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On a Solution of Quantized Non-Linear Equation with 4-Space Time Dimensions

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Abstract

Through the construction of the free scalar quantum field models, the relationships among quantized equations, Euclidean quantum field models and corresponding stochastic partial differential equations (SPDE) are reviewed. Then, a new non-linear quantized equation and an Euclidean field model are introduced.

MSC 2000:

1 Quantum field, quantized equation, Euclidean quantum field and SPDE

In this section we review the relationships among the scalar quantum field models, quantized equations, Euclidean quantum field models and corresponding stochastic partial differential equations (SPDE).

The relativistic scalar quantum fields with the space time dimensions $d+1$ are the quadruple $< \mathcal{H}, U, \phi, \mathcal{D} >$ that satisfy the following *Garding-Wightman axioms* (cf. [SW], [WG], [Si], [RS]):

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(GW.1) (Hilbert space) $\mathcal{H}$ is a separable Hilbert space having a distinguished unit vector ψ_0.

(GW.2) (Fields and Temperedness) $\mathcal{D}$ is a dense subset of $\mathcal{H}$ and for each $f \in \mathcal{S}(\boldsymbol{R}^{d+1} \to \boldsymbol{C})$ the linear operator $\phi(f)$ with the domain $\mathcal{D}$ satisfies
(a) For any $\psi_1, \psi_2 \in \mathcal{D}$, the map $f \mapsto (\psi_1, \phi(f)\psi_2)$ is a tempered distribution, also for fixed $\psi \in \mathcal{D}$ the map $f \mapsto \phi(f)\psi$ is linear;
(b) For real f, the operator $\phi(f)$ is symmetric;
(c) $\phi(f)$ leaves $\mathcal{D}$ invariant;
(d) $\mathcal{D}$ is spanned by the finite linear combinations of vectors of the form $\phi(f_1)\cdots\phi(f_n)\psi_0$.

(GW.3) (Poincaré invariance of the field) $U(\cdot,\cdot)$ is a strongly continuous unitary representation on $\mathcal{H}$ of the restricted Poincaré group, and for each $< a, \Lambda > \in \mathcal{P}_+^\uparrow$, $U(a,\Lambda)\mathcal{D} \subset \mathcal{D}$,

$$U(a,\Lambda)\phi(f)U(a,\Lambda)^{-1}\psi = \phi(< a, \Lambda > f)\psi, \qquad f \in \mathcal{S}(\boldsymbol{R}^{d+1} \to \boldsymbol{C}), \quad \psi \in \mathcal{D},$$

where $< a, \Lambda > f = f(\Lambda^{-1}(x-a))$.

(GW.4) (Spectrum) The projection valued measure on $\boldsymbol{R}^{d+1}$ corresponding to $U(a, I)$ has support in the closed forward light cone.

(GW.5) (Locality) If f and g in $\mathcal{S}(\boldsymbol{R}^{d+1} \to \boldsymbol{C})$ have supports which are spacelike separated, then

$$[\phi(f)\phi(g) - \phi(g)\phi(f)]\psi = 0, \qquad \psi \in \mathcal{D}.$$

(GW.6) (Uniqueness of vacuum) The only vectors in $\mathcal{H}$ left invariant by all the $U(a,1)$, $a \in \boldsymbol{R}^{d+1}$ are the multiples of ψ_0.

Suppose that we are given a QF satisfying above *Garding-Wightman axioms*, then for each $n \geq 0$, the Wightman distribution $\mathcal{W}_n$ for given QF is defined by

$$\mathcal{W}_n(f_1,\ldots,f_n) \equiv (\psi_0, \phi(f_1)\cdots\phi(f_n)\psi_0), \quad f_i \in \mathcal{S}(\boldsymbol{R}^{d+1} \to \boldsymbol{C}),\ i = 1,\ldots,n. \tag{1.1}$$

The sequence of Wightman distributions $\{\mathcal{W}_n\}_{n=0,1,\ldots}$ corresponding to a QF satisfying the *Garding-Wightman axioms* satisfies the *Wightman axioms for the Wightman distributions* (cf. section II.1 of [Si]). Conversely, by the Wightman reconstruction theorem (cf. section II.2 of [Si]), a sequence of

Wightman distributions satisfying the *Wightman axioms for the Wightman distributions* defines a QF obeying the *Garding-Wightman axioms.*

Any sequence of Wightman distributions that satisfies the *Garding-Wightman axioms* admits an analytic continuation to the domain with purely imaginary time components. Such analytic continuation is called as a sequence of Schwinger functions and is denoted by $\{S_n\}_{n=0,1,\ldots}$. Then $\{S_n\}_{n=0,1,\ldots}$ satisfies the *Osterwalder-Schrader (O-S) axioms* (cf. section II-3 of [Si]).

The most well known QF with the space time dimension $d+1$ is the free scalar quantum field with mass $m > 0$ defind through the sequence of Wightman distributions $\{\mathcal{W}_n\}_{n=0,1,\ldots}$ given as follows:
Let

$$H_m \equiv \{(\tau, \vec{\xi}) \in \boldsymbol{R}^{d+1} \mid \tau^2 - \sum_{j=1}^{d} (\vec{\xi}_j)^2 = m^2, \ \tau > 0\},$$

and define a measure Ω_m on H_m such that

$$\Omega_m(E) = \int_{I_m(E)} \frac{1}{\sqrt{|\vec{\xi}|^2 + m^2}} d\vec{\xi}, \qquad E \subset H_m, \tag{1.2}$$

where $I_m : (\tau, \vec{\xi}) \mapsto \vec{\xi} \in \boldsymbol{R}^d$.
Let

$$\Delta_+(t, \vec{x}; m^2) \equiv \Big(\mathcal{F}\Omega_m\Big)(t, \vec{x}),$$

where $\mathcal{F}$ denotes the $d+1$-dimensional Fourier transform. Finally for each $n = 0, 1, \ldots$ let

$$\mathcal{W}_n((t_1, \vec{x}_1), \ldots, (t_n, \vec{x}_n)) \equiv [(t_1, \vec{x}_1), \ldots, (t_n, \vec{x}_n)], \tag{1.3}$$

with

$$[(t_1, \vec{x}_1), \ldots, (t_n, \vec{x}_n)] = 0, \qquad \text{for an odd } n; \tag{1.4}$$

$$\begin{aligned} [(t_1, \vec{x}_1), \ldots, (t_{2n}, \vec{x}_{2n})] \ &\equiv \ \sum_{\text{pair}} \Delta_+((t_{i_1}, \vec{x}_{i_1}) - (t_{j_1}, \vec{x}_{j_1}); m^2) \times \cdots \\ &\quad \times \ \Delta_+((t_{i_n}, \vec{x}_{i_n}) - (t_{j_n}, \vec{x}_{j_n}); m^2), \end{aligned} \tag{1.5}$$

where the sum is over all $\frac{(2n)!}{2^n n!}$ ways of writing $\{1, \ldots, 2n\}$ as $i_1, \ldots, i_n, j = 1, \ldots, j_n$ with $i_1 < i_2 < \cdots < i_n; i_1 < j_1; \ldots; i_n < j_n$.

The sequence of Schwinger functions $\{S_n\}_{n=0,1,\ldots}$ corresponding to the above sequence of Wightman functions is defined through (1.3), (1.4) and (1.5) with the replacement of $\Delta_+((t_{i_k},\vec{x}_{i_k})-(t_{j_k},\vec{x}_{j_k});m^2)$ by $J^{\frac{1}{2}}((t_{i_k},\vec{x}_{i_k})-(t_{j_k},\vec{x}_{j_k}))$, where

$$J^\alpha(t,\vec{x}) \equiv \left(\mathcal{F}(|(\tau,\vec{\xi})|^2+m^2)^{-\alpha}\right)(t,\vec{x}), \qquad \alpha>0,$$

where $\mathcal{F}$ denotes the $d+1$-dimensional Fourier transform. Namely,

$$S_n((t_1,\vec{x}_1),\ldots,(t_n,\vec{x}_n)) \equiv [(t_1,\vec{x}_1),\ldots,(t_n,\vec{x}_n)]_S, \tag{1.6}$$

where $[\cdot]_S$ is defined through (1.4) and (1.5) by replacing Δ_+ by $J^{\frac{1}{2}}$.

By the Nelson reconstruction theorem (cf. [Ne1], section IV.2 of [Si]), if a random field indexed by $\mathcal{S}(\boldsymbol{R}^{d+1}\to\boldsymbol{R})$ (equivalently, a random field defined through $\mathcal{S}'(\boldsymbol{R}^{d+1}\to\boldsymbol{R})$-valued random variables) satisfies *Nelson's axioms*, then the moment functions of the $\mathcal{S}'(\boldsymbol{R}^{d+1}\to\boldsymbol{R})$-valued random variables correspond to a sequence of Schwinger functions that satisfies the *O-S axioms*.

As an example, the sequence of Schwinger functions $\{S_n\}_{n=0,1,\ldots}$ corresponding to the free QF defined by (1.6) is the moment function of the $\mathcal{S}'(\boldsymbol{R}^{d+1}\to\boldsymbol{R})$-valued random variable ϕ_N known as the Nelson's Euclidean free field such that

$$E[\phi_N(f_1)\cdots\phi_N(f_n)] = S_n(f_1,\ldots,f_n), \quad f_i\in\mathcal{S}(\boldsymbol{R}^{d+1}\to\boldsymbol{R}),\ i=1,\ldots,n,$$

where

$$\phi_N(f) \equiv \int_{\boldsymbol{R}^{d+1}}((-\Delta_{d+1}+m^2)^{-\frac{1}{2}}f)(t,\vec{x})dW_{d+1}(t,\vec{x}). \tag{1.7}$$

Where Δ_{d+1} is the $d+1$ -dimensional Laplace operator, and W_{d+1} is a $d+1$-dimensional isonormal Gaussian process such that

$$E[W_{d+1}(h)W_{d+1}(g)] = \int_{\boldsymbol{R}^{d+1}} h(\boldsymbol{x})g(\boldsymbol{x})d\boldsymbol{x}, \qquad h,\ g\in L^2(\boldsymbol{R}^{d+1}\to\boldsymbol{R}), \tag{1.8}$$

($\dot{W}_{d+1}$ is known as the $d+1$-dimensional Gaussian white noise, cf. [HKPS]. for the stochastic integral representation of ϕ_N, cf. [AY1], [AY2], [AFY], [Y]).

For the free QF defined through the sequence of Wightman functions given by (1.3), we also have an expression of the field operator ϕ (cf.

(GW1)) by means of a composition of an $\mathcal{S}'(\boldsymbol{R}^d \to \boldsymbol{R})$-valuedrandom variable with unitary operators:

$$\phi(\delta_{t_0}(t) \otimes \varphi(\vec{x})) = e^{id\Gamma((-\Delta_d+m^2)^{\frac{1}{2}})t_0}\phi_0(\varphi)e^{-id\Gamma((-\Delta_d+m^2)^{\frac{1}{2}})t_0}, \tag{1.9}$$

$$\varphi \in \mathcal{S}(\boldsymbol{R}^d \to \boldsymbol{C}),$$

where Δ_d is the d-dimensional Laplace operator,

$$\phi_0(\varphi) \equiv \int_{\boldsymbol{R}^d}((-\Delta_d + m^2)^{-\frac{1}{4}}\varphi)(\vec{x})dW_d(\vec{x}), \tag{1.10}$$

with W_d being the d-dimensional isonormal Gaussian process satisfying (1.8) with $d+1$ replaced by d, and $d\Gamma((-\Delta_d + m^2)^{\frac{1}{2}})$ is the second quantization of the pseudo-differential operator $(-\Delta_d + m^2)^{\frac{1}{2}}$.

By (1.7) and (1.9), ϕ_N and ϕ satisfy the following stochastic partial differential equation (SPDE) and quantized equation respectively:

$$(-\Delta_{d+1} + m^2)^{\frac{1}{2}}\phi_N = W_{d+1}; \tag{1.11}$$

$$(\frac{\partial^2}{\partial t^2} - \Delta_d + m^2)\phi = 0. \tag{1.12}$$

This is a brief observation of the relationships among the free QF defined through $\{\mathcal{W}_n\}_{n=0,1,\dots}$ (1.3), quantized equation (1.12) which is satisfied by the free field operator ϕ (1.9), the Euclidean free field defined through $\{S_n\}_{n=0,1,\dots}$ (1.6), the Euclidean random field (1.7) and the SPDE (1.11) which is satisfied by the Nelson's Euclidean free field operator ϕ_N.

More definite correspondence of (1.7) and (1.9) to the free quantum field will be mentioned in Remarks 1 and 2 in the next section.

There are various interesting SPDE's that have correspondences to Euclidean quantum field theory (cf. ,eg. [ARö], [AG], [AGW], [AGY], [AY1] and [Ne2]).

For the quantum field operator ϕ given by (1.9) and its modifications, there are several considerations on their interesting properties, in particular on their essential self-adjointness property (cf., for eg. [AFY], [K], [Se]). In [AFY] the essential self-adjointness of the Wick powers of the field operator ϕ is discussed, and a new result is derived by making use of the stochastic expression of ϕ (cf. (1.9)), where the dense domain of these field operators are taken as $\mathcal{D}_{\frac{1}{4}}$ (cf. Remark 1 in the next section) that is a space of finite linear combination of $\phi_0(f_1) \times \cdots \times \phi_0(f_n)$, $f_i \in \mathcal{S}(\boldsymbol{R}^d \to \boldsymbol{C})$, $n \in \boldsymbol{N}$.

However, it should be remarked that it still be very interesting to consider the essentially self-adjointness problem on the more general class of the field operators that are more singular than the Wick powers of ϕ, including the Wick exponential opetators etc., by setting their definitions domain as a domain of hyper functions (cf. [NM1], [NM2]).

In section X7 of [RS], there is a discussion concerning a quantized equation with non-linear pertervation term that is a modification of (1.12) such that

$$(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2)\phi = -4\lambda\phi^3, \tag{1.13}$$

where $\lambda > 0$ is some fixed positive number and the space time dimension is taken as $d+1 = 2$. As was mentioned in X7 of [RS] and it is well known that (1.13) is a heuristic equation. However it is a Schrödinger type equation that is expected to give a dynamics on the so called $(\phi^4)_2$ QF (for a brief definition of ϕ^4 QF, cf. Remark 2 in the next section, and for its exact definition cf. , eg. [RS], [Si], [GJ]). equation

In [Os], the quantized equation (1.13) for $d+1 = 4$ is considered by not taking ϕ to be an operator but taking it to be a form on a Hilbert space.

In the next section, we shall consider a modification of the quantized non-linear equations (1.13) with $d+1 = 4$ and a modified model of $(\phi^4)_4$ Euclidean quantum field.

2 Modified quantized equation and an Euclidean field model having correspondences with $(\phi^4)_4$ QF

`A modification of (1.13).`

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Suppose that on $(\Omega, \mathcal{F}, P)$ we are given a 3-dimensional Gaussian white noize $\dot{W}_3$ defined by (1.8). Let ϵ be a fixed number such that $0 < \epsilon \leq \frac{1}{16}$. Define the $\mathcal{S}'(\boldsymbol{R}^3 \to \boldsymbol{R})$-valued random variables $\varphi^0(\vec{x})$ and $\varphi^I(\vec{x})$ as follows:

$$\varphi^0(f) \equiv \int_{\boldsymbol{R}^3}((-\Delta_3 + m^2)^{-\epsilon+\frac{1}{4}}f)(\vec{x})dW_3(\vec{x}), \quad f \in \mathcal{S}(\boldsymbol{R}^3 \to \boldsymbol{R}),$$

$$\varphi^I(f) \equiv \int_{\boldsymbol{R}^3}((-\Delta_3 + m^2)^{-1+\epsilon+\frac{1}{4}}f)(\vec{x})dW_3(\vec{x}), \quad f \in \mathcal{S}(\boldsymbol{R}^3 \to \boldsymbol{R}).$$

Let $\mathcal{H}_{\epsilon-\frac{1}{4}}$ be the Hilbert space that is the complexification of the real L^2 space spanned by the linear combinations of the random variables

$$\varphi^0(f_1) \times \cdots \times \varphi^0(f_n), \qquad f_i \in \mathcal{S}(\boldsymbol{R}^3 \to \boldsymbol{R}), \quad i = 1, \ldots, n, \quad n \in \boldsymbol{N}. \tag{2.1}$$

For fixed numbers $\lambda \geq 0$, $r \geq 0$ and a given bounded region

$$\Lambda_{3,r} \equiv \{\vec{x} \in \boldsymbol{R}^3 \mid |\vec{x}| < r\},$$

define a linear operator $\Phi(\vec{x}, t)$ on $\mathcal{H}_{\epsilon-\frac{1}{4}}$ as follows:

$$\Phi(\vec{x}, t) \equiv e^{i(d\Gamma(\mu)+V_\Lambda)t}\varphi^0(\vec{x})e^{-i(d\Gamma(\mu)+V_\Lambda)t},$$

where as in the previous section $d\Gamma(\mu)$ is the second quantization of the pseudo differential operator μ such that

$$\mu \equiv (-\Delta_3 + m^2)^{\frac{1}{2}},$$

and

$$V_\Lambda \equiv \lambda \int_\Lambda : (\varphi^I(\vec{x}))^4 : d\vec{x},$$

with

$$: (\varphi^I(\vec{x}))^4 := \int_{(\boldsymbol{R}^3)^4} \prod_{j=1}^{4} J((\vec{x}) - (\vec{x}_j))dW_3(\vec{x}_1) \cdots dW_3(\vec{x}_4),$$

(the 4-th Wick power of φ^I defined by the multiple stochastic integral cf. [AY], [Y]), with the integral kernel J of the pseudo differential operator

$$(-\Delta_3 + m^2)^{-\frac{3}{4}+\epsilon}$$

(i.e. J is the Fourier transform of the symbol $(|\vec{\xi}|^2 + m^2)^{-\frac{3}{4}+\epsilon}$).

We have the following Theorem (in this paper we state the main part of our results only, and of which proofs and detailed discussions will be given in a forthcoming paper).

Theorem 2.1 *Let $\mathcal{D}_{\epsilon-\frac{1}{4}}$ be the linear subspace of $\mathcal{H}_{\epsilon-\frac{1}{4}}$ such that*

$\mathcal{D}_{\epsilon-\frac{1}{4}} \equiv$ *the complexification of*

the finite linear combinations of the vectors given by (2.1).

Then the operator valued distribution $\Phi(\vec{x},t)$ on $\mathcal{S}(\boldsymbol{R}^3 \to \boldsymbol{C})$ (precisely, $\Phi(\cdot,t) : \mathcal{S}(\boldsymbol{R}^3 \to \boldsymbol{C}) \ni f \mapsto \Phi(f,t)$) satisfies the following quantized nonlinear equation:

$$(\frac{\partial^2}{\partial t^2} - \Delta_3 + m^2)\Phi(\vec{x},t) = -4\lambda : (\Phi(L_\epsilon \delta_{\vec{x}},t))^3 :, \tag{2.2}$$

where

$$L_\epsilon \equiv (-\Delta_3 + m^2)^{-1+2\epsilon}, \qquad \delta_{\vec{x}} \equiv \text{the Dirac point measure on } \vec{x},$$

and $: (\Phi(L_\epsilon \delta_{\vec{x}},t))^3 :$ *is the Wick power of the operator* $\Phi(L_\epsilon \delta_{\vec{x}},t)$ *such that*

$$\Phi(\vec{x},t) \equiv e^{i(d\Gamma(\mu)+V_\Lambda)t} : (\varphi^0(L_\epsilon \delta_{\vec{x}}))^3 : e^{-i(d\Gamma(\mu)+V_\Lambda)t},$$

with

$$: (\varphi^0(L_\epsilon \delta_{\vec{x}}))^3 :=: \varphi^I(\vec{x})^3 := \int_{(\boldsymbol{R}^3)^3} \prod_{j=1}^{3} J((\vec{x}) - (\vec{x}_j)) dW_3(\vec{x}_1) \cdots dW_3(\vec{x}_3),$$

(the 3rd Wick power of φ^0). □

Remark 1.

In the above formulation if we replace φ^0 by the usual time zero free field ϕ_0 defined by (1.10) setting $d = 3$, and define $\mathcal{H}_{\frac{1}{4}}$ and $\mathcal{D}_{\frac{1}{4}}$ as the Hilbert space and its subspace by replacing $(-\Delta_3 + m^2)^{-\epsilon+\frac{1}{4}}$ by $(-\Delta_3 + m^2)^{-\frac{1}{4}}$ in the definition of $\mathcal{H}_{\epsilon-\frac{1}{4}}$ and $\mathcal{D}_{\epsilon-\frac{1}{4}}$, then the operator ϕ defined by (1.9), the Hilbert space $\mathcal{H}_{\frac{1}{4}}$ and the dense domain $\mathcal{D}_{\frac{1}{4}}$ define the $d+1 = 4$ dimensional free scalar QF satisfying the Garding-Wightman axioms. □

A modification of the Euclidean $(\Phi^4)_4$ random field model.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Suppose that on $(\Omega, \mathcal{F}, P)$ we are given a 4-dimensional Gaussian white noize $\dot{W}_4$ defined by (1.8). Let $\phi_N(t,\vec{x})$ be the Euclidean free field (Nelson field) with $d+1 = 4$ that is an $\mathcal{S}'(\boldsymbol{R}^4 \to \boldsymbol{R})$-valuedrandom variable defined by (1.7):

$$\phi_N(f) \equiv \int_{\boldsymbol{R}^4} ((-\Delta_4 + m^2)^{-\frac{1}{2}} f)(\vec{x}) dW_4(\vec{x}), \quad f \in \mathcal{S}(\boldsymbol{R}^4 \to \boldsymbol{R}). \tag{2.3}$$

Also, define an $\mathcal{S}'(\boldsymbol{R}^4 \to \boldsymbol{R})$-valued random variable $\phi_I(t,\vec{x})$ as follows:

$$\phi_I(f) \equiv \int_{\boldsymbol{R}^4} ((-\frac{\partial^2}{\partial t^2} + (-\Delta_3 + m^2)^3)^{-\frac{1}{2}} f)(\vec{x}) dW_4(\vec{x}), \quad f \in \mathcal{S}(\boldsymbol{R}^4 \to \boldsymbol{R}). \tag{2.4}$$

For fixed numbers $\lambda \geq 0$, $r \geq 0$ and a given bounded region

$$\Lambda_{4,r} \equiv \{(t,\vec{x}) \in \boldsymbol{R}^4 \mid |(t,\vec{x})| < r\},$$

we define a probability measure ν on $\mathcal{S}'(\boldsymbol{R}^4 \to \boldsymbol{R})$ such that

$$\begin{aligned} &\int_{\mathcal{S}'(\boldsymbol{R}^4 \to \boldsymbol{R})} < \phi, f_1 > \cdots < \phi, f_n > \nu(d\phi) \\ &= \frac{1}{Z_\Lambda} E^P[\phi_N(f_1) \cdots \phi_N(f_n)\, e^{-\lambda < :\phi_I^4:, I_{\Lambda_{4,r}} >}], \\ &f_1, \ldots, f_n \in \mathcal{S}(\boldsymbol{R}^4 \to \boldsymbol{R}),\ n \in \boldsymbol{N}, \end{aligned} \tag{2.5}$$

where

$$Z_\Lambda \equiv E^P[e^{-\lambda < :\phi_I^4:, I_{\Lambda_{4,r}} >}],$$

$E^P[\cdot]$ is the expectation with respect to the measure P, $I_{\Lambda_{4,r}}$ is the indicator function of the measurable set $\Lambda_{4,r}$, and

$$:\phi_I^4: (t,\vec{x}) \equiv \int_{(\boldsymbol{R}^4)^4} \prod_{j=1}^{4} K((t,\vec{x}) - (t_j,\vec{x}_j)) dW_4(t_1,\vec{x}_1) \cdots dW_4(t_4,\vec{x}_4),$$

(the 4-th Wick power of ϕ_I defined by the multiple stochastic integral cf. [AY1], [AY2], [Y]) with the integral kernel K of the pseudo differential operator $(-\frac{\partial^2}{\partial t^2} + (-\Delta_3 + m^2)^3)^{-\frac{1}{2}}$ (i.e. K is the Fourier transform of the symbol $(\tau^2 + (|\vec{\xi}|^2 + m^2)^3)^{-\frac{1}{2}}$).

Theorem 2.2 *For each $n = 0, 1, \ldots$, let*

$$S_n(f_1, \ldots, f_n) = \int_{\mathcal{S}'(\boldsymbol{R}^4 \to \boldsymbol{R})} < \phi, f_1 > \cdots < \phi, f_n > \nu(d\phi). \tag{2.6}$$

Then $\{S_n\}_{n=0,1,\ldots}$ satisfies all the O-S axioms except the Euclidean invariance.

In particular, it satisfies the reflection positivity property (positive definiteness). □

The reflection positivity of this model follows from its Markovian property, more definitely, ν does not define a Markov field, but defines an $\mathcal{S}'(\boldsymbol{R}^4 \to \boldsymbol{R})$-valued Markov process (cf. [AH-K], [ARö]).

Remark 2.

ii) The two dimensional $(\phi^4)_2$ Euclidean field with the truncation to the self interaction term such that $\Lambda_{2,r} \equiv \{(t,x) \in \boldsymbol{R}^2 \,|\, |(t,x)| < r\}$, is defined by changing the formulas (2.3), (2.4) and (2.5) as follows:
replacing the space time dimension 4 by 2 in (2.3), (2.4) and (2.5);
and replacing the interaction term $e^{-\lambda <:\phi_I^4:, I_{\Lambda_{4,r}}>}$ in (2.5) by $e^{-\lambda <:\phi_N^4:, I_{\Lambda_{2,r}}>}$ where ϕ_N is the Nelson field defined by (2.3) for $d+1=2$.

In particular if $\lambda = 0$, then the Schwinger function defined by (2.5) and (2.6) is the Schwinger function of the free Euclidean field.

ii) If it is possible to take this model as a simple analogy with the $(\phi^4)_4$ QF that has a correspondence with the quantized nonlinear equation (1.13), then the Eucldean field defined by ν is expected to have some correspondence with a QF operator Φ satisfying the quantized equation such that

$$(\frac{\partial^2}{\partial t^2} + (-\Delta_3 + m^2)^3)\Phi = -4\lambda\Phi^3.$$

$\square$

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