the fixed endpoint self-repelling process has been known [4] before the stochastic chain [5].)

The parameter u can be extrapolated to $u > 1$ and all the results in the previous sections hold. Naturally, we expect the resulting chain to be self-attractive.

Since all the results in the previous sections hold for the self-repelling walks, the generalized law of iterated logarithms Thm. 7 also hold.

Another typical asymptotic property, the displacement exponent deals with expectations; $\mathrm{E}[\,|W_k|^s\,] \doteqdot k^{s\nu}$, $s > 0$. An upper bound for $\mathrm{E}[\,|W_k|^s\,]$ has similar implication to that for the law of iterated logarithms, in that, a typical path moves back and forth, thus it cannot go much far compared to its path length. In fact, the upper bound is proved in the general framework of the previous sections for all the chains constructed in §3 [1, §5]. A lower bound, on the other hand, has different meanings from that for the law of iterated logarithms. While the latter is an estimate for how far a typical path can go, the former is an estimate for averages, hence paths which are accidentally close to the origin at specified step must also be considered, and it seems (at present) that it cannot be proved without further assumptions. For the self-repelling walks, a geometric consideration similar to the reflection principle can be applied.

Theorem 8 ([5]) *For any $u > 0$, the self-repelling walk $W_k = W_{u,k}$, $k \in \mathbb{Z}_+$, has a displacement exponent $\nu = \nu_u$ given by (43), in the following sense;*

$$\lim_{k\to\infty} \frac{1}{\log k} \log \mathrm{E}[\,|W_k|^s\,] = s\nu, \quad s \geqq 0\,.$$

◇

The known proof is techincally involved and we leave it to the original paper [5].

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