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# Parabolic isometries of CAT(0) spaces and CAT(0) dimensions

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I gave a talk on the paper "Parabolic isometries of CAT(0) spaces and CAT(0) dimensions", [FSY].

Let $(X, d)$ be a geodesic space. Let $\Delta(a, b, c) \subset X$ be a geodesic triangle with three vertices, $a, b, c$, and three geodesics, $[a, b], [b, c], [c, a]$, joining them. A geodesic triangle, $\overline{\Delta}(\bar{a}, \bar{b}, \bar{c})$, in the Euclidean plane is called a *comparison triangle* if $d(a, b) = d(\bar{a}, \bar{b}), d(b, c) = d(\bar{b}, \bar{c}), d(c, a) = d(\bar{c}, \bar{a})$. Comparison triangles always exist. For a point, $x$, on one of the sides of $\Delta$, say, $[a, b]$, a point $\bar{x} \in [\bar{a}, \bar{b}]$ is called the comparison point if $d(a, x) = d(\bar{a}, \bar{x})$. $X$ is called a *CAT(0) space* if for any two points, $x, y$, on the sides of $\Delta$, we have the following inequality for the comparison points $\bar{x}, \bar{y}$ in $\overline{\Delta}$:

$$d(\bar{x}, \bar{y}) \leq d(x, y).$$

Let $X$ be a metric space. The space is said *proper* if for any point $x \in X$ and $r > 0$, the closed metric ball centered at $x$, of radius $r$ is compact. Suppose a group $G$ is acting on $X$ by isometries. The action is said *proper* if for any point $x \in X$ there exists a number $r > 0$ such that there are only a finite number of elements $g \in G$ with $d(x, gx) \leq r$.

A very informative reference on CAT(0) spaces is [BH]. Standard examples of CAT(0) spaces are simply-connected, complete, Riemannian manifolds of sectional curvature at most 0, and trees. Metric product of two CAT(0) spaces is CAT(0). It is an easy but important fact that any two points in a CAT(0) space is uniquely joined by a geodesic. There is a notion of the ideal boundary, $X(\infty)$, which gives a compactification of a proper CAT(0) space, $X$. Any point $x \in X$ and any point $p \in X(\infty)$ is uniquely joined by a geodesic in a proper CAT(0) space.

Each isometry, $g$, of a complete CAT(0) space $X$ is classified as elliptic, hyperbolic, or parabolic. It is *elliptic* if and only if $g$ fixes a point in $X$; *hyperbolic* if and only if it is not elliptic and there exists a bi-infinite geodesic in $X$ which is $g$-invariant; or else *parabolic.* Elliptic and hyperbolic ones are called *semi-simple.*

In this note, the dimension of a topological space means its covering dimension, which is sometimes called the topological dimension as well.

We state a key proposition from [FSY].

**Proposition 1.** *Let $n$ be a positive integer. Suppose $\mathbb{Z}^n$ acts on a proper $CAT(0)$ space, $X$, of dimension $n$ by isometries, properly. Then each non-trivial element of $\mathbb{Z}^n$ acts as a hyperbolic isometry. And there exists a Euclidean space of dimension $n$, $\mathbb{E}^n$, in $X$ which is convex and invariant by the group action.*

The proof is given in [FSY]. We argue by contradiction. If there is a parabolic isometry, then there is a point, $p$, in the ideal boundary of $X$ which is fixed by the group action. Moreover, each horosphere, $H$, at $p$ is invariant too. The dimension of $H$ is at most $n-1$. From this we can conclude that the cohomological dimension of the group is at most $n-1$ as well, which is impossible because the cohomological dimension of $\mathbb{Z}^n$ is $n$. Once we know the action is by semi-simple isometries, we can apply the flat torus theorem (cf. [BH]) and obtain an invariant subspace which is convex and isometric to the Euclidean space of dimension $n$. The nearest point projection from $X$ to the Euclidean space gives a deformation retract, which is equivariant by the group action.

Note that $\mathbb{Z}^2$ acts on the hyperbolic space of dimension 3, $\mathbb{H}^3$, by isometries, properly such that any non-trivial element acts as a parabolic isometry. It fixes a point in the ideal boundary, and leaves each horosphere at the point invariant.

For integers $n, m$ consider the group given by the following presentation.

$$BS(n,m) =< a, b | ab^n a^{-1} = b^m > .$$

Those groups are called Baumslag-Solitar groups.

We are interested in $BS(1, m)$, which is solvable. There are several facts of interest from our viewpoint about this group (cf. [FSY]). Let $m \geq 2$.

- There is a finite simplicial complex of dimension 2 such that its fundamental group is $BS(1, m)$ and its universal cover is contractible. Therefore the cohomological dimension of the group is 2.
- $BS(1, m)$ acts on the hyperbolic plane, $\mathbb{H}^2$, by isometries, faithfully. But the action can not be proper.
- There exists a CAT(0) space of dimension 3 on which $BS(1, m)$ acts by isometries, properly.

It would be interesting to answer the following question.
**Question.** Let $m \geq 2$. Suppose $BS(1, m)$ acts on some CAT(0) space, $X$, by isometries, properly. Then $\dim X \geq 3$ ?

# 参考文献

[BH] Martin R. Bridson, Andre Haefliger, "Metric spaces of non-positive curvature". Grundlehren der Mathematischen Wissenschaften 319. Springer, 1999.

[FSY] K.Fujiwara, T.Shioya, S.Yamagata. Parabolic isometries of CAT(0) spaces and CAT(0) dimensions. preprint. GT/0308274. 2003.